Can. J. Math. Vol. **46** (4), 1994 pp. 772–792.

FORMULAS FOR BROWN-PETERSON OPERATIONS

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ABSTRACT. We introduce a new method to calculate compositions of Brown-Peterson operations. We derive a formula for $\mathbf{r}_1^n$ and a formula for commutators.

1. **Introduction.** The Quillen theorem [Q] determines the Brown-Peterson algebra of operations $\mathrm{BP}^*(\mathrm{BP})$ as a Hopf algebra over the coefficient ring $\pi_*(\mathrm{BP})$. But the calculation of the composition of any two specific Brown-Peterson operations and the nature of further algebraic structure remains difficult. Few results have been obtained, although one has the Zahler method [Z] which still has disadvantages in some aspects. For example, it involves recursive steps and calculates only up to a certain filtration level.

In this paper we will describe a new method to calculate the composition of any two specific Brown-Peterson operations. A full description of this method is presented in §3.2. As will be seen, it is purely combinatorial and easy to implement. Examples are given in §4. The theoretical justification of this method is a composition law which is stated in §3.1 and proved in §7. Furthermore, we are able to show that the composition of basic Brown-Peterson operations is a finite sum over the basic operations. We will also introduce a set of rational generators $\{w_i\}$ in §5 to formulate the iterated composition $\mathbf{r}_1^n$. In §6, we will introduce a set of rational Brown-Peterson operations $\{\mathbf{q}_W\}$ to prove a formula which generalizes one by Zahler in [Z].

This paper is a revision of part of my dissertation written under the supervision of Professor Richard M. Kane. I would like to express my deepest gratitude and the most sincere thanks to him for his able guidance and financial support during my Ph.D. candidacy. I would like to thank Professor Rick Jardine for his generous and precious help over the last few years. Finally, I would also like to thank the referee for his valuable encouragement and suggestions, and Professor Denis Sjerve for his careful reading of the English in this paper.

2. **Background.** We review the Quillen theorem and Zahler method for the Brown-Peterson algebra.

2.1. *Basic notation.* (1). p is assumed to be a fixed prime integer throughout the paper. $\mathbb{Z}$ is the set of integers, $\mathbb{Z}_{(p)}$ is the set of p-adic integers, and $\mathbb{Q}$ is the set of rational numbers.

The author is currently an Izaak Walton Killam Memorial Post-Doctoral Fellow.
Received by the editors December 7, 1992.
AMS subject classification: 55S20, 55N99, 55P42, 55T20.
Key words and phrases: Composition of Brown-Peterson operations, Quillen theorem, Zahler method.

(2). By an exponential sequence R we mean a sequence of non-negative integers $R = (r_1, r_2, r_3, \ldots)$ such that $r_i \neq 0$ for only a finite number of i's. The length $\ell(R)$ is the largest i such that $r_i \neq 0$. An exponential sequence can be multiplied by a non-negative integer and two exponential sequences can be added component-wise. The following notation is standard:

$$\begin{aligned} |R| &:= r_1 + pr_2 + p^2 r_3 + \cdots, \\ \|R\| &:= \sum_{i \geq 1} 2r_i(p^i - 1), \\ R' &:= (r_2, r_3, r_4, \ldots), \\ R_{(l)} &:= (0, \ldots, 0, \underset{l}{r_1}, r_2, \ldots), \\ \Delta_l &:= (0, \ldots, 0, \underset{l}{1}, 0, \ldots). \end{aligned}$$

(3). By a sequence of scalar parameters ξ we mean a sequence $\xi = (\xi_1, \xi_2, \xi_3, \ldots)$ which is independent of the Brown-Peterson spectrum BP. We write ξ^R for the monomial $\xi_1^{r_1} \xi_2^{r_2} \xi_3^{r_3} \cdots$.

(4). We will use the notation $\begin{bmatrix} A \\ B \end{bmatrix}$ to denote the coefficient of the monomial B in the expansion of the polynomial A in terms of a basis including B. Multinomial coefficients are denoted by $[k_1, k_2, \ldots, k_n]$ throughout the paper. For example,

$$\begin{bmatrix} (x_1 + x_2 + \cdots + x_n)^{k_1 + k_2 + \cdots + k_n} \\ x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n} \end{bmatrix} = [k_1, k_2, \ldots, k_n].$$

2.2. *Quillen theorem.* The Brown-Peterson algebra $\mathrm{BP}^*(\mathrm{BP})$ is the algebra of cohomology operations for the spectrum BP, the localization at p of the Thom spectrum **MU**. The coefficient ring $\pi_*(\mathrm{BP})$ has no torsion and is embedded in homology:

$$\begin{aligned} \pi_*(\mathrm{BP}) = \mathbb{Z}_{(p)}[v_1, v_2, v_3, \ldots] &\subset H_*(\mathrm{BP}) = \mathbb{Z}_{(p)}[m_1, m_2, m_3, \ldots], \\ \pi_*(\mathrm{BP}) \otimes \mathbb{Q} = H_*(\mathrm{BP}) \otimes \mathbb{Q} &= \mathbb{Q}[v_1, v_2, v_3, \ldots] = \mathbb{Q}[m_1, m_2, m_3, \ldots], \end{aligned}$$

where the $m_k = \frac{[\mathbb{C}P^{p^k - 1}]}{p^k}$, $k \geq 1$, are a set of rational generators, and the v_i are the Hazewinkel generators. Let $m_0 = 1, v_0 = p$. The degrees are $\deg(v_k) = \deg(m_k) = 2(p^k - 1)$ for $k \geq 0$. The following are the Hazewinkel relations [W]:

$$v_k = pm_k - \sum_{h=1}^{k-1} m_h v_{k-h}^{p^h}, \quad (k \geq 1).$$

D. Quillen [Q] and J. F. Adams [A] determined the Hopf algebra structure of $\mathrm{BP}_*(\mathrm{BP})$. As an algebra over $\pi_*(\mathrm{BP})$, $\mathrm{BP}_*(\mathrm{BP})$ has generators $t_i \in \mathrm{BP}_{2(p^i - 1)}(\mathrm{BP})$, $i \geq 1$, such that

$$\mathrm{BP}_*(\mathrm{BP}) = \pi_*(\mathrm{BP})[t_1, t_2, t_3, \ldots].$$

This also describes the left module structure over $\pi_*(\mathrm{BP})$. The coalgebra structure

$$\psi\colon \mathrm{BP}_*(\mathrm{BP}) \longrightarrow \mathrm{BP}_*(\mathrm{BP}) \bigotimes_{\pi_*(\mathrm{BP})} \mathrm{BP}_*(\mathrm{BP})$$

is given by the following inductive formulas:

$$\begin{cases} t_0 = 1, \\ \sum_{i+j=k} m_i(\psi t_j)^{p^i} = \sum_{h+i+j=k} m_h t_i^{p^h} \otimes t_j^{p^{h+i}}, \quad k \geq 1. \end{cases}$$

$\mathrm{BP}^*(\mathrm{BP})$ is the dual Hopf algebra of $\mathrm{BP}_*(\mathrm{BP})$ over $\pi_*(\mathrm{BP})$. For each exponential sequence R, there is a basic Brown-Peterson operation $\mathbf{r}_R \in \mathrm{BP}^*(\mathrm{BP})$ dual to the monomial t^R. The degree of the operation $\mathbf{r}_R$ is given by $\deg(\mathbf{r}_R) = \|R\|$. Any Brown-Peterson operation is a sum, having a possibly infinite number of terms, of the form:

$$\sum_R c_R \mathbf{r}_R, \quad c_R \in \pi_*(\mathrm{BP}) = \mathrm{BP}^{-*}(\mathrm{pt}) \subset \mathrm{BP}^{-*}(\mathrm{BP}).$$

The following theorem determines the composition of any two basic Brown-Peterson operations:

THEOREM 2.1 (QUILLEN). *Let $\xi = (\xi_1, \xi_2, \xi_3, \ldots)$ and $\eta = (\eta_1, \eta_2, \eta_3, \ldots)$ be two sequences of scalar parameters ($\xi_0 = \eta_0 = 1$). Define the Quillen polynomials $\mathbf{\Phi} = (\mathbf{\Phi}_1, \mathbf{\Phi}_2, \mathbf{\Phi}_3, \ldots)$ inductively by the Quillen relations:*

$$\begin{cases} \mathbf{\Phi}_0 = 1, \\ \sum_{i+j=k} m_i \mathbf{\Phi}_j^{p^i} = \sum_{h+i+j=k} m_h \xi_i^{p^h} \eta_j^{p^{h+i}}, \quad k \geq 1. \end{cases}$$

Then

$$\sum_{R,S} \xi^R \eta^S \cdot \mathbf{r}_R \mathbf{r}_S = \sum_T \mathbf{\Phi}^T \mathbf{r}_T.$$

2.3. *Zahler method.* Because the Quillen relations are inductive, the actual expression of compositions of Brown-Peterson operations is very difficult to explicitly write down. R. Zahle [Z] made a key observation in his calculation of the compositions. The key to his method is the following

LEMMA 2.2 (ZAHLER).

$$\mathbf{r}_E m^F = \begin{cases} 0 & \textit{if } \|E\| > \|F\|; \\ 0 & \textit{if } \|E\| = \|F\|, \textit{ and } E \neq F; \\ 1 & \textit{if } E = F. \end{cases}$$

The Zahler method to calculate $\mathbf{r}_L \cdot \mathbf{r}_R$ for any two exponential sequences is:

1. $\mathbf{r}_L \mathbf{r}_R$ is a finite linear sum of distinct basic operations:

$$\mathbf{r}_L \cdot \mathbf{r}_R = \sum_E c_E \mathbf{r}_E, \quad c_E \in \pi_*(\mathrm{BP}).$$

where $\deg(\mathbf{r}_E) - \deg(c_E) = \deg(\mathbf{r}_L) + \deg(\mathbf{r}_R)$. Alternatively, one has $\|E\| \geq \|L\| + \|R\| = \|L + R\|$. The c_E are the coefficients to be determined.

