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ON MAXI-QUASIPROJECTIVE MODULES

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Abstract

We have defined a mini-injective module and given some structures of self mini-injective rings and certain relationships between such rings and QF-rings in [8] and [9].

In this short note we shall study the modules dual to mini-injective modules, which we call maxi-quasiprojective modules. We shall give a characterization and some structures, in terms of the above modules, of those rings whose every injective module has the lifting property of direct decompositions modulo the Jacobian radical (see [5], [6] and [7]). Furthermore, we shall show that the above rings are closely related to QF-rings (see [8] and [9]).

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Throughout this note, we assume that a ring R contains an identity and every module is a unitary right R-module. We always assume that R is a right artinian ring unless otherwise stated. However, some of the first part of this note is valid without this assumption.

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1. Maxi-quasiprojective modules

Let M be an R-module. We denote the Jacobson radical of M by $J(M)$. We put $\overline{M} = M/J(M)$. If N is a direct summand of M, $N/J(N)$ may be regarded as an R-submodule of $\overline{M}$. Hence $\overline{N} = N/J(N) \subseteq \overline{M}$. An R-module T is called *hollow* if

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$J(T)$ is a unique maximal submodule of T. Since R is artinian, $\bar{M}$ is semi-simple. Let $\bar{M} = \Sigma_I \oplus N_\alpha$, where the N_α are simple. If there exists a direct decomposition $\Sigma_I \oplus M_\alpha$ of M with $\bar{M}_\alpha = N_\alpha$ for each $\alpha \in I$, we say that the direct decomposition $\bar{M} = \Sigma_I \oplus N_\alpha$ is *lifted* to M. If M has the above property for any direct decomposition of $\bar{M}$, we say that M has the *lifting property of direct decompositions* of $\bar{M}$ [6]. In this case $M = \Sigma_I \oplus M_\alpha$ and the M_α are hollow modules. Hence $M_\alpha \approx e_\alpha R/e_\alpha A$, where the e_α is a primitive idempotent and A is a right ideal in R.

It is well known that every projective module has the lifting property of direct decompositions modulo the radical [12]. We note that if $N_\alpha \not\approx N_\beta$ for each pair α, β, every direct summand of $\bar{M}$ is of the form $\Sigma_K \oplus N_\delta$ $(K \subseteq I)$ and so M has trivially the lifting property of direct decompositions of $\bar{M}$. In order to avoid this trivial case, we assume that

($\sharp$) *each N_α is isomorphic to another N_β* [7].

Now, we shall define a new class of modules. For any maximal submodule N of M, we consider a diagram

$$\begin{array}{ccccc} 0 \to N \to M & \overset{\nu}{\to} & M/N & \to 0 \\ & \nwarrow_{h} & \uparrow f & \\ & & M & \end{array}$$

where ν is the natural epimorphism. If, for any f in $\mathrm{Hom}_R(M, M/N)$, there exists an element h in $\mathrm{End}_R(M)$ which makes the above diagram commute, we say that M is a *maxi-quasiprojective module*. It is clear that every quasi-projective module is maxi-quasiprojective and the converse is not true in general. For instance, let R be a local algebra over a field K such that $R/J(R) \approx K$ and let A be a right ideal of R. Then R/A is maxi-quasiprojective, but not quasi-projective, provided that A is not a two-sided ideal (see Remarks 2 and 3 below).

Let N and N' be two maximal submodules of M. Then the definition above is equivalent to the fact:

$$(\mathrm{Hom}_R(M, M) \supseteq)\ \mathrm{Hom}_R(M, M)^* \to \mathrm{Hom}_R(M/N, M/N')$$

is an epimorphism via natural epimorphisms ν and ν', where $\mathrm{Hom}_R(M, M)^* = \{f \in \mathrm{Hom}_R(M, M) \mid f(N) \subseteq N'\}$.

We put $S = \mathrm{End}_R(M)$, $\bar{S} = \mathrm{End}_R(\bar{M})$ and $J_0(S) = \mathrm{Hom}_R(M, J(M))$. Then we have the natural monomorphism θ: $S/J_0(S) \to \bar{S}$ (see [7]).

THEOREM 1. *Let M be an R-module. Assume $M = \Sigma_I \oplus M_\alpha$ and the M_α are completely indecomposable; that is, $\mathrm{End}_R(M_\alpha)$ is local. We further assume ($\sharp$). Then the following conditions are equivalent*:

1) *θ is an epimorphism.*

2) *M is maxi-quasiprojective and each M_α is hollow.*

3) *M has the lifting property of direct decompositions of $\overline{M}$.*

PROOF. 1) → 2). We note that every element in S (resp. $\overline{S}$) is expressed as a column summable matrix with entries $f_{\alpha\beta}$, where the $f_{\alpha\beta}$ are elements in $\mathrm{Hom}_R(M_\beta, M_\alpha)$ (resp. $\mathrm{Hom}_R(\overline{M}_\beta, \overline{M}_\alpha)$). Hence it is clear that θ induces an epimorphism θ_α: $\mathrm{End}_R(M_\alpha) \to \mathrm{End}_R(\overline{M}_\alpha)$. Since $\mathrm{End}_R(M_\alpha)$ is local, so is $\mathrm{End}_R(\overline{M}_\alpha)$. Furthermore, $\overline{M}_\alpha$ is semi-simple and so $\overline{M}_\alpha$ is simple. Therefore M_α is hollow. Let N_1 and N_2 be two maximal submodules of M. Then $N_i \supseteq J(M) \approx \overline{M}/N_1$ is a direct summand of $\overline{M}$. Accordingly, M is maxi-quasiprojective.

