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STABILITY OF HÖLDER ESTIMATES FOR $\bar{\partial}$ ON PSEUDOCONVEX DOMAINS OF FINITE TYPE IN $\mathbb{C}^2$

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Abstract. Let Ω be a smoothly bounded pseudoconvex domain in $\mathbb{C}^2$ and let $b\Omega$ be of finite type m. Then we prove the stability of Hölder estimates for $\bar{\partial}$ under some perturbations of $b\Omega$. As an application, we prove the Mergelyan property with respect to $C^\alpha(\bar{\Omega})$ norms for $0 \leq \alpha < 1/m$.

§1. Introduction

Methods of integral representations for estimating solutions for $\bar{\partial}$-equation in several complex variables have been successfully used for strongly pseudoconvex domains [G-L, H, R1]. For weakly pseudoconvex domains of finite type in $\mathbb{C}^2$, Range [R2] has introduced a method for constructing integral kernels on smoothly bounded pseudoconvex domains. This method was based on Skoda's L^2 estimates [S] for holomorphic solutions $h_j(p,z)$, $j = 1, 2$, of the division problem

$$h_1(p,z)(z_1 - p_1) + h_2(p,z)(z_2 - p_2) = 1, \quad p \in b\Omega, \ z \in \Omega.$$

He has used the detailed geometric analysis of Catlin [C], near a boundary point $p_0 \in b\Omega$ of finite type to get pointwise estimates of $h_j(p,z)$, $z \in \Omega$, $j = 1, 2$. The result was:

THEOREM. ([R2]) *Let Ω be a smoothly bounded pseudoconvex domain in $\mathbb{C}^2$ of finite type m, and let $f \in C^1_{0,1}(\bar{\Omega})$ be $\bar{\partial}$-closed. Then for every $\eta > 0$, there is a solution $u^{(\eta)}$ of $\bar{\partial}u = f$ on Ω which satisfies*

$$|u^{(\eta)}(z) - u^{(\eta)}(w)| \leq C_\eta \|f\|_{L^\infty(\Omega)} |z - w|^{\frac{1}{m} - \eta} \tag{1.1}$$

for $z, w \in \Omega$. The constant C_η is independent of f.

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Throughout this paper, Ω will be a smoothly bounded pseudoconvex domain in $\mathbb{C}^2$ and $b\Omega$ is of finite type. In this paper, we will prove that the estimates in (1.1) are stable under suitable perturbations of the domain Ω.

DEFINITION 1.1. Let $\Omega \subset \mathbb{C}^n$ be a smoothly bounded pseudoconvex domain with C^∞ defining function r. By a smooth bumping family of Ω we mean a family of smoothly bounded pseudoconvex domains $\{\Omega^t\}_{0\le t\le 1}$ satisfying the following properties:

(1) $\Omega^0 = \Omega$

(2) $\Omega^{t_1} \Subset \Omega^{t_2}$ if $t_1 < t_2$,

(3) $\{b\Omega^t\}_{0\le t\le 1}$ is a C^∞ family of real hypersurfaces in $\mathbb{C}^n$,

(4) the boundary defining functions r^t of Ω^t varies smoothly with respect to t and $r^t \to r$ as $t \to 0$ in C^∞ topology.

Remark 1.2. In [Ch], the first author constructed a smooth bumping family of Ω if $b\Omega$ is of finite type. Also he showed that there is a family of diffeomorphisms $\{\phi_t\}$, $\phi_t : \Omega \to \Omega^t$, such that $\phi_0 : \Omega \to \Omega$ is an identity and the complex structure on Ω^t is C^∞ close to the complex structure on Ω as $t \to 0$.

Now we state our main result:

THEOREM 1.3. *Let Ω be a smoothly bounded pseudoconvex domain in $\mathbb{C}^2$ of finite type m, and let $\{\Omega^t\}_{0\le t\le 1}$ be a smooth bumping family of Ω. Then there is $t_\Omega > 0$, depending only on Ω, such that for each $\eta > 0$ and $t < t_\Omega$, and for each $\overline{\partial}$-closed function $f^t \in C^1_{0,1}(\overline{\Omega}^t)$, there is a solution $u_t^{(\eta)}$ of $\overline{\partial}u = f^t$ on Ω^t which satisfies*

$$|u_t^{(\eta)}(z) - u_t^{(\eta)}(w)| \le C_\eta \|f^t\|_{L^\infty(\Omega^t)} |z-w|^{\frac{1}{m}-\eta}, \tag{1.2}$$

for $z, w \in \Omega^t$. The constant C_η is independent of f and t.

For each $\alpha \ge 0$, we denote $\|\ \|_{C^\alpha(\overline{\Omega})}$ the Hölder norm of order α on $\overline{\Omega}$. Here $\|\ \|_{C^0(\overline{\Omega})}$ is the supremum norm on $\overline{\Omega}$. As an application of Theorem 1.3, we prove the following Mergelyan property:

THEOREM 1.4. *Let Ω be a smoothly bounded pseudoconvex domain in $\mathbb{C}^2$ of finite type m, and let $f \in H(\Omega) \cap C^{\alpha'}(\overline{\Omega})$, $0 \le \alpha' \le 1/m$. Then for each $\alpha < \alpha'$ ($\alpha = \alpha'$ if $\alpha' = 0$), there is a sequence $\{g_n\} \subset H(\overline{\Omega})$ such that*

$$\lim_{n\to\infty} \|g_n - f\|_{C^\alpha(\overline{\Omega})} = 0.$$

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§2. Stability of local geometry under perturbations

In this section we will investigate how the local geometry of Ω changes under small perturbations of the boundary of Ω near a point $p_0 \in b\Omega$ of finite type m.