2. Applying Lemma 2.2 to the monomial m^F with $\|F\| = \|L+R\|$, one has

$$c_F = \mathbf{r}_L(\mathbf{r}_R m^F) \in \mathbb{Z}_{(p)}.$$

3. By induction, one supposes that all coefficients c_E with $\|E\| < \|F\|$ are calculated. One can then calculate the coefficient c_F by applying Lemma 2.2:

$$c_F = \mathbf{r}_L \mathbf{r}_R m^F - \sum_{\|E\|<\|F\|} c_E \mathbf{r}_E m^F.$$

4. The above procedure terminates and one has a complete result for the composition, if one finishes the calculation at all levels of the following filtration of the algebra $\mathrm{BP}^*(\mathrm{BP})$:

$$\mathrm{F}^s\,\mathrm{BP}^n(\mathrm{BP}) := \left\{ \sum_E c_E \mathbf{r}_E \;\middle|\; \begin{array}{l} c_E \in \pi_*(\mathrm{BP}),\ \mathbf{r}_E \in \mathrm{BP}^*(\mathrm{BP}) \\ \deg(\mathbf{r}_E) - \deg(c_E) = n, \\ \deg(\mathbf{r}_E) \geq s,\ \text{for all } E \end{array} \right\}.$$

REMARK. Zahler used this method in his paper [Z] to produce a composition table of Brown-Peterson operations up to filtration 14 for $p = 2$. One thing which should be noted is that he used a different set of integral generators than the set of Hazewinkel generators.

3. **Composition law.** We state our composition law for Brown-Peterson operations and describe our method in calculating the composition of any two specific operations. Examples are given in the next section. The proof is deferred to §7 to avoid tedious details for the moment. Due to the complex nature of the composition itself, we have to use a great deal of notation in order to state our method.

3.1. *Notation and statement.* There are two sets of notation used for the composition law, both mimicking the Milnor notation in [M] for exponential sequences. We use [,] as a superscript for weighted sums and (,) for un-weighted sums.

3.1.1. Reduced cubic exponential matrices. Assume X is a "reduced cubic exponential matrix", that is, a triply-indexed sequence $(x_{h,i,j})_{h,i,j\geq 0}$ of non-negative integers such that: (a.)(reduced): $x_{h,0,0} = 0$, for $h \geq 0$; (b.)(exponential): $x_{h,i,j} \neq 0$ for only a finite number of triples (h,i,j). The length $\ell(X)$ is then defined to be the maximal integer $h+i+j$ such that $x_{h,i,j} \neq 0$. Associated with X are the following:

- Row or Column Sums:

$$S^{[1],(3)}(X)_i = \sum_{h,j} p^h x_{h,i,j}, \quad S^{[1],[2]}(X)_j = \sum_{h,i} p^{h+i} x_{h,i,j}, \quad S^{(2),(3)}(X)_h = \sum_{i,j} x_{h,i,j}$$

- Diagonal Sums:

$$T(X)_k = \sum_{h+i+j=k} x_{h,i,j}$$

- Multinomial Coefficients:

$$B(X)_k = \frac{\big(T(X)_k\big)!}{\prod_{h+i+j=k} x_{h,i,j}!}, \quad B(X) = \prod_{k\geq 1} B(X)_k.$$

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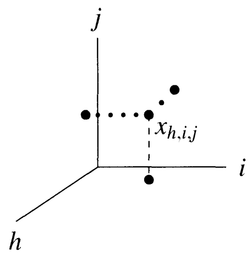
A reduced cubic exponential matrix X can be represented by a lattice in the first octant of h-i-j-space, where the number $x_{h,i,j}$ is located at the point (h, i, j):



Therefore row and column sums correspond to the weighted or unweighted sums of numbers on the vertical or horizontal planes; and the diagonal sums and the multinomial coefficients are determined by the numbers on the diagonal planes. Given exponential sequences L and R we say a reduced cubic exponential matrix X is (L, R)-feasible if $S^{[1],(3)}(X) = L$ and $S^{[1],[2]}(X) = R$.

3.1.2. Reduced exponential matrices. Assume Y is a "reduced exponential matrix", that is, an infinite matrix $(y_{\lambda,\mu})_{\lambda,\mu\geq 0}$ of non-negative integers, such that: (a.)(reduced): $y_{\lambda,0} = 0$, and $y_{0,\mu} = 0$ for $\lambda, \mu \geq 0$. (b.)(exponential): $y_{\lambda,\mu} \neq 0$ for only a finite number of pairs (λ, μ). The length $\ell(Y)$ is defined to be the maximal integer $\lambda + \mu$ such that $y_{\lambda,\mu} \neq 0$. Associated with Y are the following

- Weighted Column Sums:

$$S^{[1]}(Y)_\mu = \sum_\lambda p^\lambda y_{\lambda,\mu}$$

- Un-Weighted Row Sums:

$$S^{(2)}(Y)_\lambda = \sum_\mu y_{\lambda,\mu}$$

- Diagonal Sums:

$$T(Y)_k = \sum_{\lambda+\mu=k} y_{\lambda,\mu}$$

- Combinatorial Coefficients:

$$B(Y)_k = \frac{T(Y)_k!}{\prod_{\lambda+\mu=k} y_{\lambda,\mu}!}, \quad B(Y) = \prod_{k\geq 1} B(Y)_k$$

$$B(X, Y)_k = B(X)_k \cdot [T(X)_k, T(Y)_k] \cdot B(Y)_k, \quad [T(X), T(Y)] = \prod_k [T(X)_k, T(Y)_k]$$

$$B(X, Y) = B(X) \cdot [T(X), T(Y)] \cdot B(Y).$$

For exponential sequences T and W we can represent the pair (W, Y) by a table of the following form:

$$
\begin{array}{c|cccc}
 & w_1 & w_2 & w_3 & \cdots \\
\hline
 & y_{1,1} & y_{1,2} & y_{1,3} & \cdots \\
 & y_{2,1} & y_{2,2} & y_{2,3} & \cdots \\
 & \vdots & \vdots & \vdots & \ddots \\
\hline
 & & & &
\end{array}
$$

The pair (W, Y) is called *T-feasible* if $T + T(Y) = W + S^{[1]}(Y)$; or equivalently, from the table, if the weighted column sums of the (W, Y)-part, minus the diagonal sums of the Y-part, are the numbers for the sequence T.

3.1.3. Theorem. Our main theorem in this paper is the following composition law for Brown-Peterson operations.

THEOREM 3.1. *Given exponential sequences L and R,*

$$\mathbf{r}_L \cdot \mathbf{r}_R = \sum_{\substack{X,Y \\ S^{[1],(3)}(X)=L \\ S^{[1],[2]}(X)=R}} B(X,Y)\cdot m^{S^{(2),(3)}(X)} \cdot (-m)^{S^{(2)}(Y)} \cdot \mathbf{r}_{T(X)+T(Y)-S^{[1]}(Y)}.$$

The proof of this theorem will be given in §7. It is not readily apparent that the summation on the right hand side of the above formula is finite; but we prove this in Lemma 7.1. As a corollary we get the following theorem of Kane [K].

COROLLARY 3.2 (KANE). *Let L, R be exponential sequences. Let* $R(A)$, $S(A)$, $b(A)$ *and* $T(A)$ *be the Milnor notation in [M]. Then*

$$\mathbf{r}_L \cdot \mathbf{r}_R = \sum_{\substack{S(A)=L \\ R(A)=R}} b(A)\cdot \mathbf{r}_{T(A)} \operatorname{mod}(v_1, v_2, v_3, \ldots).$$

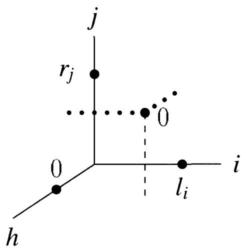
3.2. *Our method.* The composition law 3.1 is very complicated in its appearance. In practice, we adopt the following steps and use the rational operations $\mathbf{q}_T$ in §7 as a bridge in our calculation to work out a complete result.

Suppose we are given exponential sequences L and R and we want to calculate the composition $\mathbf{r}_L \cdot \mathbf{r}_R$ in terms of the basic operations $\mathbf{r}_W$.

STEP 1. Calculate $\mathbf{r}_L \cdot \mathbf{r}_R$ in terms of the rational operations $\mathbf{q}_T$. By Theorem 7.2,

$$\mathbf{r}_L \cdot \mathbf{r}_R = \sum_{(L,R)\text{-feasable } X} B(X)\cdot m^{S^{(2),(3)}(X)} \mathbf{q}_{T(X)}.$$

An obvious (L, R)-feasible reduced cubic exponential matrix is



where L is on the i-axis,
R is on the j-axis, and
0 is elsewhere.