2) → 3). Since M_α is hollow, $J(M_\alpha) \oplus \Sigma_{\beta\neq\alpha} \oplus M_\beta$ is a maximal submodule of M. Hence $\mathrm{Hom}_R(M_\alpha, M_\beta) \to \mathrm{Hom}_R(\overline{M}_\alpha, \overline{M}_\beta)$ is an epimorphism for $\alpha, \beta \in I$. We assume $\overline{M}_\alpha \approx \overline{M}_\beta$. Then M_α, M_β being hollow, there exist epimorphisms f: $M_\alpha \to M_\beta$, g: $M_\beta \to M_\alpha$ by the above. Since R is artinian and so the M_α are of finite length, $M_\alpha \approx M_\beta$. Hence $\{M_\alpha\}_I$ is (semi-) T-nilpotent (see [11]). Therefore M has the lifting property of direct decompositions of M by [7], Corollary 1 to Theorem 2.

3) → 1). This is clear from [7], Theorem 2.

THEOREM 2 (the dual to [8], Theorem 3). *Let R be a right artinian ring. Then the following two conditions are equivalent*:

1) *Every injective E has the lifting property of direct decompositions of $\overline{E}$.*

2) *An injective cogenerator is maxi-quasiprojective and a direct sum of hollow submodules*: *that is*, *right* QF-2* [5].

PROOF. Every injective is a direct sum of completely indecomposable modules. Hence the theorem is clear from Theorem 1 and [7], Theorem 2 and its remark (note that we do not use the assumption (♯) for the implication 2) → 3) in the proof).

REMARKS. 1. We can define an essentially quasi-projective module as the dual to uni-injective [8], when we replace a maximal submodule by an essential submodule. We note that if M is essentially quasi-projective, M is maxi-quasiprojective and if M is uniform and essentially quasi-projective, M is quasi-projective.

2. We take the ring defined in [8], Example 2. Let $L \supseteq K$ be two field satisfying the following conditions: $[L : K] = 2$ and there exists an isomorphism σ of L onto K. Put $R = L \oplus Lu$ a vector space over L. We define a product on R as $(x_1 + x_2u)(y_1 + y_2u) = x_1y_1 + (x_2\sigma(y_1) + x_1y_2)u$, where the x_i and the y_i are in L. Then R is mini-injective as a right R-module [8]. $R^* = \mathrm{Hom}_K(R, K)$ as right K-modules is a left R-K bimodule and $R^{**} \approx R$ as right R-K bimodules. Then R^*

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is an indecomposable and left R-maxi-quasiprojective module, which is not hollow.

3. Let K be a field and

$$R = \begin{pmatrix} K & 0 & K \\ 0 & K & K \\ 0 & 0 & K \end{pmatrix}.$$

Put $e = e_{11}$ and $f = e_{22}$. Then $eR \not\approx fR$ and $S(eR) \approx S(fR)$ $(= S)$, where S( ) means the socle of ( ). Put $M = (eR \oplus fR)/S'$, where $S' = \{s + s \mid s \in S\}$. Then M is maxi-quasiprojective, since $\overline{M} = e\overline{R} \oplus f\overline{R}$ and $e\overline{R} \not\approx f\overline{R}$ and M is an indecomposable module, which is not hollow (see [1]).

2. Lifting property of injectives

We shall study the right artinian rings whose every injective module has the lifting property of direct decompositions modulo the radical (see [5]).

First we shall give the dual to 1) of Theorem 5 in [8].

LEMMA 3. *Let M_1 and M_2 be indecomposable modules of finite length.*

1) *If $M = M_1 \oplus M_2$ is maxi-quasiprojective, and $M_1 \not\approx M_2$, then no simple submodule in $\overline{M}_1$ is isomorphic to a submodule in $\overline{M}_2$.*

2) *If M_1 is maxi-quasiprojective, and $M_1/N_1 \approx M_1/N_2$ for maximal submodules N_i in M_1, then there exists an automorphism f of M_1 with $f(N_1) = N_2$.*

PROOF. 1) Assume there exists a maximal submodule N_i of M_i such that $M_1/N_1 \approx M_2/N_2$. Then there exist f_1 in $\mathrm{Hom}_R(M_1, M_2)$ and f_2 in $\mathrm{Hom}_R(M_2, M_1)$ which induce the given isomorphism and satisfy $f_1(N_1) \subseteq N_2$ and $f_2(N_2) \subseteq N_1$. Put $h = f_2 f_1 \in \mathrm{End}_E(M_1)$. Then $h(N_1) \subsetneq N_1$ and $M_1 = h(M_1) + N_1$. If h is not an isomorphism, h is nilpotent, for M_1 has finite composition length. Hence $M_1 = h^n(M_1) + N_1 = N_1$ for some n, which is a contradiction. Therefore h is an isomorphism and f_i is also an isomorphism, a contradiction.

