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ON A NEW SINGULAR DIRECTION OF ALGEBROID FUNCTIONS

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In this paper, we prove that for an algebroid function $w(z)$ with finite lower order, satisfying $\limsup\limits_{r\to\infty}\big(T(r,w)/(\log r)^2\big) = \infty$, there exists a T direction dealing with multiple values.

1. INTRODUCTION AND MAIN RESULTS

The ν-valued algebroid function $w = w(z)(z \in \mathbf{C})$ is defined by the irreducible equation(see [**5, 6**])

$$A_\nu(z)w^\nu + A_{\nu-1}(z)w^{\nu-1} + \ldots + A_0(z) = 0, \tag{1}$$

where $A_\nu(z), \ldots, A_0(z)$ are all entire functions without any common zeros. In particular, $w(z)$ is a meromorphic function when $\nu = 1$. In 1930s, Valiron [**13**], Ullrich [**12**] and Seleberg [**10, 11**] generalised the value distribution theory of meromorphic functions(see [**4**]) to the corresponding theory of algebroid functions. The singular direction for $w(z)$ is one important objects studied in the theory of value distribution of algebroid functions. Many people, such as Valiron [**14**], Lü [**7**], Lănd Gu [**8**] and Gao and Wang [**2**], have studied the Julia direction and Borel direction of algebroid functions.

Recently, Zheng [**16**] introduced a new singular direction of meromorphic functions, called the T direction. Then Guo, Zheng and Ng [**3**] proved that a meromorphic function $f(z)$ must have at least one T direction, provided that $\limsup\limits_{r\to\infty}\big(T(r,f)/(\log r)^2\big) = \infty$. Thus a natural question is whether there exists a T direction for an algebroid function $w(z)$. In this paper, we investigate the existence of the T direction dealing with its multiple values for an algebroid function with finite lower order (that is, $\liminf\limits_{r\to\infty}\Big(\big(\log^+ T(r,w)\big)/\log r\Big) < \infty$), which implies that the algebroid function has a T direction, under the condition $\limsup\limits_{r\to\infty}\big(T(r,w)/(\log r)^2\big) = \infty$.

Let $w(z)$ is a ν-valued algebroid function, defined by (1). Then the single-valued domain $\widetilde{R}_z$ of its definition is a Riemann surface, which is a ν-sheet covering of the z-plane. A point in $\widetilde{R}_z$ is denoted by $\widetilde{z}$ if its projection in the z-plane is z. The open set

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which lies over $|z| < r$ is denoted by $|\widetilde{z}| < r$, and its boundary is denoted by $|\widetilde{z}| = r$. Let $n(r,a)$ be the number of a-points of $w(z)$ in $|\widetilde{z}| < r$, counted according to their multiplicities, and $\overline{n}^{l)}(r,a)$ be the number of distinct a-points of $w(z)$ with multiplicity $\leqslant l$ in $|\widetilde{z}| < r$. In addition, $\overline{n}(r,a)$ denotes the number of distinct a-points of $w(z)$ in $|\widetilde{z}| < r$, ignoring the multiplicity. Let

$$S(r,w) = \frac{1}{\pi}\int\int_{|\widetilde{z}|<r} \frac{|w'(\widetilde{z})|^2}{(1+|w(\widetilde{z})|^2)^2} d\omega = \frac{1}{\pi}\sum_{j=1}^{\nu}\int\int_{|z|<r}\frac{|w_j'(z)|^2}{(1+|w_j(z)|^2)^2}dxdy,$$

$$T(r,w) = \frac{1}{\nu}\int_0^r \frac{S(r,w)}{t}dt, \qquad N(r,a) = \frac{1}{\nu}\int_0^r \frac{n(t,a)-n(0,a)}{t}dt + \frac{n(0,a)}{\nu}\log r,$$

$$\overline{N}^{l)}(r,a) = \frac{1}{\nu}\int_0^r \frac{\overline{n}^{l)}(t,a)-\overline{n}^{l)}(0,a)}{t}dt + \frac{\overline{n}^{l)}(0,a)}{\nu}\log r,$$

$$\overline{N}(r,a) = \frac{1}{\nu}\int_0^r \frac{\overline{n}(t,a)-\overline{n}(0,a)}{t}dt + \frac{\overline{n}(0,a)}{\nu}\log r,$$