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To find all (L, R)-feasible X, we start with this reduced cubic exponential matrix, and perform the following algorithm:

- (Phase I): Suppose $X = (x_{h,i,j})$ is (L, R)-feasible and $x_{h,i,j} = 0$ for all $h \neq 0$. Select (i_0, j_0), $j_0 \neq 0$, such that $x_{0,i_0,j_0} \geq p$. If $x_{0,i_0+1,0} \geq 1$, then define $X' = (x'_{h,i,j})$ by:

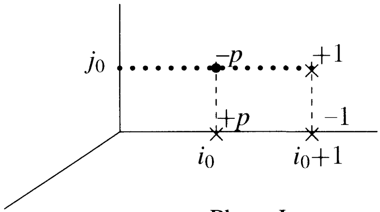
$$x'_{h,i,j} = \begin{cases} x_{h,i,j} - p & \text{if } (h, i, j) = (0, i_0, j_0), \\ x_{h,i,j} + p & \text{if } (h, i, j) = (0, i_0, 0), \text{ and } i_0 \neq 0, \\ x_{h,i,j} + 1 & \text{if } (h, i, j) = (0, i_0 + 1, j_0), \\ x_{h,i,j} - 1 & \text{if } (h, i, j) = (0, i_0 + 1, 0), \\ x_{h,i,j} & \text{otherwise.} \end{cases}$$

  Then X' is (L, R)-feasible.

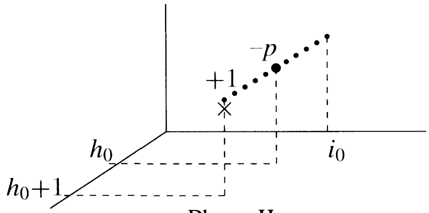
- (Phase II): Suppose $X = (x_{h,i,j})$ is (L, R)-feasible. Select a triple (h_0, i_0, j_0), $i_0 + j_0 \neq 0$, such that $x_{h_0,i_0,j_0} \geq p$. Define $X' = (x'_{h,i,j})$ by:

$$x'_{h,i,j} = \begin{cases} x_{h,i,j} - p & \text{if } (h, i, j) = (h_0, i_0, j_0), \\ x_{h,i,j} + 1 & \text{if } (h, i, j) = (h_0 + 1, i_0, j_0), \\ x_{h,i,j} & \text{otherwise.} \end{cases}$$

  Then X' is (L, R)-feasible.



Phase I



Phase II

Now for each (L, R)-feasible X, read off $T(X)$ and $B(X)$ from the diagonal planes, and $S^{(2),(3)}(X)$ from the planes perpendicular to the h-axis. Calculate $B(X) \cdot m^{S^{(2),(3)}(X)} \cdot \mathbf{q}_{T(X)}$.

STEP 2. Calculate $\mathbf{q}_T$ in terms of the basic operations $\mathbf{r}_W$ for each $T = T(X)$ obtained in Step 1. By Theorem 7.3,

$$\mathbf{q}_T = \sum_{T\text{-feasable}\,(W,Y)} [T, T(Y)] \cdot B(Y) \cdot (-m)^{S^{(2)}(Y)} \cdot \mathbf{r}_W.$$

An obvious T-feasible pair (W, Y) is given by

$$\begin{array}{c|cccc} & t_1 & t_2 & t_3 & \cdots \\ \hline & 0 & 0 & 0 & \cdots \\ & 0 & 0 & 0 & \cdots \\ & \vdots & \vdots & \vdots & \ddots \\ \hline \end{array}$$

All T-feasible pairs (W, Y) are generated from the above pair by the following process:

(1). Expand t_1 to the first column in base p:

$$t_1 = w_1 + py_{1,1} + p^2 y_{2,1} + \cdots, \qquad w_1, y_{\lambda,\mu} \geq 0.$$

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(2). Modify the remaining numbers t_k by setting $t'_k := t_k + y_{k,1}$, for all $k \geq 2$.

(3). Expand t'_2 to the second column in base p as in (1):

$$t'_2 = w_2 + py_{1,2} + p^2 y_{2,2} + \cdots, \qquad w_2, y_{\lambda,\mu} \geq 0.$$

(4). Modify the remaining numbers t'_k, $k \geq 3$ as in (2). Proceed with the next t_k.

(5). If we reach a zero column, let W be the sequence in the first row and Y be the remaining matrix. Then (W, Y) is T-feasible. The diagonal and the row sums can be easily calculated for the coefficient of $\mathbf{r}_W$.

$$\begin{array}{|cccc}
w_1 & t'_2 & t'_3 & \cdots \\ \hline
y_{1,1} & 0 & 0 & \cdots \\
y_{2,1} & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots \\ \hline
\end{array}
\quad \longrightarrow \quad
\begin{array}{|cccc}
w_1 & w_2 & t''_3 & \cdots \\ \hline
y_{1,1} & y_{1,2} & 0 & \cdots \\
y_{2,1} & y_{2,2} & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots \\ \hline
\end{array}
\quad \longrightarrow \quad \cdots$$

STEP 3. Calculate the total sum for the composition $\mathbf{r}_L \cdot \mathbf{r}_R$. If needed, the result can be transformed to an expression where all the coefficients are polynomials in the Hazewinkel generators $v_1, v_2, v_3, \ldots$. For example, the Hazewinkel relations imply that,

$$m_1 = \frac{v_1}{p}, \quad m_2 = \frac{v_2}{p} + \frac{v_1^{1+p}}{p^2}, \quad m_3 = \frac{v_3}{p} + \frac{v_1 v_2^p + v_1^{p^2} v_2}{p^2} + \frac{v_1^{1+p+p^2}}{p^3}.$$

On the other hand, if we need the result only up to a certain degree, the following formula will help to eliminate many terms in the generating process:

$$\deg(\mathbf{r}_W) = \sum_k 2(p^k - 1)T(X)_k + \sum_{\lambda,\mu} 2(p^\lambda - 1)y_{\lambda,\mu}.$$

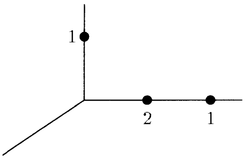
This completes the description of our method.

4. Examples.

We now present two examples in calculating compositions of Brown-Peterson operations. We assume $p = 2$ for convenience. It is clear that the case when p is an odd prime integer can be handled in the exact same way. The first example is simple so as to show the basic ideas and techniques of computation. Whereas, in the second example, there are 12 feasible reduced cubic exponential matrices X, 31 feasible pairs (W, Y), and 9 distinct basic operations in the final expression of the composition. This is truly marvelous.

4.1. *Example:* $\mathbf{r}_{2,1} \cdot \mathbf{r}_1$.

X:

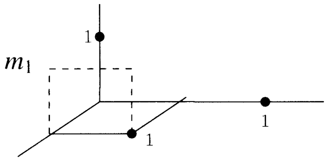


$T(X) = (3, 1)$,
coeff. of $\mathbf{q}_{T(X)} = \frac{3!}{2!\,1!}\frac{1!}{1!\,0!} \cdot m^0$.

	$W:$	$3\ 1\ 0$	$1\ 2\ 0$	$1\ 0\ 1$
$-m_1$	$Y:$		1	$1\ 1$
$-m_2$				
$[T(X), T(Y)]B(Y)(-m)^{S^{(2)}(X)}\mathbf{r}_W:$		$\mathbf{r}_{3,1}$	$-2m_1\mathbf{r}_{1,2}$	$2m_1^2\mathbf{r}_{1,0,1}$

So, $\mathbf{q}_{3,1} = \mathbf{r}_{3,1} - 2m_1\mathbf{r}_{1,2} + 2m_1^2\mathbf{r}_{1,0,1}$.

X:



$T(X) = (1, 2)$,
coeff. of $\mathbf{q}_{T(X)} = \frac{1!}{0!\,1!}\frac{2!}{1!\,1!} \cdot m_1^1$.

	$W:$	$1\ 2\ 0$	$1\ 0\ 1$
$-m_1$	$Y:$		1
$-m_2$			
$[T(X), T(Y)]B(Y)(-m)^{S^{(2)}(X)}\mathbf{r}_W:$		$\mathbf{r}_{1,2}$	$-m_1\mathbf{r}_{1,0,1}$

So,

$$\mathbf{q}_{1,2} = \mathbf{r}_{1,2} - m_1\mathbf{r}_{1,0,1}.$$

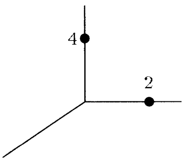
Hence, the product

$$\mathbf{r}_{2,1} \cdot \mathbf{r}_1 = 3\mathbf{r}_{3,1} - 4m_1\mathbf{r}_{1,2} + 4m_1^2\mathbf{r}_{1,0,1} = 3\mathbf{r}_{3,1} - 2v_1\mathbf{r}_{1,2} + v_1^2\mathbf{r}_{1,0,1}.$$

4.2. *Example:* $\mathbf{r}_2 \cdot \mathbf{r}_4$.