2) If we apply the above argument for $M_1 = M_2$, we have 2).

LEMMA 4. *Let M_1 be an indecomposable module of finite length. We put $M = M_1^{(I)}$ a direct sum of $|I|$-copies of M_1. We assume that M_1 is maxi-quasiprojective and N a maximal submodule of M. Then there exists a decomposition $M = \Sigma_I \oplus M'_\alpha$ of M such that $N = M_1 \oplus \Sigma_{I-\{1\}} \oplus M'_\alpha$, where $M'_\alpha \approx M_1$ for all $\alpha \in I$ and N_1 is a maximal submodule of M'_1, where $|I|$ means the cardinal of I.*

PROOF. We shall show that N contains a non-zero direct summand of M if $|I| \geq 2$. Since N is a maximal submodule, $N \supseteq J(M)$. We denote $M/J(M)$ by $\bar{M}$ (note that, for a submodule K of M, $\bar{K} = (K + J(M))/J(M)$). Let $M = \Sigma_I \oplus M_\alpha$, and put $M_\alpha \cap N = N_\alpha$. If $M_\alpha \subseteq N$ for some α, we are done. Hence we may assume $M = N + M_\alpha$ for any $\alpha \in I$. Then $M/N = (N + M_\alpha)/N \approx M_\alpha/N_\alpha$ and so N_α is a maximal submodule of M_α. Now, since $\bar{M}_\alpha$ is semi-simple,

(1) $\bar{M}_\alpha = \bar{N}_\alpha \oplus \bar{A}_\alpha$, where $J(M_\alpha) \subseteq A_\alpha \subseteq M_\alpha$ and $\bar{A}_\alpha$ is a simple submodule.

Since $N \supseteq \Sigma_I \oplus N_\alpha \supseteq J(M)$, $\bar{N} = \Sigma_I \oplus \bar{N}_\alpha \oplus \bar{N}_0$ and $\bar{N}_0 \neq 0$, for N is maximal and $|I| \geq 2$. Let $\bar{N}_0^*$ be a simple submodule of $\bar{N}_0$; $N_0^* \supseteq J(M)$. We take the decomposition

$$(2) \qquad \bar{M} = \left(\Sigma_I \oplus \bar{N}_\alpha\right) \oplus \left(\Sigma_I \oplus \bar{A}_\alpha\right).$$

Let p and p_α be the projection of $\bar{M}$ onto $\Sigma_I \oplus \bar{N}_\alpha$ and $\bar{A}_\alpha$, respectively. Since $\bar{N}_0^*$ is simple, there exists a finite subset I' of I such that $p_\alpha(\bar{N}_0^*) = \bar{A}_\alpha$ for $\alpha \in I'$ and $p_\beta(\bar{N}_0^*) = 0$ for $\beta \in I - I'$. We may assume $I' = \{1, 2, \ldots, n\}$. Hence there exists a set of isomorphisms $g_i\colon \bar{A}_1 \to \bar{A}_i$ $(i \geq 2)$ such that $\bar{N}_0^* = \{p(n) + a_1 + g_2(a_1) + \cdots + g_n(a_1) \mid n \in \bar{N}_0^*,\ a_1 = p_1(n) \in \bar{A}_1\}$. On the other hand, there exists f_i: $M_1 \to M_i$ such that $f_i(N_1) = N_i$ and f_i induces g_i on $\bar{A}_1$ by Lemma 3. We put $M_1(f) = \{m_1 + f_2(m_1) + \cdots + f_n(m_1) \mid m_1 \in M_1\}$. Then it is clear that $M_1(f) \approx M_1$ and $M = M_1(f) \oplus \Sigma_{I-\{1\}} \oplus M_\alpha$. Let $\bar{m}_1 = \bar{n}_1 + \bar{a}_1$, where $\bar{n}_1 \in \bar{N}_1$ and $\bar{a}_1 \in \bar{A}_1$. Then there exists n in N_0^* such that $\bar{n} = p(\bar{n}) + \bar{a}_1 + g_2(\bar{a}_1) + \cdots + g_n(\bar{a}_1)$. Hence $\overline{m_1 + f_2(m_1) + \cdots + f_n(m_1)} = \bar{n}_1 + f_2(\bar{n}_1) + \cdots + f_n(\bar{n}_1) + \bar{a}_1 + f_2(\bar{a}_1) + \cdots + f_n(\bar{a}_1) = \bar{n}_1 + f_2(\bar{n}_1) + \cdots + f_n(\bar{n}_1) + \bar{n} - p(\bar{n}) \in \Sigma_1 \oplus \bar{N}_\alpha \oplus N_0^* \subseteq \bar{N}$, and so $M_1(f) \subseteq N$. Let **F** be the set of all direct sums of indecomposable modules K_α isomorphic to M_1, which are contained in N and are locally direct summands of M; that is, any finite sum of the K_α is a direct summand of M. Then **F** is non-empty by the above, and we can find a maximal member in **F** with respect to the relation to the members of direct components by Zorn's lemma, say $\Sigma_J \oplus M'_\alpha$ $(\subseteq N)$. Since M_1 has the finite length, $\{M_\alpha\}_I$ is a semi-T-nilpotent set [11]. Hence $\Sigma_J \oplus M'_\alpha$ is a direct summand of M by [11], Theorem, say $M = \Sigma_J \oplus M'_\alpha \oplus M^*$. Hence $N = \Sigma_J \oplus M'_\alpha \oplus (N \cap M^*)$ and $N \cap M^*$ is a maximal submodule of M^*. M^* is also a direct sum of submodules isomorphic to M_1 by [16]. Therefore $|I - J| = 1$ by the above and the maximality of $\Sigma_J \oplus M'_\alpha$.