Let $r(z)$ be a defining function for Ω. We may assume that there is a coordinate functions z_1, z_2 defined near p_0 such that $|\frac{\partial r}{\partial z_2}(z)| \geq c > 0$ for all z in a small neighborhood U of p_0, for some constant $c > 0$. Let us fix $p \in U$ for a moment. Then we have the following special coordinates $\zeta = \zeta(p) = (\zeta_1, \zeta_2)$.

PROPOSITION 2.1. ([C , Proposition 1.1]) *For each $p \in U$, there is a biholomorphism $\Phi_p : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$, $\Phi_p(0) = p$, $\Phi(\zeta) = z$, such that the domain $D_p = \Phi_p^{-1}(\Omega)$ has a defining function given by*

$$\rho(\zeta) = r \circ \Phi_p(\zeta) = r(p) + \operatorname{Re} \zeta_2 + \sum_{\substack{j+k \leq m \\ j,k>0}} a_{j,k}(p) \zeta_1^j \bar{\zeta}_1^k + \mathcal{O}(|\zeta_1|^{m+1} + |\zeta_2||\zeta|).$$

Here Φ_p and the coefficients $a_{j,k}(p)$ depend smoothly on $p \in U$. For $l = 2, \ldots, m$, and $\delta > 0$, set

$$A_l(p) = \max\{|a_{j,k}(p)| : j + k = l\},$$

and

$$\tau(p, \delta) = \min\{(\delta / A_l(p))^{\frac{1}{l}} : 2 \leq l \leq m\}.$$

Since p_0 is of finite type m, it follows that $A_m(p_0) \neq 0$, and hence $A_m(p) \neq 0$, for $p \in U$. So

$$\delta^{\frac{1}{2}} \lesssim \tau(p, \delta) \lesssim \delta^{\frac{1}{m}}, \quad p \in U,$$

provided that U is sufficiently small. Now consider a smooth bumping family $\{\Omega^t\}_{0 \leq t \leq 1}$ of Ω with defining functions $\{r^t\}$. Let us investigate how the quantity $\tau(p, \delta)$ changes as t (and hence r^t) varies.

If we apply Proposition 2.1 at the point p with r replaced by r^t, then we obtain another special coordinates $\zeta = \zeta^t(p) = (\zeta_1, \zeta_2)$ about p, defined by the biholomorphic map $\Psi_p^t : \mathbb{C}^2 \to \mathbb{C}^2$, $z = \Psi_p^t(\zeta)$, $\Psi_p^t(0) = p$, and the defining function $\rho^t = r^t \circ \Psi_p^t$, for the domain $D_p^t = (\Psi_p^t)^{-1}(\Omega^t)$, has the form

$$\rho^t(\zeta) = r^t(p) + \operatorname{Re} \zeta_2 + \sum_{\substack{j+k \leq m \\ j,k>0}} a_{j,k}^t(p) \zeta_1^j \bar{\zeta}_1^k + \mathcal{O}(|\zeta_1|^{m+1} + |\zeta_2||\zeta|). \tag{2.1}$$

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Note that Ψ_p^t is given by

$$\Psi_p^t = \Psi^1 \circ \cdots \circ \Psi^m,$$

where

$$\Psi^1(u) = (p_1 + u_1, p_2 + \left(\frac{\partial r^t}{\partial z_2}(p)\right)^{-1} \left(\frac{u_2}{2} - \frac{\partial r^t}{\partial z_1}(p)u_1\right)),$$
$$\Psi^\nu(\zeta) = (\zeta_1, \zeta_2 - \frac{2}{\nu!}\frac{\partial^\nu s_\nu^t}{\partial u_1^\nu}(0)\zeta_1^\nu), \quad \nu = 2, \ldots, m,$$

and where

$$s_{\nu+1}^t = s_\nu^t \circ \Psi^\nu, \text{ and } s_1^t = r^t. \tag{2.2}$$

Therefore Ψ_p^t can be written as

$$\Psi_p^t(\zeta) = (p_1 + \zeta_1, p_2 + d_0(t,p)\zeta_2 + \sum_{k=1}^m d_k(t,p)\zeta_1^k),$$

where

$$\begin{aligned} d_0(t,p) &= \frac{1}{2}\left(\frac{\partial r^t}{\partial z_2}(p)\right)^{-1}, \text{ and} \\ d_k(t,p) &= -\frac{1}{k!}\left(\frac{\partial r^t}{\partial z_2}(p)\right)^{-1} \frac{\partial^k s_k^t}{\partial u_1^k}(0), \quad k = 1, 2, \ldots, m. \end{aligned} \tag{2.3}$$

Set $r = r^0, \rho = \rho^0, s_\nu = s_\nu^0, \Phi_p = \Psi_p^0$. To investigate how the quantity $\tau(p, \delta)$ change as t varies, we have to compare $a_{j,k}^t(p)$ with $a_{j,k}(p)$. The defining function r^t of Ω^t can be written as

$$r^t = r + tG,$$

where G is a smooth function of z. Therefore we get

$$\rho^t = r^t \circ \Psi_p^t = r \circ \Psi_p^t + t(G \circ \Psi_p^t). \tag{2.4}$$

Throughout this paper, we denote by $\mathcal{O}(1)$ the bounded functions or bounded vector valued funtions in $\overline{U}$. We need two lemmas.

