*J. Aust. Math. Soc.* **91** (2011), 275–288
doi:10.1017/S1446788711001479

# IDEALS IN OPERATOR SPACE PROJECTIVE TENSOR PRODUCT OF $C^*$-ALGEBRAS

**RANJANA JAIN and AJAY KUMAR** ✉

(Received 10 January 2011; accepted 4 August 2011)

Communicated by G. A. Willis

**Abstract**

Let $A$ and $B$ be $C^*$-algebras. We prove the slice map conjecture for ideals in the operator space projective tensor product $A \widehat{\otimes} B$. As an application, a characterization of the prime ideals in the Banach $*$-algebra $A \widehat{\otimes} B$ is obtained. In addition, we study the primitive ideals, modular ideals and the maximal modular ideals of $A \widehat{\otimes} B$. We also show that the Banach $*$-algebra $A \widehat{\otimes} B$ possesses the Wiener property and that, for a subhomogeneous $C^*$-algebra $A$, the Banach $*$-algebra $A \widehat{\otimes} B$ is symmetric.

2010 *Mathematics subject classification*: primary 46L06; secondary 46L07, 47L25.
*Keywords and phrases*: $C^*$-algebras, operator space projective tensor norm, Haagerup tensor product.

## 1. Introduction

A systematic study of the tensor products of subspaces and subalgebras of $C^*$-algebras was initiated by Blecher and Paulsen in [6] and Effros and Ruan in [8, 9]. Analogous constructions to those for Banach spaces, for example, quotients, duals and tensor products, were defined and studied. For a Hilbert space $H$ we let $\mathcal{B}(H)$ denote the space of bounded operators on $H$. An operator space $X$ on $H$ is just a closed subspace of $\mathcal{B}(H)$. If $E$ and $F$ are operator spaces, then the operator space projective tensor product, which we denote by $E \widehat{\otimes} F$, is the completion of the algebraic tensor product $E \otimes F$ in the norm

$$\|u\|_{\wedge} = \inf\{\|\alpha\| \, \|v\| \, \|w\| \, \|\beta\| : u = \alpha(v \otimes w)\beta\}$$

where the infimum runs over all the arbitrary decompositions in which $v \in M_p(E)$, $w \in M_q(F)$, $\alpha \in M_{1,pq}$ and $\beta \in M_{pq,1}$, where $p$ and $q$ are arbitrary positive integers. Here $M_{k,l}$ denotes the space of $k \times l$ matrices over $\mathbb{C}$. If $E$ and $F$ are $C^*$-algebras, then $E \widehat{\otimes} F$ admits a Banach algebra structure with canonical isometric involution (see [14]). The main objective of this paper is to study the closed $*$-ideals of this Banach $*$-algebra.

In Section 2 we study the slice map problem for the ideals of $A \widehat{\otimes} B$. Tomiyama (see [23]) studied the slice maps on the tensor product of $C^*$-algebras with respect to the 'min'-norm. Later Wassermann (see [24]) discussed the slice map problem in greater detail. The slice map problem was then studied and used in several different contexts (see, for instance, [2, 25]). It is interesting to note that the slice map property is not held for the 'min' norm by all $C^*$-algebras. In fact, for the 'min'-norm, the slice map problem for ideals is equivalent to the problem of whether every tensor product $A \otimes_{\min} B$ has Property F of Tomiyama (see [24, Remark 24]). In 1991 Smith (see [21]) studied the slice map property for the Haagerup norm and proved that the slice map conjecture is true for all subspaces of $\mathcal{B}(H)$. We show that the slice map conjecture is true for ideals with respect to the operator space projective tensor norm.

The ideal structure for the Haagerup tensor product with the 'min'-norm has been studied extensively in [1, 3, 22]. In [14] and [12] the authors investigated some properties of the closed ideals of the projective tensor product $A \widehat{\otimes} B$. For example, they considered the sum of the product ideals, the minimal ideals and the maximal ideals. In Section 3 we give a characterization of the prime ideals, primitive ideals and maximal modular ideals of the Banach $*$-algebra $A \widehat{\otimes} B$. Finally, in Section 4 certain $*$-algebraic properties of $A \widehat{\otimes} B$, namely the Wiener property and symmetry, are studied. Throughout the paper $A$ and $B$ will denote $C^*$-algebras unless otherwise specified.

Recall that the Haagerup norm of an element $u$ of the algebraic tensor product $A \otimes B$ of two $C^*$-algebras $A$ and $B$ is defined by

$$\|u\|_h = \inf\left\{\left\|\sum_i a_i a_i^*\right\|^{1/2}\left\|\sum_i b_i^* b_i\right\|^{1/2} : u = \sum_{i=1}^n a_i \otimes b_i\right\}.$$

The Haagerup tensor product $A \otimes_h B$ is defined to be the completion of $A \otimes B$ in the norm $\|\cdot\|_h$. The Banach space projective norm of $u \in A \otimes B$ is given by

$$\|u\|_\gamma = \inf\left\{\sum_i \|a_i\|\,\|b_i\| : u = \sum_{i=1}^n a_i \otimes b_i\right\}.$$

The norms $\|\cdot\|_h$, $\|\cdot\|_\wedge$ and $\|\cdot\|_\gamma$ on the tensor product $A \otimes B$ of two $C^*$-algebras $A$ and $B$ satisfy the inequality

$$\|\cdot\|_h \le \|\cdot\|_\wedge \le \|\cdot\|_\gamma.$$

Necessary and sufficient conditions on $A$ and $B$ for the equivalence of these norms are given in [15].

## 2. The slice map property for ideals

For each $\phi \in A^*$ we define a linear map $R_\phi : A \otimes B \to B$ by

$$R_\phi\left(\sum_{i=1}^n a_i \otimes b_i\right) = \sum_{i=1}^n \phi(a_i) b_i.$$

Then it can be easily seen that $R_\phi$ is well defined. In addition, $R_\phi$ is continuous with respect to the 'min'-norm (see [24]) and hence it is also continuous with respect to the

https://doi.org/10.1017/S1446788711001479 Published online by Cambridge University Press

larger operator space projective tensor norm with $\|R_\phi\| \le \|\phi\|$. Thus $R_\phi$ can be extended to $A \mathbin{\widehat{\otimes}} B$ as a bounded linear map where it is known as the *right slice map* associated to $\phi$. One can similarly define the left slice map $L_\psi$ for each $\psi \in B^*$.