We assume that an R-module is a direct sum of indecomposable modules M_α of finite length. Then we can rearrange this decomposition as follows:

$$(3) \qquad M \approx \Sigma_{\alpha \in I} \oplus M_\alpha^{(J_\alpha)}, \qquad M_\alpha \not\approx M_\beta \quad \text{if } \alpha \neq \beta.$$

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THEOREM 5. *Let M be as in* (3). *Then M is maxi-quasiprojective if and only if the M_α are maxi-quasiprojective for all α and no simple submodule in $\overline{M}_\alpha$ is isomorphic to a simple submodule in $\overline{M}_\beta$ if $\alpha \neq \beta$ (and hence $|I|$ is finite).*

PROOF. Assume M is maxi-quasiprojective. Then so is any direct summand of M by the definition. Hence we have the property above. Since R is artinian, I is finite. Let N be a maximal submodule in M. By the assumption, $\overline{M}_\alpha^{(J_\alpha)}$ is a direct sum of homogeneous components in $\overline{M}$. Hence $\overline{N} = \overline{N}_1 \oplus \Sigma_{\alpha \neq 1} \oplus \overline{M}_\alpha^{(J_\alpha)}$ for some homogeneous component $M_1^{(J_1)}$, where N_1 is a maximal submodule in $M_1^{(J_1)}$, and so $M/N \approx M_1^{(J_1)}/N_1$. We take another maximal submodule N' of M such that $M/N \approx M/N'$. Then as above, we obtain $\overline{N}' = \overline{N}_1' \oplus \Sigma_{\alpha \neq 1} \oplus \overline{M}_\alpha^{(J_\alpha)}$, where N_1' is a maximal submodule of $M_1^{(J_1)}$. Now we obtain two decompositions of $M_1^{(J_1)}$ by Lemma 4:

$$M_1^{(J_1)} = M_1'^{(J_1-1)} \oplus M_1' \supseteq N = M_1'^{(J_1-1)} \oplus N_1' \quad \text{and}$$
$$M_1^{(J_1)} = M_1''^{(J_1-1)} \oplus M_1'' \supseteq N' = M_1''^{(J_1-1)} \oplus N_1'',$$

where $N_1' \subset M_1'$ and $N_1'' \subset M_1''$ and $M_1 \approx M_1' \approx M_1''$. Hence we obtain an automorphism f of $M_1^{(J_1)}$ by the assumption, which induces the given isomorphism between M_1'/N_1' and M_1''/N_1''. Therefore M is maxi-quasiprojective.

COROLLARY 1. *Let M be as above and let N be a submodule of M containing $J(M)$. We assume that M is maxi-quasiprojective and each M_α is cyclic hollow. Then there exists a decomposition of M such that $M = \Sigma_I \oplus M_\beta' \supset N = \Sigma_{I_1} \oplus M_\beta' \oplus \Sigma_{I_2} \oplus J(M_\gamma')$: $I = I_1 \cup I_2$ and the M_α' are indecomposable. Let N' be another submodule of M containing $J(M)$. If $M/N \approx M/N'$, there exists an automorphism f of M which induces the above isomorphism and $f(N) = N'$.*

PROOF. We take the same argument as the proof of Lemma 4. Since M_α is hollow, $\overline{A}_\alpha$ is either simple or zero. Hence, if $N \neq J(M)$, N contains a non-zero direct summand of M from the method after (1) in the proof of Lemma 4. We can use the same argument for the remainder.