LEMMA 2.2. *For each $p \in U$ and for all sufficiently small $t > 0$, we have*

$$d_j(t,p) = d_j(0,p) + t\mathcal{O}(1), \qquad j = 0, 1, \ldots, m.$$

Proof. By (2.3), we see that for all sufficiently small t,

$$d_0(t,p) = \frac{1}{2}\left(\frac{\partial r^t}{\partial z_2}(p)\right)^{-1} = \frac{1}{2}\left(\frac{\partial r}{\partial z_2}(p) + t\left(\frac{\partial G}{\partial z_2}(p)\right)\right)^{-1}$$
$$= \frac{1}{2}\left(\frac{\partial r}{\partial z_2}(p)\right)^{-1} + t\mathcal{O}(1) = d_0(0,p) + t\mathcal{O}(1),$$

because $|\frac{\partial r}{\partial z_2}| \geq c > 0$. By induction on ν in (2.2), it is easy to show that $s^t_\nu = s_\nu + t\mathcal{O}(1)$. Hence for all sufficiently small $t > 0$, we get that

$$d_k(t,p) = -\frac{1}{k!}\left(\frac{\partial r^t}{\partial z_2}(p)\right)^{-1}\frac{\partial^k s^t_k}{\partial u_1^k}(0)$$
$$= -\frac{1}{k!}\left(\frac{\partial r^t}{\partial z_2}(p)\right)^{-1}\left[\frac{\partial^k s_k}{\partial u_1^k}(0) + t\mathcal{O}(1)\right]$$
$$= -\frac{1}{k!}\left(\frac{\partial r}{\partial z_2}(p)\right)^{-1}\frac{\partial^k s_k}{\partial u_1^k}(0) + t\mathcal{O}(1) = d_k(0,p) + t\mathcal{O}(1),$$

for $k = 1, 2, \ldots, m$. □

LEMMA 2.3. *For each $p \in U$ and for all sufficiently small $t > 0$, we have*

$$a^t_{j,k}(p) = a_{j,k}(p) + t\mathcal{O}(1).$$

Proof. From (2.1) and (2.4), we can get

$$a^t_{j,k}(p) = \frac{1}{j!k!}\frac{\partial^{j+k}\rho^t}{\partial\zeta_1^j\partial\bar{\zeta}_1^k}(0) = \frac{1}{j!k!}\frac{\partial^{j+k} r\circ\Psi^t_p}{\partial\zeta_1^j\partial\bar{\zeta}_1^k}(0) + t\mathcal{O}(1).$$

Hence it suffices to show that

$$\frac{1}{j!k!}\frac{\partial^{j+k} r\circ\Psi^t_p}{\partial\zeta_1^j\partial\bar{\zeta}_1^k}(0) = \frac{1}{j!k!}\frac{\partial^{j+k} r\circ\Psi^0_p}{\partial\zeta_1^j\partial\bar{\zeta}_1^k}(0) + t\mathcal{O}(1). \tag{2.5}$$

By Lemma 2.2, we can write $\Psi^t_p = \Psi^0_p + tF$, where F is a smooth vector valued function on $\overline{U}$. Hence we have $r\circ\Psi^t_p = r\circ\Psi^0_p + t\mathcal{O}(1)$, and this proves (2.5).

For $l = 2, \ldots, m$ and $\delta > 0$, set

$$A^t_l(p) = \max\{|a^t_{j,k}(p)| : j + k = l\}, \tag{2.6}$$

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and

$$\tau^t(p,\delta) = \min\{(\delta/A_l^t(p))^{\frac{1}{l}} : 2 \le l \le m\}. \tag{2.7}$$

Note that there is a constant $c > 0$ so that $|A_m^0(p)| \ge c > 0$ for all $p \in U$. Then by virtue of Lemma 2.3, we have $|A_m^t(p)| \ge \frac{c}{2}$ for all $p \in U$ provided that t is sufficiently small. Hence $\tau^t(p,\delta)$ is well defined for all sufficiently small t and satisfies the relation

$$\frac{1}{C}\delta^{\frac{1}{2}} \le \tau^t(p,\delta) \le C\delta^{\frac{1}{m}}, \tag{2.8}$$

for some constant $C > 0$ independent of t and p. Also the definition of $\tau^t(p,\delta)$ easily implies that if $\delta' < \delta''$, then

$$(\delta'/\delta'')^{\frac{1}{2}}\tau^t(p,\delta'') \le \tau^t(p,\delta') \le (\delta'/\delta'')^{\frac{1}{m}}\tau^t(p,\delta'').$$

Now define

$$R_\delta^t(p) = \{\zeta \in \mathbb{C}^2;\ |\zeta_1| < \tau^t(p,\delta) \text{ and } |\zeta_2| < \delta\}$$

and

$$Q_\delta^t(p) = \{\Psi_p^t(\zeta); \zeta \in R_\delta^t(p)\}.$$

In the sequal we denote D_1^l any partial derivative operator of the form $\frac{\partial^l}{\partial\zeta_1^\mu\partial\bar{\zeta}_1^\nu}$, where $\mu + \nu = l$. If we combine the definitions in (2.6), (2.7) and by virtue of Taylor's theorem, we can easily get the following derivative estimates of ρ^t.

Proposition 2.4. *Let p be an arbitrary point in U. There are independent constants $C > 0$ and $t_0 > 0$ such that the function $\rho^t = r^t \circ \Psi_p^t$ satisfies*

$$|\rho^t(\zeta) - \rho^t(0)| \le C\delta,\ \zeta \in R_\delta^t(p),\ \text{and} \tag{2.9}$$

$$|D_1^l\rho^t(\zeta)| \le C(\tau^t(p,\delta))^{-l},\ \zeta \in R_\delta^t(p),\ l = 1,2,\dots,m, \tag{2.10}$$

for all $0 \le t \le t_0$.

By Lemma 2.2, it follows that there is an independent constant $C > 0$ such that $|d_0(t,p)| \le C$ for all sufficiently small $t \ge 0$ and for all $p \in U$. Thus $|\frac{\partial\rho^t}{\partial\zeta_2}| \ge \frac{1}{C}$ in U, for all sufficiently small t. This fact and (2.8), (2.9), (2.10) in Proposition 2.4 imply that all the constants in the theorems in section 1–3 in [C] are stable with respect to t. Let $\Omega^t(\epsilon) = \{z; r^t(z) < \epsilon\}$ and define

$$S^t(\epsilon) = \{z; -\epsilon < r^t(z) < \epsilon\} \text{ and}$$
$$S_-(\epsilon)^t = \{z; -\epsilon < r^t(z) \leq 0\}.$$

Let $\{L_1, L_2\}$ be the local frame on $\overline{U}$ satisfying $L_1 r^t = 0, L_2 = \frac{\partial}{\partial z_2}$. Then, in particular, the following important theorem holds.