If $J$ is a closed ideal of $B$, then $A \mathbin{\widehat{\otimes}} J$ is a closed ideal of $A \mathbin{\widehat{\otimes}} B$ (see [14]) and clearly $R_\phi(x) \in J$ for all $x \in A \mathbin{\widehat{\otimes}} J$. In Theorem 3 we prove the converse of this statement which is known as the slice map problem for ideals.

LEMMA 1. *The set $\{R_\phi \mid \phi \in A^*\}$ is total on $A \mathbin{\widehat{\otimes}} B$, that is, if $x \in A \mathbin{\widehat{\otimes}} B$ and $R_\phi(x) = 0$ for all $\phi \in A^*$, then $x = 0$.*

PROOF. For $\phi \in A^*$ and $\psi \in B^*$ we consider the map $\phi \otimes \psi : A \otimes B \to \mathbb{C}$ defined by

$$(\phi \otimes \psi)\Big(\sum_i a_i \otimes b_i\Big) = \sum_i \phi(a_i)\psi(b_i).$$

Note that, by the definition of the Banach space injective norm $\lambda$ (see [22, p. 188]),

$$\Big|\sum_i \phi(a_i)\psi(b_i)\Big| \le \|\phi\|\,\|\psi\| \,\Big\|\sum_i a_i \otimes b_i\Big\|_\lambda.$$

It follows that $\phi \otimes \psi$ is continuous with respect to larger norms, in particular, the 'min'-norm and '$\wedge$'-norm. Thus $\phi \otimes \psi$ can be extended to continuous linear functionals on $A \otimes_{\min} B$ and $A \mathbin{\widehat{\otimes}} B$. Let us denote its extensions by $\phi \otimes_{\min} \psi$ and $\phi \mathbin{\widehat{\otimes}} \psi$. We claim that the set $\{\phi \mathbin{\widehat{\otimes}} \psi \mid \phi \in A^*, \psi \in B^*\}$ is total on $A \mathbin{\widehat{\otimes}} B$.

In order to establish our claim we consider an element $x \in A \mathbin{\widehat{\otimes}} B$ such that

$$(\phi \mathbin{\widehat{\otimes}} \psi)(x) = 0 \quad \forall \phi \in A^*, \psi \in B^*.$$

Observe that if $i : A \mathbin{\widehat{\otimes}} B \to A \otimes_{\min} B$ is the canonical map, then both of the maps $\phi \mathbin{\widehat{\otimes}} \psi$ and $(\phi \otimes_{\min} \psi) \circ i$ are continuous on $A \mathbin{\widehat{\otimes}} B$ and agree on $A \otimes B$. It follows that $(\phi \otimes_{\min} \psi)(i(x)) = 0$ for all $\phi \in A^*$ and $\psi \in B^*$. Now we take faithful representations $\{\pi_A, H\}$ and $\{\pi_B, K\}$ of $A$ and $B$ respectively, and let $\xi_i \in H$ and $\eta_i \in K$ for $i = 1, 2$. We set $\phi := \langle \pi_A(\cdot)\xi_1, \xi_2\rangle \in A^*$ and $\psi := \langle \pi_B(\cdot)\eta_1, \eta_2\rangle \in B^*$. Then

$$0 = (\phi \otimes_{\min} \psi)(i(x)) = \langle (\pi_A \otimes \pi_B)(i(x))\xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2\rangle.$$

This equality holds for all $\xi_i \in H$ and for all $\eta_i \in K$ for $i = 1, 2$, which implies that $(\pi_A \otimes \pi_B)(i(x)) = 0$. We use the facts that $\pi_A \otimes \pi_B$ is faithful (see [22, Theorem IV.4.9]) and that $i$ is injective (see [11, Corollary 1]) to deduce our claim.

Finally, the relation

$$\langle x, \phi \mathbin{\widehat{\otimes}} \psi\rangle = \langle R_\phi(x), \psi\rangle = \langle L_\psi(x), \phi\rangle \quad \forall x \in A \mathbin{\widehat{\otimes}} B,$$

yields the required result. □

Recall that, for Banach spaces $X$ and $Y$, a mapping $\theta : X \to Y$ is said to be a *quotient map* if it maps the open unit ball of $X$ onto that of $Y$ (see [8]). A quotient map is clearly surjective and, for a Banach space $X$ and a closed subspace $Y$ of $X$, the canonical quotient map $\pi : X \to X/Y$ is a quotient map in the above sense. As in the case of

https://doi.org/10.1017/S1446788711001479 Published online by Cambridge University Press

the Haagerup tensor product (see [1]), the operator space projective tensor product of quotient maps behaves nicely. Although it is straightforward, we include a proof of the following lemma for the sake of convenience.

LEMMA 2. *Let $I$ and $J$ be closed ideals of the $C^*$-algebras $A$ and $B$ and let $\pi : A \to A/I$ and $\rho : B \to B/J$ be the corresponding quotient maps. Then the following hold.*

(1) *The map*
$$\pi \widehat{\otimes} \rho : A \widehat{\otimes} B \to (A/I) \widehat{\otimes} (B/J)$$
*is a quotient map with*
$$\ker(\pi \widehat{\otimes} \rho) = A \widehat{\otimes} J + I \widehat{\otimes} B.$$

(2) *If $K$ is a closed ideal of $A \widehat{\otimes} B$ containing $\ker(\pi \widehat{\otimes} \rho)$, then $(\pi \widehat{\otimes} \rho)(K)$ is a closed ideal of $(A/I) \widehat{\otimes} (B/J)$ and*
$$(\pi \widehat{\otimes} \rho)^{-1}((\pi \widehat{\otimes} \rho)(K)) = K.$$

PROOF. Part (1) follows directly from [12, Proposition 3.5].

To prove part (2) we consider an element $(\pi \widehat{\otimes} \rho)(x)$ of $(A/I) \widehat{\otimes} (B/J)$ where $x \in A \widehat{\otimes} B$ and
$$(\pi \widehat{\otimes} \rho)(x) \in \mathrm{cl}((\pi \widehat{\otimes} \rho)(K)).$$
Given an arbitrary $\epsilon > 0$, there exists $k \in K$ such that
$$\|(\pi \widehat{\otimes} \rho)(k - x)\|_{(A/I)\widehat{\otimes}(B/J)} < \epsilon.$$
By part (1) above there is an isomorphism between $(A \widehat{\otimes} B)/Z$ and $(A/I) \widehat{\otimes} (B/J)$ where $Z = \ker(\pi \widehat{\otimes} \rho)$. Therefore
$$\|(k - x) + Z\|_{(A\widehat{\otimes}B)/Z} < c\epsilon$$
for some constant $c$. So there exists some $z \in Z \subseteq K$ with
$$\|(k + z) - x\|_{(A\widehat{\otimes}B)/Z} \leq c\epsilon.$$
Since $K$ is closed and $k + z \in K$ we must have $x \in K$, which proves our claim. Finally, a routine calculation verifies the equation in the statement. □

We are now ready to present a proof of the slice map problem for ideals.