$$m(r,w) = \frac{1}{2\pi\nu}\int_{|\widetilde{z}|=r}\log^+|w(\widetilde{z})|d\theta = \frac{1}{\nu}\sum_{j=1}^{\nu} m(r,w_j) = \frac{1}{\nu}\sum_{j=1}^{\nu}\frac{1}{2\pi}\int_0^{2\pi}\log^+|w_j(re^{i\theta})|d\theta,$$

where $S(r,w)$ is called the mean covering number of $|\widetilde{z}| \leqslant r$ into the w-sphere, and we call $T(r,w)$ the characteristic function of $w(z)$. It is known from [**5**, pp. 84] that

$$T(r,w) = m(r,w) + N(r,w) + O(1).$$

Let $n(r,\widetilde{R}_z)$ be the number of the branch points of $\widetilde{R}_z$ in $|\widetilde{z}| < r$, counted with the order of the branch point. Set

$$N(r,\widetilde{R}_z) = \frac{1}{\nu}\int_0^r \frac{n(t,\widetilde{R}_z)-n(0,\widetilde{R}_z)}{t}dt + \frac{n(0,\widetilde{R}_z)}{\nu}\log r.$$

From [**5**], we know that

$$N(r,\widetilde{R}_z) \leqslant 2(\nu-1)T(r,w) + O(1). \tag{2}$$

Next, we define the angle region in the z-plane,

$$\Delta(\theta_0,\delta) = \left\{z \;\middle|\; |\arg z - \theta_0| < \delta, 0 \leqslant \theta_0 < 2\pi, 0 < \delta < \frac{\pi}{2}\right\},$$
$$\overline{\Delta}(\theta_0,\delta) = \left\{z \;\middle|\; |\arg z - \theta_0| \leqslant \delta\right\}.$$

$\widetilde{\Delta}(\theta_0,\delta)$ and $\widetilde{\overline{\Delta}}(\theta_0,\delta)$ denote the regions where $\widetilde{R}_z$ lies over $\Delta(\theta_0,\delta)$ and $\overline{\Delta}(\theta_0,\delta)$, respectively. Let $n(r,\Delta(\theta_0,\delta),a)$ be the number of a-points of $w(z)$ in $\{|\widetilde{z}| < r\} \cap \widetilde{\Delta}(\theta_0,\delta)$, counted according to their multiplicities, and $\overline{n}^{l)}(r,\Delta(\theta_0,\delta),a)$ be the number of distinct a-points of $w(z)$ with multiplicity $\leqslant l$ in $\{|\widetilde{z}| < r\} \cap \widetilde{\Delta}(\theta_0,\delta)$. Then $\overline{n}(r,\Delta(\theta_0,\delta),a)$ denotes the number of distinct a-points of $w(z)$ in $\{|\widetilde{z}| < r\} \cap \widetilde{\Delta}(\theta_0,\delta)$, ignoring multiplicity. In addition, $n\big(r,\Delta(\theta_0,\delta),\widetilde{R}_z\big)$ denotes the number of the branch points in

$\{|\tilde{z}| < r\} \cap \tilde{\Delta}(\theta_0, \delta)$, counted with the order of branch. Just like the above, we define $N(r, \Delta(\theta_0, \delta), a)$, $\overline{N}^{l)}(r, \Delta(\theta_0, \delta), a)$, $\overline{N}(r, \Delta(\theta_0, \delta), a)$ and $N(r, \Delta(\theta_0, \delta), \tilde{R}_z)$, respectively.

DEFINITION 1: Let $w(z)$ be a ν-valued algebroid function defined by (1). The radial $\arg z = \theta_0$ is called a T direction of $w(z)$, if for arbitrary $\delta > 0 (0 < \delta < \pi/2)$, in the angle region $\Delta(\theta_0, \delta)$, given any $a \in \overline{C}$, we have

$$\limsup_{r\to\infty} \frac{N(r, \Delta(\theta_0, \delta), a)}{T(r, w)} > 0,$$

possibly with the exception of at most 2ν values of a.