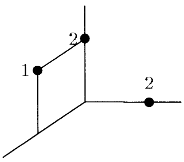
STEP 1. All possible $(2, 4)$-feasible cubic exponential matrices X are:

(A-1):



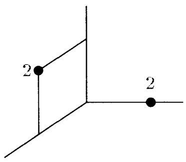
15

(A-2):



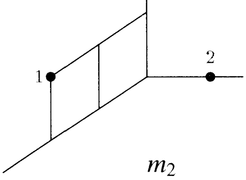
$6m_1$

(A-3):



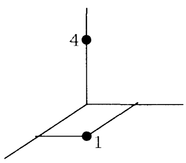
m_1^2

(A-4):



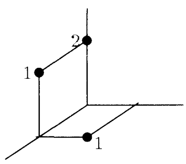
m_2

(A-5):



m_1

(A-6):



$2m_1^2$

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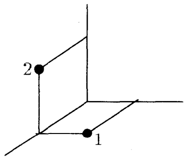
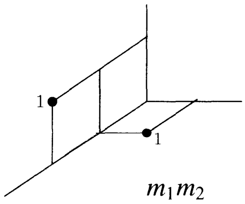
(A-7):



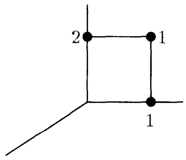
$3m_1^3$

(A-8):



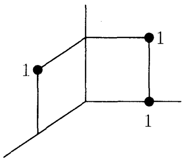
m_1m_2

(B-1):



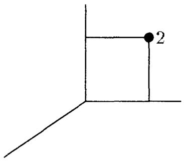
3

(B-2):



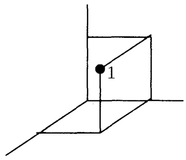
$2m_1$

(C-1):



1

(C-2):



m_1

STEP 2. The (W, Y)-tables are listed below:

(1). For (A-1):

$$\underset{\mathbf{r}_6}{\begin{array}{|l} 6\ 0\ 0 \\ \hline \\ \hline \end{array}} \qquad \underset{-m_1\mathbf{r}_{4,1}}{\begin{array}{|l} 4\ 1\ 0 \\ \hline 1 \\ \hline \end{array}} \qquad \underset{m_1^2\mathbf{r}_{2,2}}{\begin{array}{|l} 2\ 2\ 0 \\ \hline 2 \\ \hline \end{array}} \qquad \underset{-m_1^3\mathbf{r}_{2,0,1}}{\begin{array}{|l} 2\ 0\ 1 \\ \hline 2\ 1 \\ \hline \end{array}}$$

$$\underset{-m_2\mathbf{r}_{2,0,1}}{\begin{array}{|l} 2\ 0\ 1 \\ \hline 0 \\ 1 \\ \hline \end{array}} \qquad \underset{-m_1^3\mathbf{r}_{0,3}}{\begin{array}{|l} 0\ 3\ 0 \\ \hline 3 \\ 0 \\ \hline \end{array}} \qquad \underset{m_1^4\mathbf{r}_{0,1,1}}{\begin{array}{|l} 0\ 1\ 1 \\ \hline 3\ 1 \\ 0 \\ \hline \end{array}} \qquad \underset{m_1m_2\mathbf{r}_{0,1,1}}{\begin{array}{|l} 0\ 1\ 1 \\ \hline 1 \\ 1 \\ \hline \end{array}}$$

(2). For (A-2) and (A-5):

$$\underset{\mathbf{r}_{4,1}}{\begin{array}{|l} 4\ 1\ 0 \\ \hline \\ \hline \end{array}} \qquad \underset{-2m_1\mathbf{r}_{2,2}}{\begin{array}{|l} 2\ 2\ 0 \\ \hline 1 \\ \hline \end{array}} \qquad \underset{2m_1^2\mathbf{r}_{2,0,1}}{\begin{array}{|l} 2\ 0\ 1 \\ \hline 1\ 1 \\ \hline \end{array}}$$

$$\underset{3m_1^2\mathbf{r}_{0,3}}{\begin{array}{|l} 0\ 3\ 0 \\ \hline 2 \\ \hline \end{array}} \qquad \underset{-3m_1^3\mathbf{r}_{0,1,1}}{\begin{array}{|l} 0\ 1\ 1 \\ \hline 2\ 1 \\ 0 \\ \hline \end{array}} \qquad \underset{-m_2\mathbf{r}_{0,1,1}}{\begin{array}{|l} 0\ 1\ 1 \\ \hline 0 \\ 1 \\ \hline \end{array}}$$

(3). For (A-3) and (A-6):

$$\underset{\mathbf{r}_{2,2}}{\begin{array}{|ccc}2&2&0\\\hline &&\\ &&\\\hline\end{array}}\qquad \underset{-3m_1\mathbf{r}_{0,3}}{\begin{array}{|ccc}0&3&0\\\hline 1&&\\ &&\\\hline\end{array}}\qquad \underset{-m_1\mathbf{r}_{2,0,1}}{\begin{array}{|ccc}2&0&1\\\hline 0&1&\\ &&\\\hline\end{array}}\qquad \underset{3m_1^2\mathbf{r}_{0,1,1}}{\begin{array}{|ccc}0&1&1\\\hline 1&1&\\ &&\\\hline\end{array}}$$

(4). For (A-4):

$$\underset{\mathbf{r}_{2,0,1}}{\begin{array}{|ccc}2&0&1\\\hline &&\\ &&\\\hline\end{array}}\qquad \underset{-m_1\mathbf{r}_{0,1,1}}{\begin{array}{|ccc}0&1&1\\\hline 1&&\\ &&\\\hline\end{array}}$$

(5). For (A-7):

$$\underset{\mathbf{r}_{0,3}}{\begin{array}{|ccc}0&3&0\\\hline &&\\ &&\\\hline\end{array}}\qquad \underset{-m_1\mathbf{r}_{0,1,1}}{\begin{array}{|ccc}0&1&1\\\hline &1&\\ &&\\\hline\end{array}}$$

(6). For (A-8):

$$\underset{\mathbf{r}_{0,1,1}}{\begin{array}{|ccc}0&1&1\\\hline &&\\ &&\\\hline\end{array}}$$

(7). For (B-1):

$$\underset{\mathbf{r}_{3,1}}{\begin{array}{|ccc}3&1&0\\\hline &&\\ &&\\\hline\end{array}}\qquad \underset{-2m_1\mathbf{r}_{1,2}}{\begin{array}{|ccc}1&2&0\\\hline 1&&\\ &&\\\hline\end{array}}\qquad \underset{2m_1^2\mathbf{r}_{1,0,1}}{\begin{array}{|ccc}1&0&1\\\hline 1&1&\\ &&\\\hline\end{array}}$$

(8). For (B-2):

$$\underset{\mathbf{r}_{1,2}}{\begin{array}{|ccc}1&2&0\\\hline &&\\ &&\\\hline\end{array}}\qquad \underset{-m_1\mathbf{r}_{1,0,1}}{\begin{array}{|ccc}1&0&1\\\hline &1&\\ &&\\\hline\end{array}}$$

(9). For (C-1):

$$\underset{\mathbf{r}_{0,2}}{\begin{array}{|ccc}0&2&0\\\hline &&\\ &&\\\hline\end{array}}\qquad \underset{-m_1\mathbf{r}_{0,0,1}}{\begin{array}{|ccc}0&0&1\\\hline &1&\\ &&\\\hline\end{array}}$$

(10). For (C-2):

$$\underset{\mathbf{r}_{0,0,1}}{\begin{array}{|ccc}0&0&1\\\hline &&\\ &&\\\hline\end{array}}$$

Hence:

$$
\begin{aligned}
\mathbf{r}_2 \cdot \mathbf{r}_4 = {} & \text{The Total Sum} \\
= {} & 15\big(\mathbf{r}_6 - m_1\mathbf{r}_{4,1} + m_1^2\mathbf{r}_{2,2} - (m_1^3 + m_2)\mathbf{r}_{2,0,1} - m_1^3\mathbf{r}_{0,3} + (m_1^4 + m_1m_2)\mathbf{r}_{0,1,1}\big) \\
& + (6m_1 + m_1)\cdot\big(\mathbf{r}_{4,1} - 2m_1\mathbf{r}_{2,2} + 2m_1^2\mathbf{r}_{2,0,1} + 3m_1^2\mathbf{r}_{0,3} - (3m_1^3 + m_2)\mathbf{r}_{0,1,1}\big) \\
& + (m_1^2 + 2m_1^2)\cdot(\mathbf{r}_{2,2} - 3m_1\mathbf{r}_{0,3} - m_1\mathbf{r}_{2,0,1} + 3m_1^2\mathbf{r}_{0,1,1}) \\
& + m_2\cdot(\mathbf{r}_{2,0,1} - m_1\mathbf{r}_{0,1,1}) + 3m_1^3\cdot(\mathbf{r}_{0,3} - m_1\mathbf{r}_{0,1,1}) \\
& + m_1m_2\cdot\mathbf{r}_{0,1,1} + 3\cdot(\mathbf{r}_{3,1} - 2m_1\mathbf{r}_{1,2} + 2m_1^2\mathbf{r}_{1,0,1}) \\
& + 2m_1\cdot(\mathbf{r}_{1,2} - m_1\mathbf{r}_{1,0,1}) + (\mathbf{r}_{0,2} - m_1\mathbf{r}_{0,0,1}) + m_1\cdot\mathbf{r}_{0,0,1}. \\
= {} & 15\mathbf{r}_6 - 8m_1\mathbf{r}_{4,1} + 4m_1^2\mathbf{r}_{2,2} - (14m_2 + 4m_1^3)\mathbf{r}_{2,0,1} \\
& + 8m_1m_2\mathbf{r}_{0,1,1} + 3\mathbf{r}_{3,1} - 4m_1\mathbf{r}_{1,2} + 4m_1^2\mathbf{r}_{1,0,1} + \mathbf{r}_{0,2} \\
= {} & 15\mathbf{r}_6 - 4v_1\mathbf{r}_{4,1} + v_1^2\mathbf{r}_{2,2} - (7v_2 + 4v_1^3)\mathbf{r}_{2,0,1} \\
& + v_1(2v_2 + v_1^3)\mathbf{r}_{0,1,1} + 3\mathbf{r}_{3,1} - 2v_1\mathbf{r}_{1,2} + v_1^2\mathbf{r}_{1,0,1} + \mathbf{r}_{0,2}.
\end{aligned}
$$

5. **Iterated products.** We define the weighted symmetric polynomials which will be used to express the iterated products of Brown-Peterson operations. We then introduce a new set of rational generators for $\pi_*(\mathrm{BP})$, and use it to calculate the product $\mathbf{r}^n_{\Delta_{(l)}}$.