PROPOSITION 2.5. *There is a constant $c > 0$ (independent of t and δ) such that for all small $\delta > 0$ and small $t > 0$, there is a plurisubharmonic function $\lambda^t_\delta \in C^\infty(U \cap \Omega^t(\delta))$ with the following properties:*

(1) $|\lambda^t_\delta(z)| \leq 1,\ z \in U \cap \Omega^t(\delta)$,

(2) *for all $L = s_1 L_1 + s_2 L_2$ at z, where $z \in U \cap S^t(\delta)$,*

$$\partial\bar\partial\lambda^t_\delta(L, \bar L) \geq c(|s_1|^2(\tau^t(p,\delta))^{-2} + |s_2|^2\delta^{-2}),$$
$$\partial\bar\partial\lambda^t_\delta(L, \bar L) \geq c|L\lambda^t_\delta|^2, \text{ and}$$

(3) *if Ψ^t_p is the map associated with a given $p \in U \cap S^t(\delta)$, then for all $\zeta \in R^t_\delta(p)$ with $|\rho^t(\zeta)| < \delta$,*

$$|D^k_1 D^l_2(\lambda^t_\delta \circ \Psi^t_p(\zeta))| \leq C_{k,l}(\tau^t(p,\delta))^{-k}\delta^{-l},$$

where $C_{k,l}$ does not depend on t.

Now set $U' = \{\zeta : \Psi^0_p(\zeta) \in U\}$, and define

$$J^t_\delta(p,\zeta) = [\delta^2 + |\zeta_2|^2 + \sum_{k=2}^{m} A^t_k(p)^2|\zeta_1|^{2k}]^{\frac{1}{2}} \text{ and}$$
$$W^t_{s,\delta}(p) = \{\zeta \in U' : |\rho^t(\zeta)| < sJ^t_\delta(\zeta)\}.$$

PROPOSITION 2.6. *There is an independent constant $C > 0$ such that for each $p \in U \cap b\Omega^t$ and each small $\delta > 0$ and $t > 0$ there exists a smooth real valued function $H^t_{p,\delta}(\zeta)$ defined in $W^t_{s,\delta}(p)$ (where s is a small constant independent of p, δ and t) such that*

(1) $-CJ^t_\delta(p,\zeta) \leq H^t_{p,\delta}(\zeta) \leq -\frac{1}{C}J^t_\delta(p,\zeta)$,

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(2) *for any* $L = s_1 L_1' + s_2 L_2'$ *at* ζ,

$$\frac{1}{C}\partial\bar{\partial}H^t_{p,\delta}(L,\bar{L})(\zeta) \le J^t_\delta(p,\zeta)[\frac{|s_1|^2}{\tau^t(p,J^t_\delta(p,\zeta))^2} + \frac{|s_2|^2}{J^t_\delta(p,\zeta)^2}], \text{ and}$$

(3) *for any* $L = s_1 L_1' + s_2 L_2'$ *at* ζ,

$$|LH^t_{p,\delta}| \le CJ^t_\delta(p,\zeta)(\frac{|s_1|}{\tau^t(p,J^t_\delta(p,\zeta))} + \frac{|s_2|}{J^t_\delta(p,\zeta)}),$$

where $L_i' = (\Psi^t_p)^{-1}_* L_i, \quad i = 1,2.$

Proof. Note that the proof of Proposition 4.1 in [C] only uses the properties of λ^t_δ in Proposition 2.5. Since the estimates in Proposition 2.5 are stable with respect to t, we can prove Proposition 2.6 by the method similar as Proposition 4.1 in [C].

Set $\rho^{\epsilon,t}_{p,\delta} = \rho^t(\zeta) + \epsilon H^t_{p,\delta}(\zeta)$, with $\epsilon > 0$. Since $\frac{\partial\rho}{\partial x_2} \approx 1$ (where $\zeta_2 = x_2 + iy_2$), we also have $\frac{\partial\rho^t}{\partial x_2} \approx 1$ for all sufficiently small t. Thus it follows that for all sufficiently small ϵ (independent of t), $\frac{\partial\rho^{\epsilon,t}_{p,\delta}}{\partial x_2} \approx 1$. Then as Catlin did in [C], the set $S^{t,\epsilon}_\delta = \{\zeta \in W^t_{s,\delta}(p) : \rho^{\epsilon,t}_{p,\delta} = 0\}$ is a smooth pseudoconvex hypersurface (from the side $\rho^{\epsilon,t}_{p,\delta} < 0$) for all sufficiently small t. Let us fix $\epsilon = \epsilon_0$ and set $\rho^t_{p,\delta} = \rho^{\epsilon_0,t}_{p,\delta}$. For $t,\delta \ge 0$ and $a > 0$, we define the nonisotropic polydiscs $P^a_\delta(t,\zeta')$ centered at ζ' by

$$P^a_\delta(t,\zeta') = \{\zeta \in \mathbb{C}^2 : |\zeta_2 - \zeta_2'| < aJ^t_\delta(p,\zeta'),\ |\zeta_1 - \zeta_1'| < \tau^t(p, aJ^t_\delta(p,\zeta'))\}.$$

We now state the main result of this section.

PROPOSITION 2.7. *There are positive constants a and c (independent of p,δ and t) such that for each $p \in U \cap b\Omega^t$ where t is sufficiently small, there is a pseudoconvex domain D^{t*}_p with the following properties:*

(1) $0 \in bD^{t*}_p$,

(2) $\{\zeta \in \overline{D^t_p} : 0 < |\zeta| < c\} \subset D^{t*}_p$,

(3) *for* $\zeta' \in D^t_p$ *with* $|\zeta'| < a$ *one has* $P^a_0(t,\zeta') \subset D^{t*}_p$.