THEOREM 3. *Let $J$ be a closed ideal of $B$. Then*
$$A \widehat{\otimes} J = \{x \in A \widehat{\otimes} B \mid R_\phi(x) \in J \ \forall \phi \in A^*\}.$$

PROOF. Consider an element $x \in A \widehat{\otimes} B$ such that $R_\phi(x) \in J$ for all $\phi \in A^*$. By Lemma 2 there is a quotient map
$$i \widehat{\otimes} \pi : A \widehat{\otimes} B \to A \widehat{\otimes} (B/J)$$
corresponding to the quotient map $\pi : B \to B/J$. Note that $i$ is the identity map on $A$ and the kernel of $i \widehat{\otimes} \pi$ is given by
$$\ker(i \widehat{\otimes} \pi) = A \widehat{\otimes} J.$$

https://doi.org/10.1017/S1446788711001479 Published online by Cambridge University Press

Also observe that, by continuity and agreement on $A \otimes B$,

$$\pi \circ R_\phi = r_\phi \circ (i \mathbin{\widehat{\otimes}} \pi)$$

where $r_\phi : A \mathbin{\widehat{\otimes}} (B/J) \to B/J$ is the right slice map. Using the fact that $R_\phi(x) \in J$ for all $\phi \in A^*$, we see that $r_\phi(i \mathbin{\widehat{\otimes}} \pi(x)) = 0$ for all $\phi \in A^*$. Thus by Lemma 1 we may deduce that $i \mathbin{\widehat{\otimes}} \pi(x) = 0$ so that $x \in \ker(i \mathbin{\widehat{\otimes}} \pi) = A \mathbin{\widehat{\otimes}} J$. The other containment is easy to prove. □

We next give an application of Theorem 3 which will be used later to characterize the prime ideals. For the Haagerup norm such a result was proved for subspaces of $\mathcal{B}(H)$ in [21, Corollary 4.6].

Proposition 4. *Let $A_1$, $A_2$ and $B_1$, $B_2$ be closed ideals of $A$ and $B$, respectively. Then,*

$$(A_1 \mathbin{\widehat{\otimes}} B_1) \cap (A_2 \mathbin{\widehat{\otimes}} B_2) = (A_1 \cap A_2) \mathbin{\widehat{\otimes}} (B_1 \cap B_2).$$

Proof. Since $A_i \mathbin{\widehat{\otimes}} B_i$ are closed ideals of $A \mathbin{\widehat{\otimes}} B$ for $i = 1, 2$ (see [14]) it is easy to see that

$$(A_1 \cap A_2) \mathbin{\widehat{\otimes}} (B_1 \cap B_2) \subseteq (A_1 \mathbin{\widehat{\otimes}} B_1) \cap (A_2 \mathbin{\widehat{\otimes}} B_2).$$

For the reverse inclusion we consider an element $v \in (A_1 \mathbin{\widehat{\otimes}} B_1) \cap (A_2 \mathbin{\widehat{\otimes}} B_2)$. The image $R_\phi(v)$ is an element of $B_1 \cap B_2$ for all $\phi \in A^*$ and so by Theorem 3 we may deduce that $v \in A \mathbin{\widehat{\otimes}} (B_1 \cap B_2)$.

Next, consider any $\psi \in (B_1 \cap B_2)^*$ and let $\tilde{\psi}$ be an extension to $B^*$. Again $L_{\tilde{\psi}}(v) \in (A_1 \cap A_2)$ and $L_\psi(v) = L_{\tilde{\psi}}(v)$, so that $L_\psi(v) \in (A_1 \cap A_2)$. This is true for every $\psi \in (B_1 \cap B_2)^*$ and so, applying the slice map property again for the left slice map, we see that $v \in (A_1 \cap A_2) \mathbin{\widehat{\otimes}} (B_1 \cap B_2)$ which establishes our claim and our result follows. □

Using the slice map property for the right and the left slice maps and the technique of extending linear functionals used in Proposition 4, we can easily deduce the following corollary.

Corollary 5. *Let $I$ be a closed ideal of $A$ and let $J$ be a closed ideal of $B$. Then*

$$I \mathbin{\widehat{\otimes}} J = \{x \in A \mathbin{\widehat{\otimes}} B \mid R_\phi(x) \in J, L_\psi(x) \in I \ \forall \phi \in A^*, \psi \in B^*\}.$$

## 3. The ideal structure of $A \mathbin{\widehat{\otimes}} B$

This section deals with the structure of the prime ideals, primitive ideals and modular ideals of $A \mathbin{\widehat{\otimes}} B$. These ideal structures play an important role in determining the structure of a Banach $*$-algebra.

In a Banach algebra a proper closed ideal $K$ is said to be *prime* if, for any pair of closed ideals $I$ and $J$ such that $IJ \subseteq K$, either $I \subseteq K$ or $J \subseteq K$. It is well known that a proper closed ideal $K$ of a $C^*$-algebra $A$ is prime if and only if, for any pair of closed ideals $I$ and $J$ satisfying $I \cap J \subseteq K$, either $I \subseteq K$ or $J \subseteq K$. This property is also true for $A \mathbin{\widehat{\otimes}} B$, as can be explicitly deduced from the following result. The proof of the following result is largely inspired by [1].

https://doi.org/10.1017/S1446788711001479 Published online by Cambridge University Press

THEOREM 6. *A proper closed ideal $K$ of $A \widehat{\otimes} B$ is prime if and only if*

$$K = A \widehat{\otimes} F + E \widehat{\otimes} B$$

*for some prime ideals $E$ and $F$ of $A$ and $B$, respectively.*

PROOF. Let $K$ be a proper closed prime ideal of $A \widehat{\otimes} B$. Consider the family $\mathcal{S}$ of pairs $(M, N)$ such that $A \widehat{\otimes} N + M \widehat{\otimes} B \subseteq K$ where $M$ and $N$ are closed ideals of $A$ and $B$ respectively. Then $\mathcal{S}$ is a nonempty partially ordered set whose order is given by $(M, N) \leq (M_1, N_1)$ if $M \subseteq M_1$ and $N \subseteq N_1$ for all $(M, N), (M_1, N_1) \in \mathcal{S}$.