5.1. *Weighted symmetric polynomials.*

DEFINITION. Suppose $\alpha_1, \alpha_2, \ldots, \alpha_n$ are scalar parameters. The weighted symmetric polynomials for $\alpha_1, \alpha_2, \ldots, \alpha_n$ are defined to be

$$
\begin{cases}
\sigma_1 = \alpha_1 + \alpha_2 + \cdots + \alpha_n \\
\sigma_2 = \alpha_1^p\alpha_2 + \cdots + \alpha_1^p\alpha_n + \alpha_2^p\alpha_3 + \cdots + \alpha_{n-1}^p\alpha_n \\
\vdots \\
\sigma_n = \alpha_1^{p^{n-1}}\alpha_2^{p^{n-2}}\cdots\alpha_n.
\end{cases}
$$

Notice the similarity with the well-known elementary symmetric polynomials. Denote by $\sigma_i^{(p^k)}$ the weighted symmetric polynomials for the parameters $\alpha_1^{p^k}, \alpha_2^{p^k}, \ldots, \alpha_n^{p^k}$. Let $\sigma_i^{(p^k)} = 0$ if $i > n$, and $\sigma_0^{(p^k)} = 1$.

THEOREM 5.1. *Define polynomials Ω_k in the parameters α_i with coefficients in $\pi_*(\mathrm{BP})$ by:*

$$
\begin{cases}
\Omega_0 = 1 \\
\sum_{i+j=k} m_i\Omega_j^{p^i} = \sum_{i+j=k} m_i\sigma_j^{(p^i)}, \quad k \geq 1.
\end{cases}
$$

Then

$$
\sum_{r_1, r_2, \ldots, r_n \geq 0} \mathbf{r}_{r_n}\cdots\mathbf{r}_{r_2}\cdot\mathbf{r}_{r_1}\cdot\alpha_1^{r_1}\alpha_2^{r_2}\cdots\alpha_n^{r_n} = \sum_T \mathbf{r}_T\cdot\Omega^T.
$$

REMARK. The operations $\mathbf{r}_{r_i}$ are in a reverse order. There is a similar formula for iterated products of $\mathbf{r}_{r_i\Delta_l}$ for any $l \geq 1$. The weighted symmetric polynomials σ_i occur also in a formula for iterated products of Steenrod operations. We will treat this topic along with applications in a separate paper [L].

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PROOF. It is observed that the Quillen relations are equivalent to the matrix equation:

$$\begin{pmatrix} \Phi_0 & 0 & 0 & \cdots \\ \Phi_1 & \Phi_0^p & 0 & \cdots \\ \Phi_2 & \Phi_1^p & \Phi_0^{p^2} & \cdots \\ \Phi_3 & \Phi_2^p & \Phi_1^{p^2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} m_0 \\ m_1 \\ m_2 \\ m_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} \eta_0 & 0 & 0 & \cdots \\ \eta_1 & \eta_0^p & 0 & \cdots \\ \eta_2 & \eta_1^p & \eta_0^{p^2} & \cdots \\ \eta_3 & \eta_2^p & \eta_1^{p^2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \xi_0 & 0 & 0 & \cdots \\ \xi_1 & \xi_0^p & 0 & \cdots \\ \xi_2 & \xi_1^p & \xi_0^{p^2} & \cdots \\ \xi_3 & \xi_2^p & \xi_1^{p^2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} m_0 \\ m_1 \\ m_2 \\ m_3 \\ \vdots \end{pmatrix}.$$

Write $\Phi = \xi * \eta$. We then find that this star operation is associative: $(\xi * \eta) * \zeta = \xi * (\eta * \zeta)$ and has an identity $0 = (0, 0, 0, \ldots)$: $\xi * 0 = 0 * \xi = \xi$. An obvious induction from the Quillen theorem 2.1 shows that

$$\sum_{R(1), R(2), \ldots, R(n)} \mathbf{r}_{R(1)} \mathbf{r}_{R(2)} \cdots \mathbf{r}_{R(n)} \xi_{(1)}^{R(1)} \xi_{(2)}^{R(2)} \cdots \xi_{(n)}^{R(n)} = \sum_T \mathbf{r}_T (\xi_{(1)} * \xi_{(2)} * \cdots * \xi_{(n)})^T.$$

In particular, if $\xi_{(i)} = (\alpha_{n-i+1}, 0, 0, \ldots)$, $i = 1, 2, \ldots, n$, the matrix equation for the polynomials $\Omega_i := (\xi_{(1)} * \xi_{(2)} * \cdots * \xi_{(n)})_i$ is

$$\begin{pmatrix} 1 & 0 & 0 & \cdots \\ \Omega_1 & 1 & 0 & \cdots \\ \Omega_2 & \Omega_1^p & 1 & \cdots \\ \Omega_3 & \Omega_2^p & \Omega_1^{p^2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} m_0 \\ m_1 \\ m_2 \\ m_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ \alpha_1 & 1 & 0 & \cdots \\ 0 & \alpha_1^p & 1 & \cdots \\ 0 & 0 & \alpha_1^{p^2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \cdots \begin{pmatrix} 1 & 0 & 0 & \cdots \\ \alpha_n & 1 & 0 & \cdots \\ 0 & \alpha_n^p & 1 & \cdots \\ 0 & 0 & \alpha_n^{p^2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} m_0 \\ m_1 \\ m_2 \\ m_3 \\ \vdots \end{pmatrix}.$$

This can be solved for Ω_i inductively and is equivalent to the set of relations declared in the theorem.

5.2. $\mathbf{r}_{\Delta_l}^n$.

DEFINITION. Introduce elements $w_k \in \pi_*(\mathrm{BP}) \otimes \mathbb{Q}$ by:

$$\begin{cases} w_0 = 1 \\ w_k + w_{k-1}^p m_1 + w_{k-2}^{p^2} m_2 + \cdots + w_0^{p^k} m_k = 0, \quad k \geq 1. \end{cases}$$

It is easy to see that $\{w_1, w_2, w_3, \ldots\}$ is also a set of rational generators for $\pi_*(\mathrm{BP})$. We find it useful to replace the generators $\{m_1, m_2, m_3, \ldots\}$ of $\pi_*(\mathrm{BP}) \otimes \mathbb{Q}$ by this set.

THEOREM 5.2. *Let $l \geq 1$. Suppose that $s_1, s_2, \ldots, s_n$ are integers such that $0 \leq s_i \leq p-1$, $i = 1, 2, \ldots, n$. Then*

$$\mathbf{r}_{s_1 \Delta_l} \mathbf{r}_{s_2 \Delta_l} \cdots \mathbf{r}_{s_n \Delta_l} = \sum_{|T| = s_1 + s_2 + \cdots + s_n} [s_1, s_2, \ldots, s_n] \cdot w^{T'} \cdot \mathbf{r}_{T_{(l)}}.$$

In particular, we have

$$\mathbf{r}_{\Delta_l}^n = \sum_{|T| = n} n! \cdot w^{T'} \cdot \mathbf{r}_{T_{(l)}}$$

$$\mathbf{r}_{s_1} \mathbf{r}_{s_2} \cdots \mathbf{r}_{s_n} = \sum_{|T| = s_1 + s_2 + \cdots + s_n} [s_1, s_2, \ldots, s_n] \cdot w^{T'} \cdot \mathbf{r}_T.$$

PROOF. For brevity, we prove only the case where $l = 1$. This we do by applying Theorem 5.1. The general case can be proved by making the same kind of argument using a formula for iterated products of $\mathbf{r}_{r_i\Delta_l}$. In Theorem 5.1, $\mathbf{r}_{s_n}\cdots\mathbf{r}_{s_2}\mathbf{r}_{s_1}$ is the coefficient of $\alpha_1^{s_1}\alpha_2^{s_2}\cdots\alpha_n^{s_n}$ on the left hand side. On the right hand side,

$$\mathbf{r}_{s_n}\cdots\mathbf{r}_{s_2}\mathbf{r}_{s_1} = \sum_T \left[\begin{matrix}\Omega^T\\ \alpha_1^{s_1}\alpha_2^{s_2}\cdots\alpha_n^{s_n}\end{matrix}\right]\mathbf{r}_T.$$

We want to find the coefficient $\left[\begin{matrix}\Omega^T\\ \alpha_1^{s_1}\alpha_2^{s_2}\cdots\alpha_n^{s_n}\end{matrix}\right]$.