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Proof. Set $D_p^{\delta,t} = \{|\zeta| < c : \rho^t(\zeta) < 0\} \cup \{\zeta \in W_{s,\delta}^t(p) : \rho_{p,\delta}^t(\zeta) < 0\}$. Then $D_p^{\delta,t}$ is a pseudoconvex domain which satisfies

$$\overline{D_p^t} \cap \{|\zeta| < c\} \subset D_p^{\delta,t} \subset (D_p^t \cap \{|\zeta| < c\}) \cup W_{s,\delta}^t(p).$$

Define $D_p^{t*} = \operatorname{int} \cap_{0<\delta\le\delta_0} D_p^{\delta,t}$. Then D_p^{t*} is pseudoconvex and we can prove properties (1), (2) and (3) following Range's proof of Proposition 2.4 in [R2].

§3. Stability of estimates of holomorphic generating form

In this section, we will show the stability of some pointwise estimates for holomorphic L^2 functions. We will use Proposition 2.7 and Cauchy estimate on $P_0^{\frac{a}{2}}(t,z) \subset D_p^{t*}$ for a fixed $z \in D_p^t$. Set $\delta^t(z) = \operatorname{dist}(z, bD_p^{t*})$ for $z \in D_p^t$. Suppose $h \in \mathcal{O}(D_p^{t*})$ satisfies

$$(M_\eta^t)^2 = \int_{D_p^{t*}} \frac{|h(\zeta)|^2}{|\zeta|^2} (\delta_p^t)^{2\eta}(\zeta)\, dV(\zeta) < \infty \tag{3.1}$$

for some $\eta > 0$. Set $\beta = (a/2)J^t(p,z)$. We may assume $\beta \le 1$ by choosing a sufficiently small for all small t. Then from (2.8), it follows that there exists $C > 0$ so that $\frac{1}{C}\beta^{\frac{1}{2}} \le \tau^t := \tau^t(p,\beta) \le C\beta^{\frac{1}{m}}$. If we use Proposition 2.7 and Cauchy estimates on $P_0^{\frac{a}{2}}(t,z) \subset D_p^{t*}$, we obtain the following estimates as Range did in [R2].

$$\begin{aligned} |h(z)| &\le C_2 C^{1+\eta}\left[\frac{1}{\beta^{1+\eta}} + \frac{|z|}{\tau^t\beta^{1+\eta}}\right] M_\eta^t; \\ \left|\frac{\partial h}{\partial z_1}(z)\right| &\le C_2 C^{1+\eta}\left[\frac{1}{\tau^t\beta^{1+\eta}} + \frac{|z|}{(\tau^t)^2\beta^{1+\eta}}\right] M_\eta^t; \text{ and} \\ \left|\frac{\partial h}{\partial z_2}(z)\right| &\le C_2 C^{1+\eta}\left[\frac{1}{\beta^{2+\eta}} + \frac{|z|}{\tau^t\beta^{2+\eta}}\right] M_\eta^t, \end{aligned} \tag{3.2}$$

for all sufficiently small t where C_2 depends only on the dimension of the domain.

By Lemma 2.3 and Proposition 2.7, it follows that $a \cdot \min(1, \sum_{k=2}^m A_k^t(p)) \ge c > 0$ for all sufficiently small t. Hence by estimating β and τ^t from below, we see that

$$\beta \gtrsim |\rho^t(z)| + |z_2| + |z|^m \text{ and} \tag{3.3}$$

$$\tau^t \gtrsim |z|, \tag{3.4}$$

hold uniformly for all sufficiently small t. Since Ψ_p^t is a biholomorphism which changes smoothly as t varies, Proposition 2.7 implies that there are positive constants c and γ which are independent of p and t (for all small t) such that

$$r^t \geq c|z-p|^m \text{ for } |z-p| \leq \gamma \text{ and } z \notin \Psi_p^t(D_p^{t*}).$$

Since a pseudoconvex domain $\Omega \subset \mathbb{C}^n$ of finite type is regular [Ch], it follows that there is a pseudoconvex domain Ω^* with $\overline{\Omega^0} \subset \Omega^* \Subset \{z : r(z) < c\gamma^m\}$. Choose $t_1 > 0$ so that

$$0 \leq \operatorname{dist}(b\Omega^0, b\Omega_t) < \operatorname{dist}(b\Omega^0, b\Omega^*) \text{ for all } 0 \leq t \leq t_1.$$

Let $\mu = \sup\{r^t(\zeta) : \zeta \in b\Omega^*, 0 \leq t \leq t^1\}$. Then $0 < \mu < c\gamma^m$ and we can choose $\gamma' > 0$ such that $0 < \mu < c(\gamma')^m < c\gamma^m$. Define Ω_p^t by

$$\Omega_p^t := [\Omega^* \cap \{z : |z-p| < \gamma\} \cap \Psi_p^t(D_p^{t*})] \cup (\Omega^* \cap \{z : |z-p| > \gamma'\}.$$

Then Ω_p^t for $0 \leq t \leq t_1$ is still pseudoconvex.