DEFINITION 2: Let $w(z)$ be a ν-valued algebroid function defined by (1), and $l(> 2\nu)$ be a positive integer. For arbitrary $\delta > 0 (0 < \delta < \pi/2)$, if

$$\limsup_{r\to\infty} \frac{\overline{N}^{l)}(r, \Delta(\theta_0, \delta), a)}{T(r, w)} > 0 \tag{3}$$

holds for any $a \in \overline{C}$ except at most 2ν possible exceptions, then the radial $\arg z = \theta_0$ is called a T direction dealing with multiple value of $w(z)$.

REMARK. From the definitions, it is easy to see that the T direction dealing with multiple value is more precise, and the T direction dealing with multiple values must be a T direction. In addition, if $l = \infty$, (3) implies that

$$\limsup_{r\to\infty} \frac{\overline{N}(r, \Delta(\theta_0, \delta), a)}{T(r, w)} > 0.$$

THEOREM 1. *Let $w(z)$ be a ν-valued algebroid function defined by (1), satisfying*

$$\liminf_{r\to\infty} \frac{\log^+ T(r, w)}{\log r} = \mu < \infty, \quad \limsup_{r\to\infty} \frac{T(r, w)}{(\log r)^2} = \infty.$$

Then $w(z)$ must have at least one T direction dealing with multiple values.

2. LEMMAS

In order to prove the theorem, we need the following lemmas.

LEMMA 1. ([9]) *Let $T(r)$ be a positive and nondecreasing continuous function in $[0, \infty)$, satisfying*

$$\liminf_{r\to\infty} \frac{\log^+ T(r)}{\log r} = \mu < \infty, \quad \limsup_{r\to\infty} \frac{T(r)}{(\log r)^2} = \infty.$$

Then for any given $h > 0$, there must exist sequences $\{r_n\}$ and $\{R_n\}$ satisfying

$$R_n^{1-o(1)} \leqslant r_n \leqslant R_n (n \to \infty),$$

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such that

$$\lim_{n\to\infty}\frac{T(r_n)}{(\log r_n)^2}=\infty,\qquad T(e^hR_n)\leqslant e^{h\mu}T(R_n)\big(1+o(1)\big)\ (n\to\infty).$$

LEMMA 2. ([2]) *Suppose that $w=w(z)\big(|z|\leqslant R\big)$ is a ν-valued algebroid function defined by (1), and $a_1,a_2,\dots,a_q$ are $q(\geqslant 3)$ distinct points in the w-sphere such that the spherical distance of any two points is no small than $\delta\in(0,1/2)$. Then for any $r\in(0,R)$, we have*

$$\Big(q-2-\frac{2}{l}\Big)S(r,w)\leqslant\sum_{j=1}^{q}\overline{n}^{l)}(R,a_j)+\frac{l+1}{l}n(R,\widetilde{R}_z)+\frac{CR}{\delta^{10}(R-r)},$$

where $l\geqslant 3$ is a positive integer and C is a constant.

LEMMA 3. *Suppose that $w=w(z)\big(|z|\leqslant R\big)$ is a ν-valued algebroid function defined by (1) and $l(>2\nu)$ is a positive integer. Let $0<\delta<\delta_0<\pi/2, 0\leqslant\theta_0<2\pi$; that is, $\overline{\Delta}(\theta_0,\delta)\subset\Delta(\theta_0,\delta_0)$. The part of $\widetilde{R}_z$ which lies over $\overline{\Delta}(\theta_0,\delta)\cap\big\{|z|\leqslant r\big\}$ is denoted by $\widetilde{\overline{\Delta}}(\theta_0,\delta,r)$, and*

$$S\big(r,\overline{\Delta}(\theta_0,\delta),w\big)=\frac{1}{\pi}\int\int_{\widetilde{\overline{\Delta}}(\theta_0,\delta,r)}\frac{|w'(\widetilde{z})|^2}{(1+|w(\widetilde{z})|^2)^2}d\omega.$$

Then for any positive number $\lambda>1$, any positive integer α and any $q(\geqslant 3)$ distinct points $a_1,a_2,\dots,a_q$ in the w-sphere, we have

$$\begin{aligned}\Big(q-2-\frac{2}{l}\Big)&S\big(r,\overline{\Delta}(\theta_0,\delta),w\big)\\&\leqslant 2\sum_{j=1}^{q}\overline{n}^{l)}\big(\lambda^{2\alpha}r,\Delta(\theta_0,\delta_0),a_j\big)+\frac{l+1}{l}\Big(\frac{1+\alpha}{\alpha}\Big)n\big(\lambda^{2\alpha}r,\Delta(\theta_0,\delta_0),\widetilde{R}_z\big)\\&\qquad+\Big(q-2-\frac{2}{l}\Big)S\big(\lambda^{\alpha},\overline{\Delta}(\theta_0,\delta),w\big)+\frac{2A}{(1-\kappa)\log\lambda}\log^+ r,\end{aligned}$$

where A is a constant depending only on $a_i(i=1,2,\dots,q)$ and $\kappa(0<\kappa<1)$ is a constant depending only on δ,δ_0,λ and α.