By [12, Corollary 3.4], if $\{(M_i, N_i)\}_{i\in\Lambda}$ is an increasing chain in $\mathcal{S}$, then one can easily see that $(\sum M_i, \sum N_i)$, where

$$\sum M_i := \Big\{ \sum_{\text{finite}} x_i \;\Big|\; x_i \in M_i,\, i \in \Lambda \Big\},$$

is an upper bound of the chain in $\mathcal{S}$. By Zorn's lemma we can choose closed ideals $E$ and $F$ in $A$ and $B$ which are maximal with respect to the property $A \widehat{\otimes} F + E \widehat{\otimes} B \subseteq K$.

Now consider the quotient maps $\pi : A \to A/E$ and $\rho : B \to B/F$. Since we have the containment $\ker(\pi \otimes \rho) \subseteq K$ we may deduce by Lemma 2 that $(\pi \otimes \rho)(K)$ is a closed ideal of $A/E \widehat{\otimes} B/F$. We claim that $(\pi \otimes \rho)(K) = 0$. This would imply $K = A \widehat{\otimes} F + E \widehat{\otimes} B$. Suppose, for contradiction, that the ideal $(\pi \otimes \rho)(K)$ is nonzero. Then it must contain a nonzero elementary tensor, say $\pi(a) \otimes \rho(b)$, where $a \otimes b \in K$ (see [12, Proposition 3.7]). Let $E_0$ and $F_0$ be the closed ideals generated by $a$ and $b$, respectively. Then the product ideal $E_0 \widehat{\otimes} F_0$ is contained in $K$. Now consider the product ideals $M = A \widehat{\otimes} (F + F_0)$ and $N = (E + E_0) \widehat{\otimes} B$. Using Proposition 4 and [12, Proposition 3.6], we see that

$$MN \subseteq M \cap N = E \widehat{\otimes} F + E \widehat{\otimes} F_0 + E_0 \widehat{\otimes} F + E_0 \widehat{\otimes} F_0.$$

It is clear that $MN \subseteq K$ so that either $M \subseteq K$ or $N \subseteq K$. Using the maximality property of $E$ and $F$, either $E_0 \subseteq E$ or $F_0 \subseteq F$. Thus either $\pi(a) = 0$ or $\rho(b) = 0$, contradicting the fact that $(\pi \otimes \rho)(a \otimes b) \neq 0$.

Next we prove that $E$ and $F$ are prime ideals. Note that $E$ and $F$ are both proper ideals since $K$ is proper. Let $I$ and $J$ be closed ideals of $A$ such that $I \cap J \subseteq E$. Then

$$(I \widehat{\otimes} B)(J \widehat{\otimes} B) \subseteq (I \cap J) \widehat{\otimes} B \subseteq K$$

and so we must either have $I \widehat{\otimes} B \subseteq K$ or $J \widehat{\otimes} B \subseteq K$. Without loss of generality, suppose that $I \widehat{\otimes} B \subseteq K$. Consider any $\phi \in E^{\perp} \subseteq A^*$ and $0 \neq \psi \in F^{\perp}$. Then $(\phi \otimes \psi)(K) = 0$, which further implies that $(\phi \otimes \psi)(I \widehat{\otimes} B) = 0$. Since this is true for any $\phi \in E^{\perp}$ we must have $I \subseteq E$. Thus $E$ is prime. A similar argument shows that $F$ is also prime.

For the converse we assume that $K = A \widehat{\otimes} F + E \widehat{\otimes} B$ for some prime ideals $E$ and $F$ of $A$ and $B$, respectively. Let $I$ and $J$ be closed ideals of $A \widehat{\otimes} B$ such $IJ \subseteq K$. We define the closed ideals $M$ and $N$ by

$$M := \mathrm{cl}(I + K), \quad N := \mathrm{cl}(J + K).$$

https://doi.org/10.1017/S1446788711001479 Published online by Cambridge University Press

Then $K \subseteq M \cap N$ and $MN \subseteq K$. We claim that either $M = K$ or $N = K$, which further implies that either $I \subseteq K$ or $J \subseteq K$. Suppose, to the contrary, that both the containments $K \subseteq M$ and $K \subseteq N$ are strict. We now claim that $M$ contains a product ideal $M_1 \widehat{\otimes} N_1$ which is not contained in $K$. As before, $(\pi \otimes \rho)(M)$ is a nonzero closed ideal of $A/E \widehat{\otimes} B/F$ and

$$(\pi \otimes \rho)^{-1}((\pi \otimes \rho)(M)) = M$$

since $K \subsetneq M$. So $(\pi \otimes \rho)(M)$ contains a nonzero elementary tensor, say, $\pi(a) \otimes \rho(b)$. Define $M_1$ and $N_1$ to be the closed ideals generated by $a$ and $b$. Then $M_1 \widehat{\otimes} N_1$ is contained in $M$ but not in $K$. Similarly, $N$ contains a product ideal $M_2 \widehat{\otimes} N_2$ which is not contained in $K$. By routine calculations it is easy to see that

$$M_1 M_2 \widehat{\otimes} N_1 N_2 = \mathrm{cl}((M_1 \widehat{\otimes} N_1)(M_2 \widehat{\otimes} N_2)) \subseteq \mathrm{cl}(MN) \subseteq K,$$

which further implies that

$$\pi(M_1 M_2) \otimes \rho(N_1 N_2) \subseteq (\pi \otimes \rho)(M_1 M_2 \widehat{\otimes} N_1 N_2) = \{0\}.$$

So either $M_1 M_2 \subseteq \ker \pi = E$ or $N_1 N_2 \subseteq \ker \rho = F$. Now both $E$ and $F$ are prime and so at least one of the following containments must hold:

$$M_1 \subseteq E, \quad M_2 \subseteq E, \quad N_1 \subseteq F, \quad N_2 \subseteq F.$$

In all of these cases either $M_1 \widehat{\otimes} N_1$ or $M_2 \widehat{\otimes} N_2$ is contained in $K$ which is a contradiction. Thus $K$ is prime. □

A closed ideal $I$ of a Banach $*$-algebra $E$ is said to be *primitive* if it is the kernel of an irreducible $*$-representation of $E$ on some Hilbert space. The following gives a characterization of the primitive ideals of $A \widehat{\otimes} B$.