For a sufficiently small $b > 0$, we set $U_0 = U \cap S^0(b) = \{z \in U : -b < r(z) < b\}$ and choose $t_2 > 0$ so that $U \cap b\Omega^t \subset U \cap S^0(\frac{b}{2})$ for all $0 \leq t < t_2$. We set $\delta_p^t(z) = \operatorname{dist}(z, b\Omega_p^t)$ and given $\eta > 0$, we define the weighted L^2 norm $I_{p,\eta}^t$ on Ω_p^t by

$$I_{p,\eta}^t(h) = \left[\int_{\Omega_p^t} \frac{|h(z)|^2}{|z-p|^2}(\delta_p^t(z))^{2\eta} dV(z)\right]^{\frac{1}{2}}.$$

Also we let $g^t(p,\cdot)$ denote the second component of the inverse of the biholomorphic map Ψ_p^t and $\{L_1^t, L_2^t\}$ be a fixed orthonormal frame for $T^{(1,0)}$ on a neighborhood of $\overline{U}_0$ which satisfies $L_1^t r^t(p) = 0$. If we use the relations in (3.3) and (3.4), we obtain the following stable estimates.

PROPOSITION 3.1. *There are constants C' and a such that for all $p \in U_0 \cap b\Omega^t$ and for all sufficiently small t, the following holds: If $h \in \mathcal{O}(\Omega_p^t)$ and $I_{p,\eta}^t(h)$ is finite for some $\eta > 0$, then*

(1) $|h(z)| + |dh(z)| \leq C' I_{p,\eta}^t(h)$ *for $z \in \Omega^t$ with $|z-p| \geq a$, and if $z \in \Omega^t$ with $|z-p| < a$, then*

(2)

$$|h(z)| \leq C' \frac{I_{p,\eta}^t(h)}{(|r^t(z)| + |g^t(p,z)| + |z-p|^m)^{1+\eta}},$$

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(3)

$$|L_1^t h(z)| \leq C' \frac{I_{p,\eta}^t(h)}{|z-p|(|r^t(z)| + |g^t(p,z)| + |z-p|^m)^{1+\eta}}, \quad \text{and}$$

(4)

$$|L_2^t h(z)| \leq C' \frac{I_{p,\eta}^t(h)}{(|r^t(z)| + |g^t(p,z)| + |z-p|^m)^{2+\eta}},$$

where C' is independent of t and h.

Proof. Note that for $z \in \Omega^t$ with $|z-p| \geq a$, one has $\delta_p^t(z) \geq \operatorname{dist}(\overline{\Omega}^t, b\Omega^*) > 0$. So (1) follows from the Cauchy estimates. For all sufficiently small $t \leq \min\{t_0, t_1, t_2\}$, we replace β and τ^t with $|\rho^t(z)| + |z_2| + |z|$ and $|z|$ respectively. Note that $\rho^t = r^t \circ \Psi_p^t$ and Ψ_p^t are smooth functions in p and t. So the Jacobian determinants of Ψ_p^t and $(\Psi_p^t)^{-1}$ with respect to z are uniformly bounded by a constant as t and p vary slightly. So we pull back the estimates given by (3.2) via the map $(\Psi_p^t)^{-1}$ to get the estimates in (2), (3) and (4). ∎

Next, we will prove the existence of a holomorphic generating form with stable estimates. For the existence, one can refer Theorem 4.2 in [R2]. Let $\eta > 0$ be given. We apply Theorem 1 in [S] to pseudoconvex domains $\Omega_p^t \subset \mathbb{C}^2$ with $\alpha = 1 + \eta/2, q = 1, f = 1, g = z - p$, and $\psi = -2\eta \log \delta_p^t$. Note that $e^{-\psi(z)} = (\delta_p^t(z))^{2\eta} \geq |z-p|^{2\eta}$. Thus

$$\begin{aligned} I_{p,\eta}^t(h) &\leq \frac{2+\eta}{\eta} \int_{\Omega_p^t} 1 \cdot |z-p|^{-2\alpha-2} e^{-\psi} dV \\ &\leq \frac{2+\eta}{\eta} \int_{\Omega^*} |z-p|^{-4+\eta} dV \\ &\leq \frac{2+\eta}{\eta} \int_B |z-p|^{-4+\eta} dV \leq C'_\eta < \infty, \end{aligned} \tag{3.5}$$

where B is a large ball containing Ω^*, and C'_η depends only on η. Let us fix $t \geq 0$ for a moment. Then there are holomorphic functions $h_{\eta,j}^t(p,\cdot)$ which satisfies the equation

$$h_{\eta,1}^t(p,z)(z_1 - p_1) + h_{\eta,2}^t(p,z)(z_2 - p_2) = 1, \quad z \in \Omega_p^t.$$

Set $h_j^t = h_{\eta,j}^t$ for convenience. In general, we can not guarantee the smoothness of $h_j^t(p,z)$ on $p \in b\Omega^t$. We need some modification. For $\epsilon > 0$, let $\Omega_\epsilon^t = \{z : r^t(z) < -\epsilon\} \Subset \Omega^t$. For each $p \in U_0 \cap b\Omega^t$, define $\phi_p^t(\zeta, z) = \sum_{j=1}^2 h_j^t(p,z)(z_j - \zeta_j)$. Then $\phi_p^t \in C^\infty(\mathbb{C}^2 \times \Omega_p^t)$ is holomorphic in z and $\phi_p^t(p,z) \equiv 1$ on Ω_p^t. Thus, given $\epsilon > 0$, there is a neighborhood $V_{p,\epsilon}^t$ of p such that

$$|\phi_p^t(\zeta,z)| \geq \frac{1}{2} \quad \text{on} \quad (V_{p,\epsilon}^t \cap b\Omega^t) \times \overline{\Omega}_\epsilon^t. \tag{3.6}$$