COROLLARY 2 (the dual to [8], Theorem 5). *Let $E = \Sigma_{j=1}^n \oplus E_i$ be a minimal injective cogenerator with E_i indecomposable. We assume that E is finitely generated and maxi-quasiprojective. Then*

1) *All simple submodules in $\overline{E}_i$ are isomorphic to one another and are not isomorphic to any one in $\overline{E}_j$ for $i \neq j$.*

2) *If* $\bar{E}_i$ *is simple for all* i, *every primitive idempotent* e *in* R *is non-small* [3]; *that is*, eR *is not a small submodule in the injective envelope* $\mathrm{E}(eR)$ *of* eR *and* R *contains all simple submodules up to isomorphism and* $\mathrm{r}(J) \subseteq \mathrm{l}(J)$, *where* $J = J(R)$, $\mathrm{l}(J) = \{x \in R \mid xJ = 0\}$ *and* $\mathrm{r}(J) = \{x \in R \mid Jx = 0\}$.

PROOF. 1) Since E is a minimal cogenerator, $E_i \not\approx E_j$ for $i \neq j$ and $\Sigma_{i=1}^{n} \oplus E_i$ contains all simple R-modules up to isomorphism by Lemma 3. Hence we obtain 1) from Lemma 3.

2) If $\bar{E}_i$ is simple for all i, R is right QF-2* [5]. Hence we obtain the non-isomorphic representative set of indecomposable and injective modules $\{e_1R/e_1A_1, e_2R/e_2A_2, \ldots, e_nR/e_nA_n\}$ from Theorem 2 and [5], Theorem 3, where $\{e_i\}$ is the set of mutually orthogonal and non-isomorphic primtive idempotents and $\{A_i\}$ is a set of right ideals. We assume that e_i is small. Then $\tilde{E} = \mathrm{E}(e_iR) \supset \mathrm{J}(\tilde{E}) \supseteq e_iR \supseteq e_iA_i$. Since e_iR/e_iA_i is injective, $\tilde{E}/e_iA_i$ contains a direct summand e_iR/e_iA_i contained in $J(\tilde{E}/e_iA_i)$, which is a contradiction. Let $\mathrm{E}(R) \approx \Sigma_K \oplus (e_jR/e_jA_j)^{m_j}$, where $(e_jR/e_jA_j)^{m_j}$ is a direct sum of m_j-copies of e_jR/e_jA_j, and $K \subseteq \{1, 2, \ldots, n\}$. Since e_i is non-small, K contains i. Hence $\mathrm{E}(R)$ is a cogenerator, and so R contains all simple modules up to isomorphism. We may assume that $R \subseteq \Sigma_{i=1}^{k} \oplus e_iR/e_iA_i$ and $\mathrm{r}(J) \subseteq \Sigma_{i=1}^{k} \oplus \bar{a}_i\mathrm{r}(J)$, where $a_i \in e_iR$, $\bar{a}_i = a_i + e_iA_i$ and e_i may equal e_j for some j. We assume that $a_1\mathrm{r}(J) \not\subseteq$ the socle $\mathrm{S}(e_1R/e_1A_1)$ of e_1R/e_1A_1. Since e_1R/e_1A_1 is uniform, $\bar{a}\mathrm{r}(J) \supseteq S_1 = \mathrm{S}(e_1R/e_1A_1)$. $\mathrm{E}(R)$ being an injective cogenerator, there exists f in $\mathrm{Hom}_R(e_1R/e_1A_1, e_iR/e_iA_i)$ for some i such that $f(S_1) = 0$ and $f(\bar{a}_1\mathrm{r}(J)) \neq 0$. f is given by the left-sided multiplication of an element b in e_iRe_1. Since $f(S_1) = 0$, f is not an isomorphism. Hence $b \in e_iJe_1$ by the construction of e_iR/e_iA_i (note that $e_iR/e_iA_i \not\approx e_jR/e_jA_j$ if $i \neq j$). $f(\bar{a}_1\mathrm{r}(J)) = b\bar{a}_1\mathrm{r}(J) \subseteq \bar{e}_iJ\mathrm{r}(J) = 0$, a contradiction. Therefore $\bar{a}_1\mathrm{r}(J) \subseteq S_1$ and $\bar{a}_1\mathrm{r}(J)J \subseteq S_1J = 0$. Similarly, we have $\bar{a}_i\mathrm{r}(J)J = 0$ for all i and so $\mathrm{r}(J)J = 0$. Thus, $\mathrm{r}(J) \subseteq \mathrm{l}(J)$.

The following theorem is the dual to [8], Theorem 13.

THEOREM 6. *Let* R *be a right artinian ring. Then the following conditions are equivalent*:

1) R *is a* QF-*ring*.

2) R *is right* QF-2 *and* QF-2* *and a minimal injective cogenerator is maxi-quasi-projective* (*see Theorem* 9 *below*).

3) *Every injective* E *has the lifting property of direct decompositions of* $\bar{E}$ *and* $\mathrm{l}(J) \subseteq \mathrm{r}(J)$.

4) *Every injective* R*-module and every injective left* R*-module have the lifting property of direct decompositions modulo the radical.*

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PROOF. 1) → 2). Since R is an injective cogenerator and a projective module as a right R-module by [2], we obtain 2).