THEOREM 7. *For $C^*$-algebras $A$ and $B$ the following statements are true.*

(1) *If $E$ and $F$ are primitive ideals of $A$ and $B$ respectively, then $A \widehat{\otimes} F + E \widehat{\otimes} B$ is also a primitive ideal of $A \widehat{\otimes} B$.*
(2) *If $K$ is a primitive ideal of $A \widehat{\otimes} B$, then $K = A \widehat{\otimes} F + E \widehat{\otimes} B$ where $E$ and $F$ are prime ideals of $A$ and $B$.*
(3) *If $A$ and $B$ are separable, then $K$ is primitive if and only if $K = A \widehat{\otimes} F + E \widehat{\otimes} B$ for some primitive ideals $E$ and $F$ of $A$ and $B$, respectively.*

PROOF. First we prove part (1). Since $E$ and $F$ are primitive ideals there exist irreducible $*$- representations $\pi_1 : A \to \mathcal{B}(H_1)$ and $\pi_2 : B \to \mathcal{B}(H_2)$ such that $E = \ker \pi_1$ and $F = \ker \pi_2$. Define $\pi : A \otimes B \to \mathcal{B}(H_1 \otimes H_2)$ by

$$\pi(a \otimes b) = \pi_1(a) \otimes \pi_2(b).$$

Then, by the definition of 'min'-norm (see [22]), $\pi$ is bounded with respect to the 'min'-norm and hence also with respect to the '$\wedge$'-norm. Thus $\pi$ can be extended to $A \widehat{\otimes} B$ as a bounded $*$-representation. We first claim that $\pi$ is irreducible,

https://doi.org/10.1017/S1446788711001479 Published online by Cambridge University Press

or equivalently, $\pi(A \widehat{\otimes} B)' = \mathbb{C}I$. Since $\pi(A \widehat{\otimes} B) \supset \pi_1(A) \otimes \pi_2(B)$, we have

$$\pi(A \widehat{\otimes} B)' \subseteq (\pi_1(A) \overline{\otimes} \pi_2(B))'$$

where $\overline{\otimes}$ denotes the weak closure. Further, since $\pi_1$ and $\pi_2$ are irreducible, $\pi_1(A)$ and $\pi_2(B)$ are nondegenerate $*$-subalgebras of $B(H_1)$ and $B(H_2)$, respectively. It then follows by the Double Commutant theorem that $\pi_1(A)$ and $\pi_2(B)$ are weakly dense in $\pi_1(A)''$ and $\pi_2(B)''$. In particular,

$$\pi_1(A) \overline{\otimes} \pi_2(B) = \pi_1(A)'' \overline{\otimes} \pi_2(B)''$$

and so by Tomita's Commutation theorem we may then deduce that

$$(\pi_1(A) \overline{\otimes} \pi_2(B))' = \pi_1(A)' \overline{\otimes} \pi_2(B)' \subseteq \mathbb{C}I,$$

which shows that $\pi$ is irreducible.

Next we claim that

$$\ker \pi = A \widehat{\otimes} F + E \widehat{\otimes} B = K,$$

say. Clearly $A \widehat{\otimes} F$ and $E \widehat{\otimes} B$ are both contained in $\ker \pi$ and so $K \subseteq \ker \pi$. For the other containment, consider the quotient map

$$\theta : A \widehat{\otimes} B \to A/E \widehat{\otimes} B/F$$

with $\ker \theta = K$. Since $\ker \pi$ contains $\ker \theta$, it follows by Lemma 2 that $\theta(\ker \pi)$ is a closed ideal of $A/E \widehat{\otimes} B/F$ with $\theta^{-1}(\theta(\ker \pi)) = \ker \pi$. If $\theta(\ker \pi) \neq 0$, then it must contain a nonzero elementary tensor, say, $(a + E) \otimes (b + F)$ (see [12, Proposition 3.7]). Now $a \otimes b \in \ker \pi$ implies that $\pi_1(a) \otimes \pi_2(b) = 0$, which further implies that either $a \in E$ or $b \in F$ so that $(a + E) \otimes (b + F) = 0$, which is a contradiction. Thus $\ker \pi \subseteq \ker \theta = K$ and our claim is established.

Now we prove part (2). Let $K = \ker \pi$ for some irreducible $*$-representation $\pi$ of $A \widehat{\otimes} B$ on $H$. By [22, Lemma IV.4.1] there exist commuting $*$-representations $\pi_1 : A \to \mathcal{B}(H)$ and $\pi_2 : B \to \mathcal{B}(H)$ such that

$$\pi(a \otimes b) = \pi_1(a)\pi_2(b) \quad \forall a \in A, b \in B.$$

Now $\pi(A \otimes B) = \pi_1(A)\pi_2(B)$ and so $\pi(A \widehat{\otimes} B) \subseteq \mathrm{cl}(\pi_1(A)\pi_2(B))$. Thus we see that

$$(\pi_1(A)\pi_2(B))' = \mathrm{cl}(\pi_1(A)\pi_2(B))' \subseteq \pi(A \widehat{\otimes} B)' = \mathbb{C}I.$$

Also, note that $\pi_1$ and $\pi_2$ are both factor representations since for $P = \pi_1(A)''$ and $Q = \pi_2(B)''$ we have

$$P \cap P' = (\pi_1(A)' \cup \pi_1(A))' \subseteq (\pi_2(B) \cup \pi_1(A))' \subseteq \{\pi_1(A)\pi_2(B)\}' = \mathbb{C}I$$

as $\pi_1(A)$ and $\pi_2(B)$ commute. Now let $E = \ker \pi_1$ and $F = \ker \pi_2$. Then since $E$ and $F$ are both kernels of factor representations they are both prime ideals (see [4, Proposition II.6.1.11]). Also, by the definition of $\pi$, we see that $A \widehat{\otimes} F + E \widehat{\otimes} B \subseteq K$.

https://doi.org/10.1017/S1446788711001479 Published online by Cambridge University Press

For the reverse containment we consider $a \otimes b \in K$. Then we have $\pi_1(a)\pi_2(b) = 0$. Since $\pi_1(A)''$ is a factor and $\pi_2(B)'' \subseteq \pi_1(A)'$, we may deduce by [22, Proposition IV.4.20] that either $\pi_1(a) = 0$ or $\pi_2(b) = 0$, that is, $a \otimes b$ belongs to either $A \mathbin{\widehat{\otimes}} F$ or $E \mathbin{\widehat{\otimes}} B$. In both cases, $a \otimes b \in A \mathbin{\widehat{\otimes}} F + E \mathbin{\widehat{\otimes}} B$. Finally, a similar argument to that given for part (1) enables us to conclude that $K \subseteq A \mathbin{\widehat{\otimes}} F + E \mathbin{\widehat{\otimes}} B$.