Let $\sigma := \sigma_1^{(1)} = \alpha_1 + \alpha_2 + \cdots + \alpha_n$. Then $\Omega_1 = \sigma$.

We consider polynomials in the α_i with coefficients in $\pi_*(\mathrm{BP})\otimes\mathbb{Q}$. Denote by $I = \langle\alpha_1^p,\ldots,\alpha_n^p\rangle$ the ideal generated by $\alpha_1^p, \alpha_2^p, \ldots, \alpha_n^p$. Trivially, we have

$$\sigma_1^{(p^k)} \equiv 0 \pmod I, \quad k \geq 1;\ \sigma_i^{(p^k)} \equiv 0 \pmod I, \quad k \geq 0,\ i > 1.$$

Hence,

$$\begin{aligned}\Omega_2 + \Omega_1^p m_1 &\equiv 0\\ \Omega_3 + \Omega_2^p m_1 + \Omega_1^{p^2} m_2 &\equiv 0\\ &\vdots\\ \Omega_k + \Omega_{k-1}^p m_1 + \cdots + \Omega_1^{p^{k-1}} m_{k-1} &\equiv 0\\ &\vdots \qquad\qquad \pmod I\end{aligned}$$

A simple induction shows that

$$\Omega_k \equiv \sigma^{p^{k-1}} w_{k-1} \pmod I, \quad k \geq 1.$$

Thus

$$\begin{aligned}\Omega^T &= \Omega_1^{t_1}\Omega_2^{t_2}\cdots\Omega_k^{t_k}\cdots\\ &\equiv \sigma^{t_1+pt_2+p^2t_3+\cdots}\cdot w_0^{t_1}w_1^{t_2}\cdots w_{k-1}^{t_k}\cdots\\ &= \sigma^{|T|}w^{T'} \pmod I.\end{aligned}$$

Let $0 \leq s_1, s_2, \ldots, s_n \leq p-1$.

$$\left[\begin{matrix}\Omega^T\\ \alpha_1^{s_1}\alpha_2^{s_2}\cdots\alpha_n^{s_n}\end{matrix}\right] = \begin{cases}[s_1, s_2, \ldots, s_n]\cdot w^{T'} & \text{if } |T| = s_1 + s_2 + \cdots + s_n,\\ 0 & \text{otherwise.}\end{cases}$$

This gives the formula for the product $\mathbf{r}_{s_n}\cdots\mathbf{r}_{s_2}\mathbf{r}_{s_1}$. We see that the $\mathbf{r}_s$, $s = 0, 1, \ldots, p-1$, commute with each other. Therefore the formula in the theorem follows immediately.

REMARK. We see from this theorem that the set $\{\mathbf{r}^n_{\Delta_l}\}_{n\geq 0}$ is linearly-independent over $\pi_*(\mathrm{BP})$. Also, we know that the coefficient $[s_1, s_2, \ldots, s_n] \cdot w^{T'} \in \pi_*(\mathrm{BP})$ for any T, since any composition of Brown-Peterson operations can be written as a sum of basic Brown-Peterson operations with coefficients in $\pi_*(\mathrm{BP})$. We can make this convincing by noting two facts:

(1). The largest power of p in the prime decomposition of the integer $m!$ for the case of $m = |T| = s_1 + s_2 + \cdots + s_n$, is greater than or equal to

$$t_2 + (1+p)t_3 + (1+p+p^2)t_4 + \cdots + (1+p+\cdots+p^{k-1})t_{k+1} + \cdots.$$

(2). An inductive argument shows, by the Hazewinkel relations, that

$$p^k m_k \in \pi_*(\mathrm{BP}),\ p^{1+p+\cdots+p^{k-1}} \cdot w_k \in \pi_*(\mathrm{BP}), \quad k \geq 1.$$

6. **Formulas for commutators.** Let $[x,y] = xy - yx$ denote the commutator in $\mathrm{BP}^*(\mathrm{BP})$. In [Z], Zahler showed that $[\mathbf{r}_1, \mathbf{r}_N] = \mathbf{r}_{(N-p,1,0,\ldots)}$. In this section, we generalize this result and introduce a new set of rational Brown-Peterson operations $\{\mathbf{q}_T\}$.

6.1. *Rational Brown-Peterson operations.* Recall that $\mathrm{BP}_*(\mathrm{BP}) = \pi_*(\mathrm{BP})[t_1, t_2, t_3, \ldots]$. Let

$$s_k = t_k + m_1 t_{k-1}^p + m_2 t_{k-2}^{p^2} + \cdots + m_{k-1} t_1^{p^{k-1}}, \quad k \geq 1.$$

Then $\mathrm{BP}\,\mathbb{Q}_*(\mathrm{BP}) = \pi_*(\mathrm{BP}) \otimes \mathbb{Q}[s_1, s_2, s_3, \ldots]$. One knows that $\mathrm{BP}\,\mathbb{Q}^*(\mathrm{BP})$ is dual to $\mathrm{BP}\,\mathbb{Q}_*(\mathrm{BP})$ over $\pi_*(\mathrm{BP})\otimes\mathbb{Q}$. We define the operation $\mathbf{q}_T \in \mathrm{BP}\,\mathbb{Q}^*(\mathrm{BP})$ by taking the dual element of $s^T \in \mathrm{BP}\,\mathbb{Q}_*(\mathrm{BP})$. Hence, any operation from $\mathrm{BP}\,\mathbb{Q}^*(\mathrm{BP})$ can then be written as a (possibly infinite) sum of elements from $\{\mathbf{q}_T\}_T$ with coefficients in $\pi_*(\mathrm{BP}) \otimes \mathbb{Q}$.

We define the polynomials Ψ_k by splitting the Quillen relations:

$$\begin{cases} \Psi_k = \Phi_k + m_1 \Phi_{k-1}^p + \cdots + m_{k-1}\Phi_1^{p^{k-1}}, \\ \Psi_k = \sum_{h+i+j=k,\, i+j\geq 1} m_h \xi_i^{p^h} \eta_j^{p^{h+i}}, \end{cases} \qquad k \geq 1.$$

Since $\Phi_1 = \Psi_1$, we can solve for each Φ_k as a polynomial in the Ψ_k. In turn, each Φ_k is a polynomial in the ξ_i and η_j.

THEOREM 6.1. *(1).* $\mathbf{r}_W = \sum_T \begin{bmatrix} \Psi^T \\ \Phi^W \end{bmatrix} \mathbf{q}_T$,

(2). $\mathbf{r}_L \cdot \mathbf{r}_R = \sum_T \begin{bmatrix} \Psi^T \\ \xi^L \eta^R \end{bmatrix} \mathbf{q}_T$.

PROOF. Since the relations of the Ψ_k in terms of the Φ_i are the same as that of the s_k in terms of the t_i by definition, $\begin{bmatrix} s^T \\ t^W \end{bmatrix} = \begin{bmatrix} \Psi^T \\ \Phi^W \end{bmatrix} \in \pi_*(\mathrm{BP})\otimes\mathbb{Q}$. We have $s^T = \sum_W \begin{bmatrix} s^T \\ t^W \end{bmatrix} t^W$. Dually,

$$\mathbf{r}_W = \sum_T \begin{bmatrix} s^T \\ t^W \end{bmatrix} \mathbf{q}_T = \sum_T \begin{bmatrix} \Psi^T \\ \Phi^W \end{bmatrix} \mathbf{q}_T.$$

Hence (1) follows.

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(2). Theorem 2.1 implies that

$$\mathbf{r}_L \cdot \mathbf{r}_R = \sum_W \begin{bmatrix} \Phi^W \\ \xi^L \cdot \eta^R \end{bmatrix} \cdot \mathbf{r}_W.$$

Since

$$\sum_T \begin{bmatrix} \Phi^W \\ \Psi^T \end{bmatrix} \cdot \begin{bmatrix} \Psi^T \\ \xi^L \eta^R \end{bmatrix} = \begin{bmatrix} \Phi^W \\ \xi^L \eta^R \end{bmatrix}$$

$$\sum_W \begin{bmatrix} \Phi^W \\ \Psi^T \end{bmatrix} \cdot \begin{bmatrix} \Psi^S \\ \Phi^W \end{bmatrix} = \begin{cases} 1 & \text{if } T = S \\ 0 & \text{otherwise,} \end{cases}$$

$$\begin{aligned}
\mathbf{r}_L \cdot \mathbf{r}_R &= \sum_{W,T} \begin{bmatrix} \Phi^W \\ \Psi^T \end{bmatrix} \begin{bmatrix} \Psi^T \\ \xi^L \eta^R \end{bmatrix} \mathbf{r}_W \\
&= \sum_T \begin{bmatrix} \Psi^T \\ \xi^L \eta^R \end{bmatrix} \Big(\sum_W \begin{bmatrix} \Phi^W \\ \Psi^T \end{bmatrix} \mathbf{r}_W \Big) \\
&= \sum_T \begin{bmatrix} \Psi^T \\ \xi^L \eta^R \end{bmatrix} \Big(\sum_{W,S} \begin{bmatrix} \Phi^W \\ \Psi^T \end{bmatrix} \cdot \begin{bmatrix} \Psi^S \\ \Phi^W \end{bmatrix} \mathbf{q}_S \Big) \quad \text{by (1),} \\
&= \sum_T \begin{bmatrix} \Psi^T \\ \xi^L \eta^R \end{bmatrix} \Bigg(\sum_S \Big(\sum_W \begin{bmatrix} \Phi^W \\ \Psi^T \end{bmatrix} \cdot \begin{bmatrix} \Psi^S \\ \Phi^W \end{bmatrix} \Big) \mathbf{q}_S \Bigg) \\
&= \sum_T \begin{bmatrix} \Psi^T \\ \xi^L \eta^R \end{bmatrix} \cdot \mathbf{q}_T.
\end{aligned}$$

6.2. *Commutators.* We now prove a generalization of Lemma 5.10 in [Z].