PROOF: Set $r_\mu=\lambda^{\alpha\mu}, r_{\mu,k}=\lambda^{\alpha\mu+k}(\mu=0,1,2,\dots,0\leqslant k\leqslant\alpha-1)$, where $r_{\mu,0}=r_\mu, r_{\mu,\alpha}=r_{\mu+1,0}$. Let $\Omega_{\mu,k}=\big\{z\mid r_{\mu,k}\leqslant|z|\leqslant r_{\mu,k+1}\big\}\cap\Delta(\theta_0,\delta_0)$. For a given positive integer n, we have

$$\sum_{k=0}^{\alpha-1}\sum_{\mu=0}^{n}\Omega_{\mu,k}=\big\{z\mid 1=r_0\leqslant|z|\leqslant r_{n+1}\big\}\cap\Delta(\theta_0,\delta_0),$$

for which the component is non-empty. Without loss of generality, we assume that $k=0$ such that

$$\sum_{\mu=0}^{n}n(\Omega_{\mu,0},\widetilde{R}_z)\leqslant\frac{1}{\alpha}n(r_{n+1},\Delta(\theta_0,\delta_0),\widetilde{R}_z),$$

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where $n(\Omega_{\mu,0}, \widetilde{R}_z)$ denotes the number of branch points in $\widetilde{\Omega}_{\mu,0}$, counted with the order of the branch point. Let

$$\Delta_\mu = \left\{z \mid r_{\mu,0} < |z| < r_{\mu+1,1}\right\} \cap \Delta(\theta_0, \delta_0),$$
$$\overline{\Delta}_\mu = \left\{z \mid \frac{r_{\mu,0} + r_{\mu,1}}{2} \leqslant |z| \leqslant \frac{r_{\mu+1,0} + r_{\mu+1,1}}{2}\right\} \cap \overline{\Delta}(\theta_0, \delta).$$

By the Riemann Mapping Theorem, there exists an univalent function $f(z)$ mapping Δ_μ onto the unit disk $|\zeta| < 1$. At the same time, the point

$$z = \left((r_{\mu,0} + r_{\mu,1} + r_{\mu+1}, 0 + r_{\mu+1,1})/4\right)e^{i\theta_0}$$

can be mapped to $\zeta = 0$, and $\overline{\Delta}_\mu$ can be mapped into $|\zeta| \leqslant \kappa$, where $\kappa (0 < \kappa < 1)$ is a constant depending only on δ, δ_0 and λ^α, independent of ν. Then from the fact that

$$S(\overline{\Delta}_\mu, w) = \frac{1}{\pi} \int\int_{\widetilde{\overline{\Delta}}_\mu} \frac{|w'(\widetilde{z})|^2}{(1 + |w(\widetilde{z})|^2)^2} d\omega$$

is a conformal invariant and Lemma 2, we deduce that

$$\begin{aligned}\left(q - 2 - \frac{2}{l}\right) S(\overline{\Delta}_\mu, w) &\leqslant \left(q - 2 - \frac{2}{l}\right) S\left(\kappa, w(f^{-1})\right) \\ &\leqslant \sum_{j=1}^{q} \overline{n}^{l)}\left(|\zeta| \leqslant 1, a_j\right) + \frac{l+1}{l} n\left(|\zeta| \leqslant 1, \widetilde{R}_z\right) + \frac{A}{1-\kappa} \\ &= \sum_{j=1}^{q} \overline{n}^{l)}(\Delta_\mu, a_j) + \frac{l+1}{l} n(\Delta_\mu, \widetilde{R}_z) + \frac{A}{1-\kappa}.\end{aligned}$$