Finally, we prove part (3). If $A$ and $B$ are separable, then every prime ideal is a primitive ideal. So the result follows from parts (1) and (2). □

In particular, if $H$ is an infinite-dimensional separable Hilbert space $H$, then there are five proper closed ideals of $\mathcal{B}(H) \mathbin{\widehat{\otimes}} \mathcal{B}(H)$ (see [12, Theorem 3.12]), namely $\{0\}$, $\mathcal{B}(H) \mathbin{\widehat{\otimes}} \mathcal{K}(H)$, $\mathcal{K}(H) \mathbin{\widehat{\otimes}} \mathcal{B}(H)$, $\mathcal{B}(H) \mathbin{\widehat{\otimes}} \mathcal{K}(H) + \mathcal{K}(H) \mathbin{\widehat{\otimes}} \mathcal{B}(H)$ and $\mathcal{K}(H) \mathbin{\widehat{\otimes}} \mathcal{K}(H)$. Of these the first four are prime as well as primitive.

We now discuss the modular ideals of $A \mathbin{\widehat{\otimes}} B$. In a Banach algebra $A$ an ideal $I$ is said to be *modular* (or *regular*) if there exists an $e \in A$ such that $xe - x, ex - x \in I$ for all $x \in A$, or equivalently, if $A/I$ is unital. It is clear that every proper ideal in a unital Banach algebra is modular. Also the ideal $\{0\}$ is modular if and only if $A$ is unital.

If $I$ is a closed modular ideal of $A$, then the product ideal $I \mathbin{\widehat{\otimes}} I$ need not be modular in $A \mathbin{\widehat{\otimes}} A$. This can be seen by considering $A = C_0(X)$ where $X$ is a locally compact Hausdorff space (noncompact). Any closed modular ideal of $C_0(X)$ has the form $I(E) = \{f \in A \mid f(E) = 0\}$ where $E$ is a compact subset of $X$ (see [13]). We now consider a closed modular ideal $I = I(E)$ of $A$. The ideal $I \mathbin{\widehat{\otimes}} I$ consists of functions vanishing on the set $(E \times X) \cup (X \times E)$ which is not compact and so $I \mathbin{\widehat{\otimes}} I$ is not modular. By contrast, the larger ideal $I \mathbin{\widehat{\otimes}} A + A \mathbin{\widehat{\otimes}} I$ consists of functions vanishing on the smaller compact set $E \times E$ and is modular. In fact we have the following result which characterizes the modular product ideals.

THEOREM 8. *Let $I$ and $J$ be closed modular ideals of $A$ and $B$, respectively. Then $I \mathbin{\widehat{\otimes}} J$ is modular in $A \mathbin{\widehat{\otimes}} B$ if and only if both $A$ and $B$ are unital.*

PROOF. If $A$ and $B$ are both unital, then so is $A \mathbin{\widehat{\otimes}} B$ and so every ideal is modular. Conversely, suppose that $I \mathbin{\widehat{\otimes}} J$ is a modular ideal. Since $A \mathbin{\widehat{\otimes}} J$ and $I \mathbin{\widehat{\otimes}} B$ both contain $I \mathbin{\widehat{\otimes}} J$, both are modular ideals of $A \mathbin{\widehat{\otimes}} B$. By Lemma 2 we have an isomorphism between $(A \mathbin{\widehat{\otimes}} B)/(A \mathbin{\widehat{\otimes}} J)$ and $A \mathbin{\widehat{\otimes}} (B/J)$ and similarly an isomorphism between $(A \mathbin{\widehat{\otimes}} B)/(I \mathbin{\widehat{\otimes}} B)$ and $(A/I) \mathbin{\widehat{\otimes}} B$. Therefore $A \mathbin{\widehat{\otimes}} (B/J)$ and $(A/I) \mathbin{\widehat{\otimes}} B$ are unital. It then follows that $A$ and $B$ are both unital by [18, Theorem 1]. □

In particular, $\mathcal{K}(H) \mathbin{\widehat{\otimes}} \mathcal{K}(H)$ is a closed modular ideal of $\mathcal{B}(H) \mathbin{\widehat{\otimes}} \mathcal{B}(H)$ but is not modular in $\mathcal{B}(H) \mathbin{\widehat{\otimes}} \mathcal{K}(H)$. However, the maximal modular ideals behave well in $A \mathbin{\widehat{\otimes}} B$ as we see in the following result.

THEOREM 9. *A closed ideal $K$ of $A \mathbin{\widehat{\otimes}} B$ is a maximal modular ideal if and only if we can write $K = A \mathbin{\widehat{\otimes}} N + M \mathbin{\widehat{\otimes}} B$ for some maximal modular ideals $M$ and $N$ of $A$ and $B$, respectively.*

PROOF. Let $K$ be a maximal modular ideal of $A \mathbin{\widehat{\otimes}} B$. Since every maximal modular ideal is also a maximal ideal, $K$ is of the form $K = A \mathbin{\widehat{\otimes}} N + M \mathbin{\widehat{\otimes}} B$ for some maximal

https://doi.org/10.1017/S1446788711001479 Published online by Cambridge University Press

ideals $M$ and $N$ of $A$ and $B$, respectively by [12, Theorem 3.11]. Now $(A \widehat{\otimes} B)/K$ is unital and is isomorphic to $A/M \widehat{\otimes} B/N$ by Lemma 2. Therefore the latter space is unital. But this implies that $A/M$ and $B/N$ are both unital by [18, Theorem 1]. Thus $M$ and $N$ are also modular ideals of $A$ and $B$, respectively.