THEOREM 6.2. *For any* $k, l, N \geq 1$,

$$[\mathbf{r}_{\Delta_k}, \mathbf{r}_{N\Delta_l}] = \begin{cases} \mathbf{r}_{(N-p^k)\cdot\Delta_l + \Delta_{k+l}} & \text{if } N \geq p^k, \\ 0 & \text{if } N < p^k. \end{cases}$$

As a result,

$$[\mathbf{r}_1, \mathbf{r}_p] = \mathbf{r}_{0,1}, [\mathbf{r}_{0,1}, \mathbf{r}_{p^2}] = \mathbf{r}_{0,0,1}, \ldots [\mathbf{r}_{\underbrace{0,\ldots,0,1}_{k}}, \mathbf{r}_{p^k}] = \mathbf{r}_{\underbrace{0,\ldots,0,1}_{k+1}}, \ldots.$$

PROOF. From Theorem 6.1 we know that

$$\mathbf{r}_{\Delta_k} \cdot \mathbf{r}_{N\Delta_l} = \sum_T \begin{bmatrix} \Psi^T \\ \xi_k \eta_l^N \end{bmatrix} \mathbf{q}_T, \quad \mathbf{r}_{N\Delta_l} \cdot \mathbf{r}_{\Delta_k} = \sum_T \begin{bmatrix} \Psi^T \\ \xi_l^N \eta_k \end{bmatrix} \mathbf{q}_T$$

$$\mathbf{r}_{(N-p^k)\Delta_l + \Delta_{k+l}} = \sum_T \begin{bmatrix} \Psi^T \\ \Phi_l^{N-p^k} \Phi_{k+l} \end{bmatrix} \mathbf{q}_T, \quad \text{if } N \geq p^k.$$

We need to compute the coefficients. On the one hand, if we work modulo the ideal

$$\langle \Phi_1, \ldots, \Phi_{l-1}, \Phi_{l+1}, \ldots, \Phi_{k+l-1}, \Phi_{k+l}^2, \Phi_{k+l+1}, \ldots \rangle$$

then

$$\Psi_n \equiv \begin{cases} 0 & 1 \le n < l \\ m_{n-l}\Phi_l^{p^{n-l}} & l \le n < k+l \\ m_k\Phi_l^{p^k} + \Phi_{k+l} & n = k+l \\ m_{n-l}\Phi_l^{p^{n-l}} & n > k+l. \end{cases}$$

Hence, if $N \ge p^k$,

$$\begin{bmatrix} \Psi^T \\ \Phi_l^{N-p^k}\Phi_{k+l} \end{bmatrix} = \begin{cases} t_{k+l} \cdot \frac{\prod_{i\ge 0} m_i^{t_{i+l}}}{m_k} & \text{if } \sum_{j\ge l} p^{j-l} t_j = N, \text{ and } t_i = 0, \text{ for } i < l, \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, we can reduce coefficients modulo the ideal

$$\langle \xi_1, \ldots, \xi_{k-1}, \xi_{k+1}, \ldots, \eta_1, \ldots, \eta_{l-1}, \eta_{l+1}, \ldots \rangle$$

in the polynomial algebra $\pi_*(\mathrm{BP}) \otimes \mathbb{Q}[\xi_1, \xi_2, \ldots, \eta_1, \eta_2, \ldots]$. Then, for $n \ge 1$,

$$\Psi_n \equiv m_{n-l}\eta_l^{p^{n-l}} + m_{n-k}\xi_k^{p^{n-k}} + m_{n-k-l}\xi_k^{p^{n-k-l}}\eta_l^{p^{n-l}}.$$

Define polynomials F_n in α and β by:

$$F_n = m_{n-l}\beta^{p^{n-l}} + m_{n-k}\alpha^{p^{n-k}} + m_{n-k-l}\alpha^{p^{n-k-l}}\beta^{p^{n-l}}, \quad n \ge 1.$$

Then, for any positive integers M and N, we have

$$\begin{bmatrix} \Psi^T \\ \xi_k^M \eta_l^N \end{bmatrix} = \begin{bmatrix} F^T \\ \alpha^M \beta^N \end{bmatrix}.$$

Likewise, if we define the polynomials G_n in α and β by

$$G_n = m_{n-l}\beta^{p^{n-l}} + m_{n-k}\alpha^{p^{n-k}} + m_{n-k-l}\alpha^{p^{n-k}}\beta^{p^{n-k-l}}, \quad n \ge 1,$$

we will have

$$\begin{bmatrix} \Psi^T \\ \xi_l^N \eta_k^M \end{bmatrix} = \begin{bmatrix} G^T \\ \alpha^M \beta^N \end{bmatrix}.$$

Furthermore,

$$F_n = G_n + m_{n-k-l}(\alpha^{p^{n-k-l}}\beta^{p^{n-l}} - \alpha^{p^{n-k}}\beta^{p^{n-k-l}}).$$

Modulo the ideal $\langle \alpha^2 \rangle$ generated by α^2 in $\pi_*(\mathrm{BP}) \otimes \mathbb{Q}[\alpha, \beta]$,

$$\begin{aligned} F^T - G^T &\equiv \frac{G^T}{G_{k+l}^{t_{k+l}}}[(G_{k+l} + \alpha\beta^{p^k})^{t_{k+l}} - G_{k+l}^{t_{k+l}}] \\ &\equiv \frac{G^T}{G_{k+l}^{t_{k+l}}}[t_{k+l} \cdot (\alpha\beta^{p^k}) \cdot G_{k+l}^{t_{k+l}-1}] \\ &= \frac{G^T}{G_{k+l}} \cdot t_{k+l} \cdot \alpha\beta^{p^k}. \end{aligned}$$

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Also, if $k < l$,

$$G_n \equiv \begin{cases} 0 & 1 \le n < l, n \ne k \\ \alpha & n = k \\ m_{n-l}\beta^{p^{n-l}} & n \ge l; \end{cases}$$

if $k = l$,

$$G_n \equiv \begin{cases} 0 & 1 \le n < k \\ \beta + \alpha & n = k \\ m_{n-k}\beta^{p^{n-k}} & n > k, n \ne 2k \\ m_k\beta^{p^k} + \alpha\beta^{p^k} & n = 2k; \end{cases}$$

and if $k > l$,

$$G_n \equiv \begin{cases} 0 & 1 \le n < l \\ m_{n-l}\beta^{p^{n-l}} & n \ge l, n \ne k \\ m_{k-l}\beta^{p^{k-l}} + \alpha & n = k. \end{cases}$$

Therefore in all cases,

$$\begin{bmatrix} \Psi^T \\ \xi_k\eta_l^N \end{bmatrix} - \begin{bmatrix} \Psi^T \\ \xi_l^N\eta_k \end{bmatrix} = \begin{bmatrix} F^T \\ \alpha\beta^N \end{bmatrix} - \begin{bmatrix} G^T \\ \alpha\beta^N \end{bmatrix} = \begin{bmatrix} F^T - G^T \\ \alpha\beta^N \end{bmatrix}.$$

So for any T,

$$\begin{bmatrix} \Psi^T \\ \xi_k\eta_l^N \end{bmatrix} - \begin{bmatrix} \Psi^T \\ \xi_l^N\eta_k \end{bmatrix} = \begin{cases} \begin{bmatrix} \Psi^T \\ \Phi_l^{N-p^k}\Phi_{k+l} \end{bmatrix} & \text{if } N \ge p^k, \\ 0 & \text{if } N < p^k. \end{cases}$$

This proves the formula in the theorem.

7. **Proof of the composition law.**

Finally, we prove the composition law for Brown-Peterson operations that was stated in §3.

7.1. *Lemma on finiteness.* It is important to know that the composition of any two basic Brown-Peterson operations is a finite sum over the basic Brown-Peterson operations. In other words, the summation on the right hand side of the formula 3.1 is finite. This is the consequence of the following

LEMMA 7.1. *(1). For given exponential sequences L and R, we have only a finite number of (L, R)-feasible reduced cubic exponential matrices X.*

(2). For a fixed exponential sequence T, we have only finite number of T-feasible pairs (W, Y).