Set $h_{j,p}^t(\zeta,z) = h_j^t(p,z)/\phi_p^t(\zeta,z)$ and shrink U_0 so that all the above hold for every $q \in U_0 \cap b\Omega^t$ and denote it again by U_0. Let us cover $\overline{U}_0 \cap b\Omega^t$ by finitely many sets $\{V_{q_1,\epsilon}^t, \dots, V_{q_l,\epsilon}^t\}$. Then there is a partition of unity $\{\chi_\nu^t \in C_0^\infty(V_{q_\nu,\epsilon}^t) : \nu = 1, \dots, l\}$ which is subordinated to the covering $\{V_{q_1,\epsilon}^t, \dots, V_{q_l,\epsilon}^t\}$. Define $h_j^{t,\epsilon}(\zeta,z) = \sum_{\nu=1}^l \chi_\nu^t(\zeta) h_{j,q_\nu}^t(\zeta,z)$. Then for each sufficiently small ϵ and t, the following hold:

$$h_j^{t,\epsilon} \in C^\infty((U_0 \cap b\Omega^t) \times \Omega_\epsilon^t), \quad j = 1,2; \tag{3.7}$$

$$h_j^{t,\epsilon}(\zeta, \cdot) \in \mathcal{O}(\Omega_\epsilon^t) \text{ for } \zeta \in U_0 \cap b\Omega^t; \tag{3.8}$$

$$h_1^{t,\epsilon}(\zeta,z)(z_1 - \zeta_1) + h_2^{t,\epsilon}(\zeta,z)(z_2 - \zeta_2) = 1 \text{ on } (U_0 \cap b\Omega^t) \times \Omega_\epsilon^t. \tag{3.9}$$

Note that if t and ϵ are sufficiently small, then $|r^t(z)| - \epsilon \gtrsim \operatorname{dist}(z, b\Omega_\epsilon^t)$ uniformly for t and $z \in \Omega_\epsilon^t$. Combining Proposition 3.1 , (3.5) and (3.6), we get the following proposition.

PROPOSITION 3.2. *For all sufficiently small $t \geq 0$ and $\epsilon > 0$, above $h_j^{t,\epsilon}$ satisfies the following stable estimates:*

(1) $|h_j^{t,\epsilon}(\zeta,z)| + |d_z h_j^{t,\epsilon}(\zeta,z)| \leq C_\eta$ *for $z \in \Omega_\epsilon^t$ with $|z-\zeta| \geq a$;*

and for $z \in \Omega_\epsilon^t$ with $|z - \zeta| < a$,

(2) $|h_j^{t,\epsilon}(\zeta,z)| \leq \dfrac{C_\eta}{[\Gamma_\epsilon^t(\zeta,z)]^{1+\eta}}$;

(3) $|L_1^t h_j^{t,\epsilon}(\zeta,z)| \leq \dfrac{C_\eta}{|z-\zeta|[\Gamma_\epsilon^t(\zeta,z)]^{1+\eta}}$;

(4) $|L_2^t h_j^{t,\epsilon}(\zeta,z)| \leq \dfrac{C_\eta}{[\Gamma_\epsilon^t(\zeta,z)]^{2+\eta}}$,

where $\Gamma_\epsilon^t(\zeta,z) := \operatorname{dist}(z,\Omega_\epsilon^t) + |g^t(\zeta,z)| + |z-\zeta|^m$, and the functions $h_j^{t,\epsilon}$ depend on η and t, but the constant C_η is independent of $\epsilon > 0$ and $t \geq 0$.

The functions $h_j^{t,\epsilon}$ in (3.7) are locally defined. For the globally defined holomorphic generating form on $b\Omega^t \times \Omega_\epsilon^t$, we will patch together the function $h_j^{t,\epsilon}$ using a partition of unity. Note that for all sufficiently small $0 \le t < t_4$, U_0 is a neighborhood of $q \in b\Omega^t \cap (\frac{1}{2}U_0)$. Since $b\Omega^t$ is compact, finitely many neighborhoods, $U_0, U_1, \ldots, U_k$ (independent of t), will cover the set $\{b\Omega^t : 0 \le t < t_4\}$. Thus we may choose a partition of unity in ζ, subordinated to $\{U_j : j = 0, \ldots, k\}$, to patch the locally defined functions $h_j^{t,\epsilon}$ to obtain a globally defined smooth functions $w_j^{t,\epsilon}$ and we obtain a holomorphic generating form $W_\eta^{t,\epsilon} = \sum_{j=1}^2 w_j^{t,\epsilon} d\zeta_j$ on $b\Omega^t \times \Omega_\epsilon^t$.

§4. Proof of Main Theorems

In this section, we prove Theorem 1.3 and Theorem 1.4 using the estimates on the holomorphic generating form in Section 3.

Proof of Theorem 1.3. Integral operator for Ω_ϵ^t can be written as $T_{\eta,\epsilon}^t = S_1 + S_2$, where

$$S_1(f) = c_2 \int_{b\Omega^t} f \wedge W_\eta^{t,\epsilon} \wedge \partial_\zeta \log |z - \zeta|^2$$

and $S_2(f)$ involves integration of Bochner-Martinelli-Koppelman kernel over Ω^t. In standard method for Hölder estimate, the only nontrivial part is

$$c_2 \int_{b\Omega^t \cap \{|\zeta - z| < a\}} f \wedge \mathrm{d}_z(W_\eta^{t,\epsilon} \wedge \partial_\zeta \log |z - \zeta|^2). \tag{4.1}$$

From Proposition 3.2, it follows that (4.1) is uniformly bounded by

$$\begin{aligned} &C_\eta ||f||_\infty \int_{b\Omega^t \cap \{|\zeta - z| < a\}} \left[\frac{1}{|\zeta - z|^2 (\Gamma_\epsilon^t)^{1+\eta}} + \frac{1}{|z - \zeta| (\Gamma_\epsilon^t)^{2+\eta}} \right] \mathrm{d}S(\zeta) \\ &\le C_\eta ||f||_\infty \operatorname{dist}(z, b\Omega_\epsilon^t)^{-\eta} \int_{b\Omega^t \cap \{|\zeta - z| < a\}} \left[\frac{1}{|\zeta - z|^2 \Gamma_\epsilon^t} + \frac{1}{|z - \zeta| (\Gamma_\epsilon^t)^2} \right] \mathrm{d}S(\zeta). \end{aligned} \tag{4.2}$$