2) → 1). Let e be a primitive idempotent. R being QF-2 and QF-2*, $E(eR)$ is hollow. Hence $E(eR) = eR$ by 2) and Corollary 2 to Theorem 5. Therefore R is self-injective (see the proof of [8], Theorem 13).

1) → 3) and 4). Since every injective module is projective by [2] and $\mathrm{l}(J) = \mathrm{r}(J)$ by [15], we have 3) and 4).

3) → 1). We know from 3) and Theorem 2 that minimal injective cogenerator are maxi-quasiprojective and R is a QF-2* ring. Hence we shall use the same notation as in the proof of Corollary 2 to Theorem 5. Since $R \subseteq E(R) \approx \Sigma_{j=1}^{n} \oplus (e_jR/e_jA_j)^{m_j}$, $\Sigma_{i=1}^{n} \oplus eR/e_iA_i$ is faithful. We shall show that $e_1A_1 \cap \mathrm{r}(J) = 0$. We assume $e_1A_1 \cap \mathrm{r}(J) \neq 0$ and take a non-zero element x in $e_1A_1 \cap \mathrm{r}(J)$. Then $e_iRx = e_iRe_1x \subseteq Jx = 0$ if $i \neq 1$. Hence $e_1Rx \not\subseteq e_1A_1$. Let y be an element in e_1Re_1 such that $yx \notin e_1A_1$. Since $x \in \mathrm{r}(J)$, $y \notin e_1Je_1$: e_1R/e_1A_1 being maxi-quasiprojective and y inducing an element in $\mathrm{End}_R(e_1R/e_1J)$, there exists an element z in e_1Re_1 such that $y - z \in e_1Je_1$ and $z(e_1A_1) \subseteq e_1A_1$ by Lemma 3. Hence $yz = zx \in e_1A_1$, which is a contradiction. Similarly, $e_iA_i \cap \mathrm{r}(J) = 0$ for all i. Now, $\mathrm{l}(J) \subseteq \mathrm{r}(J)$ and $\mathrm{l}(J)$ is an essential right ideal in R. Hence $e_iA_i = 0$ for all i, and so $R = E(R)$.

COROLLARY 1. *Let R be a right artinian ring. Then R is a* QF-*ring if and only if every injective E and every projective P have the lifting and extending property of direct decompositions of $\bar{E}$ and* S(*P*), *respectively. Furthermore, if* $\mathrm{l}(J) = \mathrm{r}(J)$ (*for example,* $J^2 = 0$ *or R is commutative*), *we can replace the two conditions above by either one.*

PROOF. This is clear from Theorem 6 and [8], Theorem 5.

As is well known, R is a QF-ring if and only if R is self-injective as a right R-module. However, R is actually quasi-injective as a right R-module from the definition of quasi-injective, and so R is injective as a right R-module by Baer's criterion. Hence the concept dual to the above is the following: A (minimal) injective cogenerator is quasi-projective. Thus we have the following corollary.

COROLLARY 2. *Let R be as above. Then R is a* QF-*ring if and only if the minimal injective cogenerator is quasi-projective.*

PROOF. We assume that the minimal injective cogenerator is quasi-projective. Then every injective is quasi-projective by [10] and the proof of [4], Proposition 2.4. Put $E = (R)$. Then $E \approx \Sigma_{i=1}^{k} \oplus e_iR/e_iA_i$, where the e_i are primitive