Thus

$$\left(q - 2 - \frac{2}{l}\right) \sum_{\mu=1}^{n-1} S(\overline{\Delta}_\mu, w) \leqslant \sum_{j=1}^{q} \sum_{\mu=1}^{n-1} \overline{n}^{l)}(\Delta_\mu, a_j) + \frac{l+1}{l} \sum_{\mu=1}^{n-1} n(\Delta_\mu, \widetilde{R}_z) + \frac{nA}{1-\kappa}. \tag{4}$$

In addition, it is easy to see that

$$\sum_{\mu=1}^{n-1} S(\overline{\Delta}_\mu, w) \geqslant S\left(r_n, \overline{\Delta}(\theta_0, \delta), w\right) - S\left(r_1, \overline{\Delta}(\theta_0, \delta), w\right). \tag{5}$$

Since $\{\Delta_\mu\}_{\mu=0}^n$ overlays $\{\Omega_{\mu,0}\}_{\mu=0}^n$ twice at most, we have

$$\sum_{\mu=1}^{n-1} \overline{n}^{l)}(\Delta_\mu, a_j) \leqslant 2\overline{n}^{l)}\left(r_{n+1}, \Delta(\theta_0, \delta_0), a_j\right), \tag{6}$$

$$\sum_{\mu=1}^{n-1} n(\Delta_\mu, \widetilde{R}_z) \leqslant \left(1 + \frac{1}{\alpha}\right) n\left(r_{n+1}, \Delta(\theta_0, \delta_0), \widetilde{R}_z\right), \tag{7}$$

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where $n(\Delta_\mu, \widetilde{R}_z)$ denotes the number of branch points in $\widetilde{\Delta}_\mu$, counted with the order of the branch point. And it follows $n = (\log r_n)/(\alpha \log \lambda)$ from $r_n = \lambda^{\alpha n}$. Then by (4)-(7), we obtain

$$(8)\quad (q-2-\frac{2}{l})S(r_n, \overline{\Delta}(\theta_0,\delta), w) \leqslant 2\sum_{j=1}^{q} \overline{n}^{l)}(r_{n+1}, \Delta(\theta_0,\delta_0), a_j) + \frac{l+1}{l}(1+\frac{1}{\alpha})n(r_{n+1}, \Delta(\theta_0,\delta_0), \widetilde{R_z}) + \left(q-2-\frac{2}{l}\right)S(r_1, \overline{\Delta}(\theta_0,\delta), w) + \frac{A}{\alpha(1-\kappa)\log\lambda}\log^+ r_n.$$

If $r \geqslant r_1 = \lambda^\alpha$, there must exist a $n > 1$ such that $r_{n-1} \leqslant r < r_n$. Thus we have $r_{n+1} = \lambda^{2\alpha} r_{n-1} \leqslant \lambda^{2\alpha} r$ and $r_n \leqslant \lambda^\alpha r \leqslant r^2$. Then (8) becomes

$$\left(q-2-\frac{2}{l}\right)S(r, \overline{\Delta}(\theta_0,\delta), w) \leqslant 2\sum_{j=1}^{q} \overline{n}^{l)}(\lambda^{2\alpha} r, \Delta(\theta_0,\delta_0), a_j) + \frac{l+1}{l}\left(\frac{1+\alpha}{\alpha}\right)n\left(\lambda^{2\alpha} r, \Delta(\theta_0,\delta_0), \widetilde{R}_z\right) + (q-2-\frac{2}{l})S(\lambda^\alpha, \overline{\Delta}(\theta_0,\delta), w) + \frac{2A}{(1-\kappa)\log\lambda}\log^+ r.$$

If $r < r_1$, the above inequality is obviously true. The Lemma is complete. □

Lemma 4. *Suppose that $w = w(z)\left(|z| \leqslant R\right)$, defined by (1), is a ν-valued algebroid function with finite lower order μ, and satisfies*

$$(9)\qquad \limsup_{r\to\infty} \frac{T(r,w)}{(\log r)^2} = \infty,$$

Let $l(> 2\nu)$, $m(> 1)$ be both positive integers, and

$$\eta_0 = 0, \eta_1 = \frac{2\pi}{m}, \dots, \eta_{m-1} = (m-1)\frac{2\pi}{m}, \eta_m = 0.$$

Then there must exist a $\Delta(\eta_i, (2\pi)/m)$ in $\left\{\Delta(\eta_i, (2\pi/m))\right\}_{i=0}^{m-1}$ such that

$$\limsup_{r\to\infty} \frac{\overline{N}^{l)}(r, \Delta(\eta_i, (2\pi/m)), a)}{T(r,w)} > 0$$

holds for any $a \in \overline{C}$ except at most 2ν possible exceptions.