For the converse, let $K = A \widehat{\otimes} N + M \widehat{\otimes} B$ where $M$ and $N$ are maximal modular ideals of $A$ and $B$, respectively. Then since $M$ and $N$ are maximal ideals we may deduce that $K$ is also a maximal ideal by [12, Theorem 3.11]. Also, the facts that $(A \widehat{\otimes} B)/K$ and $A/M \widehat{\otimes} B/N$ are isomorphic and that $A/M$ and $B/N$ are both unital together imply that $A \widehat{\otimes} B/K$ is unital. It follows that $K$ is modular. □

## 4. The Wiener property and symmetry

A Banach $*$-algebra is said to have *Wiener property* if every proper closed two-sided ideal is annihilated by an irreducible $*$-representation (see [20]). The Wiener property for group algebras and weighted group algebras has been studied in [10, 19] amongst others. It is well known that every $C^*$-algebra has the Wiener property.

THEOREM 10. *The Banach $*$-algebra $A \widehat{\otimes} B$ has the Wiener property.*

PROOF. Consider a proper closed two-sided ideal $J$ of $A \widehat{\otimes} B$. Let $J_{\min}$ denote the closure of $i(J)$ in $A \otimes_{\min} B$ where $i : A \widehat{\otimes} B \to A \otimes_{\min} B$ is the canonical homomorphism. By [14, Theorem 6] $J_{\min}$ is also a proper closed two-sided ideal of the $C^*$-algebra $A \otimes_{\min} B$ and so it is annihilated by an irreducible $*$-representation $\pi : A \otimes_{\min} B \to \mathcal{B}(H)$. Note that the isometry of the involution implies that $i$ is $*$-preserving so that we have a $*$-representation $\hat{\pi} := \pi \circ i$ of $A \widehat{\otimes} B$ on $H$ with $\hat{\pi}(J) = \{0\}$. Also, the relation $\hat{\pi}(A \otimes B) = \pi(A \otimes B)$ implies that

$$\hat{\pi}(A \widehat{\otimes} B)' \subseteq \pi(A \otimes B)' = \pi(A \otimes_{\min} B)' = \mathbb{C}I$$

where the equality between the middle expressions follows from the norm density of $\pi(A \otimes B)$ in $\pi(A \otimes_{\min} B)$. This further implies that $\hat{\pi}$ is irreducible and hence that $A \widehat{\otimes} B$ has the Wiener property. □

REMARK 11. The above theorem was originally proved under the assumptions that $A$ and $B$ were both unital or separable. The authors are grateful to the referee for providing the generalization presented here.

A Banach $*$-algebra is said to be *symmetric* if every element of the form $x^*x$ has positive spectrum, or equivalently, if every self-adjoint element has a real spectrum (see [20, Theorem 10.4.17]). Symmetry in group algebras has been investigated by various authors (see, for instance, [17, 19]). One can easily verify that a Banach $*$-algebra $A$ is symmetric if and only if, for every left modular ideal $I$ of $A$ with modular unit $\alpha$, the set $S_I$ of Hermitian sesquilinear forms $B : A \times A \to \mathbb{C}$ such that

$$\begin{aligned} &B_\alpha = B, && B(I, A) = \{0\}, \\ &B(u, u) \geq 0, && B(uw, vw) = B(v^*uw, w) \quad \forall u, v, w \in A \end{aligned}$$

https://doi.org/10.1017/S1446788711001479 Published online by Cambridge University Press

is nontrivial, where $B_\alpha(v, w) := B(v\alpha, w\alpha)$ for all $v, w \in A$ (see [19]). It is well known that every $C^*$-algebra is symmetric (see [20]). For $C^*$-algebras $A$ and $B$, we do not know whether the Banach $*$-algebra $A \mathbin{\widehat{\otimes}} B$ is symmetric or not, but if one of them is subhomogeneous, then we have an affirmative answer. Recall that a $C^*$-algebra $A$ is *subhomogeneous* if there exists a positive integer $n$ such that each irreducible representation of $A$ has dimension less than or equal to $n$.

We first modify a result from [14] to operator algebras. We say that a Banach algebra $A$ is an *operator algebra* if there exists a Hilbert space $H$ and a bicontinuous homomorphism of $A$ into $\mathcal{B}(H)$.

PROPOSITION 12. *If $A$ and $B$ are operator algebras, then $A \mathbin{\widehat{\otimes}} B$ is a Banach algebra. If $A$ and $B$ both have isometric involutions then $A \mathbin{\widehat{\otimes}} B$ is a Banach $*$-algebra.*

PROOF. It is well known that if $A$ is an operator algebra, then the multiplication operator $m : A \otimes_h A \to A$ given by $m(a \otimes b) = ab$ is completely bounded (see [5, Theorem 1.3]). This result gives us completely bounded operators

$$m_A : A \otimes_h A \to A, \quad m_B : B \otimes_h B \to B.$$

Now consider the canonical map $i : A \mathbin{\widehat{\otimes}} A \to A \otimes_h A$ which is a completely contractive homomorphism. Then the multiplication operator $m'_A : A \mathbin{\widehat{\otimes}} A \to A$, which can be regarded as $m'_A = m_A \circ i$, is completely bounded. Similarly, the multiplication operator $m'_B : B \mathbin{\widehat{\otimes}} B \to B$ is also completely bounded. In particular, the operator

$$m'_A \otimes m'_B : (A \mathbin{\widehat{\otimes}} A) \mathbin{\widehat{\otimes}} (B \mathbin{\widehat{\otimes}} B) \to A \mathbin{\widehat{\otimes}} B$$

is bounded. Using the commutativity of '$\wedge$', we may deduce that the operator

$$m'_A \otimes m'_B : (A \mathbin{\widehat{\otimes}} B) \mathbin{\widehat{\otimes}} (A \mathbin{\widehat{\otimes}} B) \to A \mathbin{\widehat{\otimes}} B$$

is also bounded. Hence $A \mathbin{\widehat{\otimes}} B$ is a Banach algebra. The proof for the involution emulates the one given in [14]. □

LEMMA 13. *Let $A$ and $B$ be $C^*$-algebras with either $A$ or $B$ finite-dimensional. Then $A \mathbin{\widehat{\otimes}} B$ is a symmetric operator algebra.*

PROOF. If $A$ or $B$ is finite-dimensional, then clearly $A \mathbin{\widehat{\otimes}} B$ is $*$-isomorphic to $A \otimes_{\min} B$, which gives the required result. □

LEMMA 14. *If $A$ is a commutative unital $C^*$-algebra and $B$ is a symmetric unital operator algebra with isometric involution, then $A \mathbin{\widehat{\otimes}} B$ is symmetric.*

PROOF. Note that $A \mathbin{\widehat{\otimes}} B$ is a Banach $*$-algebra by Proposition 12. Let $\Phi(A)$ denote the set of maximal ideals of $A$. Then $\Phi(A)$ is in one-to-one correspondence with the space of nonzero $*$-homomorphisms of $A$. For $M \in \Phi(A)$ define $h_M : A \otimes B \to B$ by

$$h_M\Big(\sum a_i \otimes b_i\Big) = \sum a_i(M) b_i.$$

https://doi.org/10.1017/S1446788711001479 Published online by Cambridge University Press

The map $h_M$ is bounded with respect to the '$\wedge$'-norm and so it can be extended to $A \widehat{\otimes} B$ as a $*$-homomorphism.