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idempotents, the A_i are right ideals and $e_jRe_ie_iA_i \subseteq e_jA_j$ from the proof of [6], Corollary 3 in page 790. We may assume that $R \subsetneq \Sigma_{i=1}^{k} \oplus e_iR/e_iA_i$ as a right R-module and R is basic (see [13] and [14]). We note, from Corollary 2 to Theorem 5, that the set $\{e_i\}$ contains the set of all non-isomorphic primitive idempotents. Let $1 = \Sigma \bar{a}_i$, where $a_i = e_ia_i$ and $\bar{a}_i = a_i + e_iA_i$. Then $J = \mathrm{J}(R) \subseteq \Sigma \oplus \bar{a}_iJ$. We shall show $\bar{a}_jJe\mathrm{l}(J) = 0$ for all i and j. Then, since $\mathrm{l}(J) = \Sigma_K \oplus e_i\mathrm{l}(J)$, where $K \subseteq \{1, 2, \ldots, k\}$, $\mathrm{l}(J) \subseteq \mathrm{r}(J)$. We note that $e_jRe_i = \mathrm{Hom}_R(e_iR, e_jR)$ and each element in e_jRe_i induces an element in $\mathrm{Hom}_R(e_iR/e_iA_i, e_jR/e_jA_j)$ from the above. Now, $\bar{a}_1Je_1\mathrm{l}(J) = \bar{e}_1(a_1Je_1\mathrm{l}(J)) = (e_1a_1Je_1)(\bar{e}_1\mathrm{l}(J))$, where we obtain this equality by regarding $e_1a_1Je_1 \subseteq \mathrm{End}_R(e_1R/e_1A_1)$. Since $\mathrm{l}(J)$ is semi-simple, so is $e_1\mathrm{l}(J)$. Further $e_1a_1Je_1 \subset \mathrm{J}(\mathrm{End}_R(e_1R))$ and each element in $\mathrm{J}(\mathrm{End}_R(e_1R))$ induces an element in $\mathrm{J}(\mathrm{End}_R(e_iR/e_1A_1))$. Hence $(e_1a_1Je_1)(\bar{e}_1\mathrm{l}(J)) = 0$, for e_1R/e_1A_1 is uniform. Next we consider $e_1a_1Je_2\mathrm{l}(J)$. If $e_2R \approx e_1R$, then we have $\bar{e}_1a_1Je_2\mathrm{l}(J) = 0$ from the above (note $e_1R/e_1A_1 \approx e_2R/e_2A_2$). We assume $e_2R \not\approx e_1R$. Case 1: $e_2\mathrm{l}(J) \subseteq e_2A_2$. Then $e_1A_1 \supseteq e_1Re\mathrm{l}(J) \supseteq e_1a_1Je_2\mathrm{l}(J)$, since $e_1Re_2e_2A_2 \subseteq e_1A_1$. Hence $\bar{a}Je_2\mathrm{l}(J) = 0$. Case 2: $e_2\mathrm{l}(J) \not\subseteq e_2A_2$ and $e_1a_1Je_2\mathrm{l}(J) \not\subseteq e_1A_1$. Then e_1R/e_1A_1 and e_2R/e_2A_2 contain the simple module isomorphic to $\bar{e}_2\mathrm{l}(J)$, which is a contradiction. Hence, if $e_2\mathrm{l}(J) \not\subseteq e_2A_2$, $e_1a_1Je_2\mathrm{l}(J) \subseteq e_1A_1$. Therefore $\bar{a}_1Je_2\mathrm{l}(J) = 0$. Similarly, $a_iJe_j\mathrm{l}(J) = 0$, and so $\mathrm{l}(J) \subseteq \mathrm{r}(J)$. Since quasi-projective is maxi-quasiprojective, we have the corollary from Theorems 2 and 6.

Finally, we take an algebra. Let K be a field and let R be a K-algebra of finite dimension. In this case we note that we have the duality functor $\mathrm{Hom}_K(-, K) = (-)^*$. Then every injective right R-module E has the lifting property of direct decompositions of $\bar{E}$ if and only if every projective left R-module has the extending property of direct decompositions of the socle; namely, R is left mini-injective and so R is a QF-algebra by [9] (we note that we may restrict ourselves to the cases where every module is finitely generated by [7]). Therefore the following theorem is clear from the above and [9], Theorem 1. We shall give the dual proof for the sake of completeness.

THEOREM 7. *Let R be an algebra over a field K with $[R : K]$ finite. Then the following conditions are equivalent*:

1) *R is a* QF-*ring.*

2) *A minimal injective cogenerator is maxi-quasiprojective.*

3) *R is a right self mini-injective ring.*

PROOF. 1) → 3). This is clear from [2].

3) → 2). Since R^* is an injective cogenerator as a left R-module, we obtain 2) for the left R-modules.

2) → 1). We may assume that R is basic and we use the same notations above. Since R is an algebra of finite dimension, every indecomposable injective is finitely generated and isomorphic to $(Re_i)^*$. We denote $(Re_i)^*$ by E_i. Then $S_i = \mathrm{End}_R(E_i)$ is anti-isomorphic to e_iRe_i. Let N_i be a maximal submodule of E_i and put $\tilde{S}_i = \{x \in S_i \mid x(N_i) \subseteq N_i\}$. Since $N_i = (Re_i/T_i)^*$ for some minimal left ideal T_i of Re_i and $T_i\mathrm{J}(R) = 0$ by Corollary 2 to Theorem 5, $\mathrm{J}(S_i)N_i \subseteq N_i$, and so $\mathrm{J}(S_i) \subseteq \tilde{S}_i$. Then $\mathrm{End}_R(E_i/N_i) = \tilde{S}_i/\mathrm{J}(S_i)$ from 2). Hence $\mathrm{End}_R(E_i/N_i)$ is anti-isomorphic to a K-subfield of $\overline{e_iRe_i}$. We put $E_i/N_i \approx \overline{e_{i'}R}$. Then $[\overline{e_{i'}Re_{i'}} : K] \leqslant [\overline{e_iRe_i} : K]$. Thus we obtain a chain of idempotents $\{e_1, e_2, \ldots, e_t, \ldots\}$ such that $E_i = (Re_i)^*$ and E_i contains a maximal submodule N_i with $E_i/N_i \approx \overline{e_{i+1}R}$. If $e_iR \approx e_{i+k}R$ for some i and k, $E_{i-1} \approx E_{i+k-1}$ by Lemma 3. Hence $e_{i-1} \approx e_{i+k-1}$. We know from this fact that the mapping: $i \to i'$ gives us a permutation of $\{1, 2, \ldots, n\}$, where $\Sigma_{i=1}^n \oplus (Re)^*$ is a minimal injective cogenerator. Hence $[\overline{e_{i'}Re_{i'}} : K] = [\overline{e_iRe_i} : K]$. Let N_1 and N_2 be two maximal submodules of E_i. Then $E_i/N_1 \approx E_i/N_2$ by Corollary 2 to Theorem 5. Hence there exists an automorphism x of E_i such that $x(N_1) = N_2$ by Lemma 3. On the other hand, $S_i = \tilde{S}_i$ from the argument above. Hence $N_2 = x(N_1) \subseteq N_1$, and so N_1 is a unique maximal submodule of E_i. Therefore R is right QF-2*. Accordingly, every injective E has the lifting property of direct decompositions of E by Theorem 2. Then we obtain a non-isomorphic representative set of indecomposable injectives $\{e_1R/e_1A_1, e_2R/e_2A_2, \ldots, e_nR/e_nA_n\}$ by [5], Theorem 3 and [6], Theorem 3. Hence $\{(e_iR/e_iA_i)^*\}_1^n$ is a non-isomorphic representative set of indecomposable and projective left R-modules. Therefore $\Sigma_{i=1}^n \oplus (e_iR/e_iA_i)^* \approx R$ as left R-modules. Accordingly, $[R : K] = \Sigma_{i=1}^n[(e_iR/e_iA_i)^* : K] = \Sigma_{i=1}^n[e_iR/e_iA_i : K]$. Hence $e_iA_i = 0$ for all i, and so R is self-injective.