In order to estimate (4.2), we need a coordinate change. Here the main point is the choice of coordinate system $s^t(\zeta, z) = (s_1, s_2, s_3, s_4)$, where $s_1 = r^t(\zeta), s_2 = \operatorname{Im} g^t(\zeta, z)$ (See [R3,V Lemma 3.4]). Notice that $dr^t(p) \wedge d_\zeta \operatorname{Im} g^t(p,p) \neq 0$ for all small t and the Jacobian matrix $J_{\mathbb{R}}(s^t)$ depends only on the 1st derivatives of r^t and $\operatorname{Im} g^t$. In this coordinates,

$$\int_{b\Omega^t \cap \{|\zeta - z| < a\}} \left[\frac{1}{|\zeta - z|^2 (\Gamma_\epsilon^t)^{1+\eta}} + \frac{1}{|z - \zeta| (\Gamma_\epsilon^t)^{2+\eta}} \right] \mathrm{d}S(\zeta)$$

is estimated by $\text{dist}(z, b\Omega_\epsilon^t)^{-1+(\frac{1}{m}-\eta)}$. Thus we obtain

$$|\text{d}_z S_1(f)(z)| \le C_\eta ||f||_{L^\infty} \text{dist}(z, b\Omega_\epsilon^t)^{-1+(\frac{1}{m}-\eta)},$$

and this implies that for all $z, w \in \Omega_\epsilon^t$, it follows that

$$|(T_{\eta,\epsilon}^t f)(z) - (T_{\eta,\epsilon}^t f)(w)| \le C_\eta ||f||_{L^\infty(\Omega^t)} |z-w|^{\frac{1}{m}-\eta}.$$

By a suitable limiting argument as Range did in [R2], we obtain (1.2). □

Proof of Theorem 1.4. Let U_j, $j = 0, 1, \ldots, N$ be a finite collection of open sets with the following properties:

(1) $\overline{\Omega} \subset \cup_{j=0}^N U_j$,

(2) $U_0 \subset\subset \Omega$,

(3) On each U_j, $j = 1, 2, \ldots, N$, there are holomorphic coordinates z_1^j, z_2^j with the property that $\partial r / \partial x_2^j > 0$, where $z_2^j = x_2^j + i y_2^j$.

Let ζ_j, $j = 0, 1, \ldots, N$ be a partion of unity subordinate to the covering $\{U_j\}$. For a given $f \in H(\Omega) \cap C^{\alpha'}(\overline{\Omega})$, $0 \le \alpha' \le 1/m$, and for all small $\delta > 0$, let f_δ be given by

$$f_\delta(z) = \zeta_0(z) f(z) + \sum_{j=1}^N \zeta_j(z) f(z_1^j, z_2^j - \delta).$$

Let $\alpha < \alpha'$ ($\alpha = \alpha'$ if $\alpha' = 0$) be arbitrary given. Observe that $f_\delta \in C^\infty(\overline{\Omega})$ and satisfies

$$\lim_{\delta \to 0} ||f_\delta - f||_{C^\alpha(\overline{\Omega})} = 0, \text{ and } \lim_{\delta \to 0} ||\overline{\partial} f_\delta||_{C^\alpha(\overline{\Omega})} = 0. \tag{4.3}$$

Assume that $\epsilon > 0$ is arbitrary given. We choose $\delta_0 > 0$ sufficiently small so that

$$||f_\delta - f||_{C^\alpha(\overline{\Omega})} < \epsilon/3, \tag{4.4}$$

for all $\delta \le \delta_0$. Next, for each $\delta \le \delta_0$, we solve $\overline{\partial} p_\delta = \overline{\partial} f_\delta$ on $\overline{\Omega}$. Since $\overline{\partial} f_\delta \in C^\infty(\overline{\Omega})$ and since $b\Omega$ is of finite type, it follows that $p_\delta \in C^\infty(\overline{\Omega})$ and the estimates (1.1) give $||p_\delta||_{C^\alpha(\overline{\Omega})} \le C_\alpha ||\overline{\partial} f_\delta||_{L^\infty(\Omega)}$. If δ_0 is sufficiently small, it follows from (4.3) that

$$||p_\delta||_{C^\alpha(\overline{\Omega})} < \epsilon/3, \tag{4.5}$$

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for all $\delta \le \delta_0$. Set $h_\delta = f_\delta - p_\delta$. Then $h_\delta \in H(\Omega) \cap C^2(\overline{\Omega})$. We may assume that h_δ is well defined on a smooth bumping family $\{\overline{\Omega}^t\}_{t \le t_\Omega}$ of Ω, for all $\delta \le \delta_0$. Since $h_\delta \in C^2(\overline{\Omega})$ and $\overline{\partial} h_\delta \equiv 0$ on Ω, it follows for each $\delta > 0$ that

$$\lim_{t \to 0} \|\overline{\partial} h_\delta\|_{C^\alpha(\Omega^t)} = 0. \tag{4.6}$$

Now for each $t > 0$, we solve $\overline{\partial} h^t_{\delta_0} = \overline{\partial} h_{\delta_0}$ on $\overline{\Omega}^t$. From the stability of Hölder estimates for $\overline{\partial}$ (Theorem 1.3), and from the estimates in (4.6), there exists $t_0 > 0$ so that

$$\|h^{t_0}_{\delta_0}\|_{C^\alpha(\overline{\Omega}^{t_0})} < \epsilon/3. \tag{4.7}$$

Set $g_\epsilon = h_{\delta_0} - h^{t_0}_{\delta_0}$. Then $g_\epsilon \in H(\Omega^{t_{\delta_0}})$, and from (4.4), (4.5) and (4.7) it follows that $\|g_\epsilon - f\|_{C^\alpha(\overline{\Omega})} < \epsilon$.

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