PROOF. (1) is clear. For (2), we list the equations for $T + T(Y) = W + S^{[1]}(Y)$:

(E1) $$t_1 = w_1 + py_{1,1} + p^2y_{2,1} + \cdots,$$

(E2) $$t_2 + y_{1,1} = w_2 + py_{1,2} + p^2y_{2,2} + \cdots,$$

$$\vdots$$

(E3) $$t_k + y_{k-1,1} + y_{k-2,2} + \cdots + y_{1,k-1} = w_k + py_{1,k} + p^2y_{2,k} + \cdots,$$

$$\vdots$$

Since we are assuming that Y is an exponential matrix, the sum on the right hand side of each equation is finite. By taking the weighted sum $\mathrm{E}_1 + p\mathrm{E}_2 + p^2\mathrm{E}_3 + \ldots + p^{k-1}\mathrm{E}_k$, we have

$$\sum_{\lambda=1}^{k} p^{\lambda-1}t_\lambda + \sum_{\lambda=1}^{k-1} p^{\lambda}y_{\lambda,1} + p\sum_{\lambda=1}^{k-2} p^{\lambda}y_{\lambda,2} + \cdots + p^{k-2}\sum_{\lambda=1}^{1} p^{\lambda}y_{\lambda,k-1}$$
$$= \sum_{h=1}^{k} p^{h-1}w_h + S^{[1]}(Y)_1 + pS^{[1]}(Y)_2 + \cdots + p^{k-2}S^{[1]}(Y)_{k-1}.$$

That is,

$$\sum_{\lambda=1}^{k} p^{\lambda-1}t_\lambda = \sum_{h=1}^{k} p^{h-1}w_h + p^k y_{k,1} + p^{k+1}y_{k+1,1} + p^{k+2}y_{k+2,1} + \cdots$$
$$+ p^k y_{k-1,2} + p^{k+1}y_{k,2} + p^{k+2}y_{k+1,2} + \cdots$$
$$+ p^k y_{1,k} + p^{k+1}y_{2,k} + p^{k+2}y_{3,k+3} + \cdots.$$

Hence, for any $k \geq 1$,

$$\sum_{\lambda=1}^{k} p^{\lambda-1}t_\lambda \geq \sum_{h=1}^{k} p^{h-1}w_h, \quad \sum_{\lambda=1}^{k} p^{\lambda-1}t_\lambda \geq p^k T(Y)_{k+1}.$$

Since T is an exponential sequence, the number on the left hand side of the above inequalities is bounded, and therefore we have only a finite number of pairs (W, Y) satisfying the equation $W = T + T(Y) - S^{[1]}(Y)$.

7.2. *Proof of the composition law.* The composition law 3.1 can be split into two theorems using the rational operations $\mathbf{q}_T$.

THEOREM 7.2.
$$\mathbf{r}_L \cdot \mathbf{r}_R = \sum_{\substack{S^{[1],(3)}(X)=L \\ S^{[1],[2]}(X)=R}} B(X)m^{S^{(2),(3)}(X)}\mathbf{q}_{T(X)}.$$

THEOREM 7.3.
$$\mathbf{q}_T = \sum_{\substack{Y,W \\ T+T(Y)=W+S^{[1]}(Y)}} [T, T(Y)] \cdot B(Y) \cdot (-m)^{S^{(2)}(Y)} \cdot \mathbf{r}_W.$$

PROOF OF THEOREM 7.2. By Theorem 6.1(2), we need to find the coefficient $\begin{bmatrix} \Psi^T \\ \xi^L\eta^R \end{bmatrix}$. Instead, we prove the following formula for any exponential sequence T:

$$\Psi^T = \sum_{T(X)=T} B(X) \cdot m^{S^{(2),(3)}(X)}\xi^{S^{[1],(3)}(X)}\eta^{S^{[1],[2]}(X)},$$

where X is a reduced cubic exponential matrix. For such X, $\ell(X)$ depends only on T, and equals the length $\ell(T)$ of T. The case $\ell(T) = 1$ is trivial. Let $\bar{T} = (t_1, \ldots, t_k, 0, \ldots)$ be any exponential sequence with $\ell(\bar{T}) = k \geq 2$. Suppose the above formula is true for the sequence $T = (t_1, \ldots, t_{k-1}, 0, \ldots)$ by induction:

$$\Psi^T = \sum_{T(X)=T} B(X) \cdot m^{S^{(2),(3)}(X)} \xi^{S^{[1],(3)}(X)} \eta^{S^{[1],[2]}(X)}.$$

Since

$$\Psi_k = \sum_{h+i+j=k, i+j\geq 1} m_h \xi_i^{p^h} \eta_j^{p^{h+i}},$$

we have, by Newton expansion,

$$\Psi_k^{t_k} = \sum_{\sum_{h+i+j=k} \bar{x}_{h,i,j}=t_k} \frac{t_k!}{\prod_{h+i+j=k} \bar{x}_{h,i,j}!} \cdot (m_h^{\bar{x}_{h,i,j}} \xi_i^{p^h \bar{x}_{h,i,j}} \eta_j^{p^{h+i} \bar{x}_{h,i,j}}).$$

Therefore,

$$\begin{aligned}
\Psi^{\bar{T}} &= \Psi^T \cdot \Psi_k^{t_k} \\
&= \Big(\sum_{T(X)=T} B(X) \cdot m^{S^{(2),(3)}(X)} \xi^{S^{[1],(3)}(X)} \eta^{S^{[1],[2]}(X)} \Big) \cdot \Big(\sum_{h+i+j=k, i+j\geq 1} m_h \xi_i^{p^h} \eta_j^{p^{h+i}} \Big) \\
&= \sum_{T(\bar{X})=T} B(\bar{X}) \cdot m^{S^{(2),(3)}(\bar{X})} \xi^{S^{[1],(3)}(\bar{X})} \eta^{S^{[1],[2]}(\bar{X})}
\end{aligned}$$

where $\bar{X} = (\bar{x}_{h,i,j})_{h,i,j\geq 0}$ is the matrix with $\bar{x}_{h,i,j} = x_{h,i,j}$ if $h+i+j < k$; 0 if $h+i+j > k$. ($\bar{X}$ has length $\ell(\bar{X}) = k$.) This proves the theorem.

PROOF OF THEOREM 7.3. The inverse relation to the one in Theorem 6.1(1) is $\mathbf{q}_T = \sum_W \begin{bmatrix} \Phi^W \\ \Psi^T \end{bmatrix} \mathbf{r}_W$. An inductive argument of the same kind as in the proof of Theorem 7.2 shows that

$$\Phi^W = \sum_{T+T(Y)=W+S^{[1]}(Y)} [T, T(Y)] \cdot B(Y) \cdot (-m)^{S^{(2)}(Y)} \Psi^T.$$

The theorem follows.

Finally, we prove Kane's Theorem:

PROOF OF COROLLARY 3.2. Recall that

$$\pi_*(\mathrm{BP}) = \mathbb{Z}_{(p)}[v_1, v_2, v_3, \ldots] \subset \pi_*(\mathrm{BP}) \otimes \mathbb{Q} = \mathbb{Q}[m_1, m_2, m_3, \ldots].$$

Let $I = \langle v_1, v_2, v_3, \ldots \rangle$ be the ideal of $\pi_*(\mathrm{BP})$, generated by the elements $v_1, v_2, v_3, \ldots$. Then, by the Hazewinkel relations, $I \otimes \mathbb{Q} = \langle m_1, m_2, m_3, \ldots \rangle$ is the ideal of $\pi_*(\mathrm{BP}) \otimes \mathbb{Q}$ generated by the elements $m_1, m_2, m_3, \ldots$. Modulo $I \otimes \mathbb{Q}$, the formula in Theorem 3.1 becomes:

$$\mathbf{r}_L \mathbf{r}_R \equiv \sum_{\substack{X,Y \\ S^{[1],(3)}(X)=L, S^{[1],[2]}(X)=R \\ S^{(2),(3)}(X)=0, S^{(2)}(Y)=0}} B(X, Y) \cdot \mathbf{r}_{T(X)+T(Y)-S^{[1]}(Y)}.$$

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The relations $S^{(2),(3)}(X) = 0$, and $S^{(2)}(Y) = 0$ imply that $y_{\lambda,\mu} = 0$, and $x_{h,i,j} = 0$ if $h \geq 1$. Hence, $Y = 0$; $T(Y) = 0$, $S^{[1]}(Y) = 0$. Let $a_{i,j} = x_{0,i,j}$ and $A = (a_{i,j})_{i,j\geq 0}$. Then $T(X) = T(A)$, $B(X, Y) = b(A)$ and $S^{[1],(3)}(X) = S(A)$, $S^{[1],[2]}(X) = R(A)$.

$$\mathbf{r}_L\mathbf{r}_R \equiv \sum_{\substack{A\\ S(A)=L,R(A)=R}} b(A)\cdot \mathbf{r}_{T(A)}, \operatorname{mod}(m_1, m_2, m_3, \ldots).$$

Since any product of Brown-Peterson operations can be written as a linear combination with coefficients in $\pi_*(\mathrm{BP})$, the relation above can be reduced to one modulo $(v_1, v_2, v_3, \ldots)$. The theorem follows.

REFERENCES

1. J. F. Adams, *Stable Homotopy and Generalized Homology*, The University of Chicago **30**, 1979, 455–467.
2. R. M. Kane, *Brown-Peterson Homology and Generalized Homology*, Quart. J. Math **30**(1979), 455–467.
3. Z. Li, *Product Formulas for Steenrod Operations*, Proc. Edinburgh Math. Soc., to appear.
4. J. Milnor, *The Steenrod Algebra and Its Dual*, Ann. of Math. **67**(1958), 150–171.
5. D. Quillen, *On the Formal Group Laws of Unoriented and Complex Cobordism Theory*, Bull. Amer. Math. Soc. **75**(1969), 1293–1298.
6. W. S. Wilson, *Brown-Peterson Homology, An Introduction and Sampler*, CBMS Regional Conference Series in Math. **48**, Amer. Math. Soc., 1982.
7. R. Zahler, *The Adams-Novikov Spectral Sequence for the Spheres*, Ann. of Math. **96**(1972), 480–504.

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