3. Self mini-injective rings

We shall add a characterization of right QF-2 and self mini-injective rings.

THEOREM 8. *Let R be a right artinian and basic ring. Then R is a right* QF-2 *and self mini-injective ring if and only if* $\mathrm{l}(J) = Ru = uR$ *for some u in R.*

PROOF. Let $R = \Sigma_{i=1}^n \oplus e_iR$ be as above. We assume that R is a right QF-2 and self mini-injective ring. Then $e_i\mathrm{l}(J) = u_iR$ and $u_i \in \overline{e_iRe_{i'}}$. We know from [8], Theorem 3 that $\mathrm{l}(J) \subseteq \mathrm{r}(J)$. Therefore, since u_iR is a unique minimal right ideal, $Ru_i \subseteq u_iR$. $\overline{e_{i'}Re_{i'}}$ being a division ring, a mapping: $u_i \to u_ir (r \in \overline{e_{i'}Re_{i'}})$ is extendable to an element in $\mathrm{Hom}_R(u_iR, u_iR)$. Then, since R is right self mini-injective, there exists an element x in e_iRe_i with $xu_i = u_ir$. Therefore $Ru_i = u_iR$. Put $u = \sum_{i=1}^n u_i$. Then $e_iu = ue_{i'} = u_i$. Hence $uR = \sum_{i=1}^n \oplus u_iR = \sum_{i=1}^n \oplus Ru_i = Ru = \mathrm{l}(J)$. Conversely, we assume $\mathrm{l}(J) = uR = Ru$. Then $\mathrm{l}(J)$ is a homomorphic image of R/J as right R-modules. Hence $uR \approx R/J$ from the composition length. Therefore $\mathrm{l}(J) = \sum_{i=1}^n \oplus e_i\mathrm{l}(J)$, $e_i\mathrm{l}(J)$ is a unique minimal right ideal, and so R is right QF-2. Furthermore, $e_i\mathrm{l}(J) \not\approx e_j\mathrm{l}(J)$ if $i \neq j$. Hence $e_i\mathrm{l}(J)$ is a two-sided ideal, and so $\mathrm{l}(J) = uR = Ru$ implies $Re_iu = e_iuR = e_i\mathrm{l}(J)$. Therefore R is a right self mini-injective, since $\mathrm{End}_R(e_iuR) = e_iRe_i$ as above.

Theorem 9. *Let R be a right artinian ring. Then R is a* QF-*ring if and only if R is a right* QF-2, QF-2* *and self mini-injective ring.*

Proof. We assume that R satisfies the second condition of the theorem. We may assume that R is basic. Let $R = \sum_{i=1}^n \oplus e_iR$, where the e_i are primitive idempotents and $e_iR \not\approx e_jR$ if $i \neq j$. Since R is QF-2 and QF-2*, $E(e_iR) \approx e_jR/e_jA$ for some j and some right ideal A. Then we have the diagram

$$\begin{array}{ccc} & e_iR & \\ & \downarrow i & \\ e_jR & \xrightarrow{\nu} e_jR/e_jA & \to 0 \end{array}$$

where i is the inclusion and ν is the natural epimorphism. Since e_iR is projective, there exists $f\colon e_iR \to e_jR$ such that $i = \nu f$. i being a monomorphism, f is the same. Hence $S(e_iR) \approx S(e_jR)$ by the assumption. Therefore $i = j$ by [8], Theorem 5. The fact that $e_iR \subset e_iR/e_iA$ implies $e_iA = 0$. Hence R is self-injective.

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