Proof: Suppose that the conclusion is false. Then for any $\Delta(\eta_i, (2\pi/m))(0 \leqslant i \leqslant m-1)$, there exists $q = 2\nu+1$ exceptional values $\{a_i^j\}(j = 1, \dots, 2\nu+1)$ such that

$$\limsup_{r\to\infty} \frac{\overline{N}^{l)}(r, \Delta(\eta_i, (2\pi/m)), a_i^j)}{T(r,w)} = 0.$$

Thus there exists R_0, when $r > R_0$,

$$\overline{N}^{l)}\Big(r, \Delta\Big(\eta_i, \frac{2\pi}{m}\Big), a_i^j\Big) = o\big(T(r,w)\big) \tag{10}$$

holds for i, j uniformly.

Let α is any positive integer. Put $\eta_{i,k} = (2\pi i/m) + (2\pi k)/(\alpha m)(0 \leqslant k \leqslant \alpha - 1)$. Given any constant $\lambda(> 1)$ and $r > R_0$, let $\Delta_{i,k} = \big\{z \mid |z| < \lambda^{2\alpha} r, \eta_{i,k} \leqslant \arg z < \eta_{i,k+1}\big\}$. Then

$$\sum_{k=0}^{\alpha-1}\sum_{i=0}^{m-1} \Delta_{i,k} = \big\{|z| < \lambda^{2\alpha} r\big\},$$

for which the component is non-empty. Without loss of generality, we assume that $k = 0$ such that

$$\sum_{i=0}^{m-1} n(\Delta_{i,0}, \widetilde{R}_z) \leqslant \frac{1}{\alpha} n(\lambda^{2\alpha} r, \widetilde{R}_z).$$

Set

$$\overline{\Delta}_i = \big\{z \mid (\eta_{i,0} + \eta_{i,1})/2 \leqslant \arg z \leqslant (\eta_{i+1,0} + \eta_{i+1,1})/2\big\},$$
$$\Delta_i = \big\{z \mid \eta_{i,0} < \arg z < \eta_{i+1,1}\big\}.$$

Since $\{\Delta_i\}_{i=0}^{m-1}$ overlays $\{\Delta_{i,0}\}_{i=0}^{m-1}$ only twice, we have

$$\sum_{i=0}^{m-1} n(\lambda^{2\alpha} r, \Delta_{i,0}, \widetilde{R}_z) \leqslant \Big(1 + \frac{1}{\alpha}\Big) n(\lambda^{2\alpha} r, \widetilde{R}_z). \tag{11}$$

For $\overline{\Delta}_i \subset \Delta_i$, applying Lemma 3, we have

$$\Big(q - 2 - \frac{2}{l}\Big) S(r, \overline{\Delta}_i, w) \leqslant 2\sum_{j=1}^{q} \overline{n}^{l)}(\lambda^{2\alpha} r, \Delta_i, a_j) + \frac{l+1}{l}\Big(\frac{1+\alpha}{\alpha}\Big) n(\lambda^{2\alpha} r, \Delta_i, \widetilde{R}_z)$$
$$+ \Big(q - 2 - \frac{2}{l}\Big) S(\lambda^{\alpha}, \overline{\Delta}_i, w) + \frac{2A_i}{(1-\kappa)\log\lambda} \log^+ r,$$

Adding from $i = 0$ to $m-1$, dividing both sides of this inequality by r and then integrating both sides from 0 to r, we obtain the following inequality

$$\Big(q - 2 - \frac{2}{l}\Big) T(r, w) \leqslant 2\sum_{i=0}^{m-1}\sum_{j=1}^{q} \overline{N}^{l)}(\lambda^{2\alpha} r, \Delta_i, a_j) + \frac{l+1}{l}\Big(\frac{1+\alpha}{\alpha}\Big)^2 N(\lambda^{2\alpha} r, \widetilde{R}_z)$$
$$+ \Big(q - 2 - \frac{2}{l}\Big) T(\lambda^{\alpha}, w) + O\big[(\log r)^2\big].$$

It follows that by (2) and (9)–(11)

$$\Big(q - 2 - \frac{2}{l}\Big) T(r, w) \leqslant 2(\nu - 1)\frac{l+1}{l}\Big(\frac{1+\alpha}{\alpha}\Big)^2 T(\lambda^{2\alpha} r, w) + o\big(T(r,w)\big). \tag{12}$$

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By Lemma 1, there exists sequence $\{r_n\}$ and $\{R_n\}$ satisfying $R_n^{1-o(1)} \leqslant r_n \leqslant R_n$ $(n \to \infty)$, such that

$$\lim_{n\to\infty} \frac{T(r_n, w)}{(\log r_n)^2} = \infty, \qquad T(\lambda^{2\alpha} R_n, w) \leqslant \lambda^{2\alpha\mu} T(R_n, w)\big(1 + o(1)\big)\ (n \to \infty).$$

Thus we have

$$\lim_{n\to\infty} \frac{T(R_n, w)}{(\log R_n)^2} = \infty.$$

Replacing r by R_n and dividing both side by $T(R_n, w)$ in (12), we obtain

$$q - 2 - \frac{2}{l} \leqslant 2(\nu - 1)\frac{l+1}{l}\Big(\frac{1+\alpha}{\alpha}\Big)^2 \lambda^{2\alpha\mu},$$

when $n \to \infty$. Letting $\lambda \to 1, \alpha \to \infty$, respectively, we deduce $l \leqslant 2v$ from $q = 2\nu + 1$. This contradicts $l > 2\nu$. □

3. Proof of Theorem 1

By Lemma 4, for any given positive integer m, there exists an angle region

$$\Delta\Big(\theta_m, \frac{2\pi}{m}\Big) = \Big\{z \,\Big|\, |\arg z - \theta_m| < \frac{2\pi}{m}\Big\},$$

such that

$$\limsup_{r\to\infty} \frac{\overline{N}^{l)}(r, \Delta(\theta_m, (2\pi/m), a)}{T(r, w)} > 0$$

holds for any $a \in \overline{C}$ with at most 2ν exceptions.

Let $E = \{\theta_m = \arg z_m \mid 0 \leqslant \theta_m < 2\pi, m = 1, 2, \ldots\}$, then there exists at least one accumulation point $\theta_0 (0 \leqslant \theta_0 < 2\pi)$ in E. Without loss generation, we assume $\theta_m \to \theta_0 (m \to \infty)$. Then $J : \arg z = \theta_0$ is a T direction dealing with multiple value of $w(z)$. Otherwise there exists a $\delta (0 < \delta < \pi/2)$ and $2\nu+1$ exceptional values $a \in \Delta(\theta_0, \delta)$, such that

$$\limsup_{r\to\infty} \frac{\overline{N}^{l)}(r, \Delta(\theta_0, \delta), a)}{T(r, w)} = 0.$$

For sufficiently large m, we have $\Delta\big(\theta_m, (2\pi/m)\big) \subset \Delta(\theta_0, \delta)$. Then

$$\limsup_{r\to\infty} \frac{\overline{N}^{l)}(r, \Delta(\theta_m, 2\pi/m), a)}{T(r, w)} = 0.$$

This contradicts the choice of $\{\theta_m\}$. This completes the Theorem.

Remark 1. If $w(z)$ has finite positive order ρ, that is, $\limsup_{r\to\infty} (\log^+ T(r, w))/\log r = \rho (0 < \rho < \infty)$, then its Borel direction of the largest type (see [**1**]) is also the T direction. Thus there is a sequence filling disks in any T direction (dealing with multiple value)(see [**1, 15**]).

REMARK 2. It remains open, whether the T direction(dealing with multiple value) of $w(z)$ exists, when the lower order of $w(z)$ is infinite; that is,

$$\liminf_{r\to\infty} \frac{\log^+ T(r,w)}{\log r} = \infty.$$

REMARK 3. If $a(z)$ is a function such that $T(r,a) = o\big(T(r,w)\big)$, is it true that

$$\limsup_{r\to\infty} \frac{N(r,\Delta(\theta_0,\delta),a)}{T(r,w)} > 0,$$

with the possible exception of at most 2ν small functions $a(z)$?

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