By [16, Corollary 2] an element $x$ of $A \widehat{\otimes} B$ is invertible if and only if $h_M(x)$ is invertible for each maximal ideal $M$ of $A$. Thus

$$\sigma(x) = \bigcup_{M \in \Phi(A)} \sigma(h_M(x))$$

where $\sigma(x)$ denotes the spectrum of $x$ in $A \widehat{\otimes} B$. Now consider a self-adjoint element $u$ of $A \widehat{\otimes} B$. For any $M \in \Phi(A)$ the image $h_M(u)$ is self-adjoint in $B$ since $h_M$ is $*$-preserving. But $B$ is symmetric and so

$$\sigma(u) = \bigcup_{M \in \Phi(A)} \sigma(h_M(u)) \subseteq \mathbb{R}.$$

Hence $A \widehat{\otimes} B$ is symmetric. □

Remark 15. Note that one can also prove the above lemma using an argument similar to the one given in [7, Corollary 3.3].

Theorem 16. *If $A$ is a subhomogeneous $C^*$-algebra, then for any $C^*$-algebra $B$ the Banach $*$-algebra $A \widehat{\otimes} B$ is symmetric.*

Proof. Since $A \widehat{\otimes} B$ can be isometrically embedded in $A^{**} \widehat{\otimes} B^{**}$ as a closed $*$-subalgebra it suffices to show that $A^{**} \widehat{\otimes} B^{**}$ is symmetric. Let $A$ be $n$-subhomogeneous. Then $A^{**}$ is a direct sum of type $I_m$ von Neumann algebras where $m \leq n$ (see [4, Theorem IV.1.4.6]). Also, each type $I_m$ von Neumann algebra is isomorphic to $M_m \widehat{\otimes} C$ where $M_m$ is the set of $m \times m$ complex matrices and $C$ is a commutative von Neumann algebra by [4, III.1.5.12]. Thus $A^{**} \widehat{\otimes} B^{**}$ is $*$-isomorphic (not necessarily isometrically) to a direct sum of algebras of the form $M_m(C) \widehat{\otimes} B^{**}$. For each $m$ the algebra $M_m(C)$ is isomorphic to $M_m \widehat{\otimes} C$ and so, using the commutativity and associativity of the operator space projective tensor product, we may deduce that $M_m(C) \widehat{\otimes} B^{**}$ is $*$-isomorphic to $C \widehat{\otimes} (M_m \widehat{\otimes} B^{**})$. Note that by Lemma 13 the algebra $M_m \widehat{\otimes} B^{**}$ is an operator algebra with an isometric involution and is symmetric. It follows that $M_m(C) \widehat{\otimes} B^{**}$ is symmetric by Lemma 14. Hence $A^{**} \widehat{\otimes} B^{**}$ is symmetric since it is the direct sum of symmetric Banach $*$-algebras (see [20, Theorem 11.4.2]). □

Remark 17. If $A$ is commutative and $B$ is any $C^*$-algebra, then by [7, Corollary 3.3] $A \otimes_\gamma B$ is symmetric. In addition, the symmetry of $A \otimes_\gamma B$ when $A$ is subhomogeneous and $B$ is any $C^*$-algebra may be proved as in Theorem 16.

## Acknowledgement

The authors would like to express their gratitude to the referee for his/her numerous valuable comments and suggestions.

https://doi.org/10.1017/S1446788711001479 Published online by Cambridge University Press

## References


[1] S. D. Allen, A. M. Sinclair and R. R. Smith, 'The ideal structure of the Haagerup tensor product of $C^*$-algebras', *J. reine angew. Math.* **442** (1993), 111–148.

[2] R. J. Archbold, 'A counter example for commutation in tensor products of $C^*$-algebras', *Proc. Amer. Math. Soc.* **81** (1981), 562–564.

[3] R. J. Archbold, E. Kaniuth, G. Schlichting and D. W. B. Somerset, 'Ideal space of the Haagerup tensor product of $C^*$-algebras', *Internat. J. Math.* **8** (1997), 1–29.

[4] B. Blackadar, *Operator Algebras—Theory of $C^*$-algebras and von-Neumann Algebras* (Springer, Berlin–Heidelberg–New York, 2006).

[5] D. P. Blecher and C. LeMerdy, 'On quotients of function algebras and operator algebra structures on $l_p$', *J. Operator Theory* **34** (1995), 315–346.

[6] D. P. Blecher and V. I. Paulsen, 'Tensor products of operator spaces', *J. Funct. Anal.* **99** (1991), 262–292.

[7] R. A. Bonic, 'Symmetry in group algebras of discrete groups', *Pacific J. Math* **11** (1961), 73–94.

[8] E. G. Effros and Z. J. Ruan, 'On matricially normed spaces', *Pacific J. Math.* **132** (1988), 243–264.

[9] E. G. Effros and Z. J. Ruan, 'A new approach to operator space', *Canad. Math. Bull.* **34** (1991), 329–337.

[10] W. Hauenschild, E. Kaniuth and A. Kumar, 'Ideal structure of Beurling algebras on $[FC]^-$ groups', *J. Funct. Anal.* **51** (1983), 213–228.

[11] R. Jain and A. Kumar, 'Operator space tensor products of $C^*$-algebras', *Math. Z.* **260** (2008), 805–811.

[12] R. Jain and A. Kumar, 'Operator space projective tensor product: embedding into second dual and ideal structure', arXiv:1106.2644v1 [math.OA].

[13] E. Kaniuth, *A Course in Commutative Banach Algebras* (Springer, New York, 2009).

[14] A. Kumar, 'Operator space projective tensor product of $C^*$-algebras', *Math. Z.* **237** (2001), 211–217.

[15] A. Kumar and A. M. Sinclair, 'Equivalence of norms on operator space tensor products of $C^*$-algebras', *Trans. Amer. Math. Soc.* **350** (1998), 2033–2048.

[16] A. Lebow, 'Maximal ideals in tensor products of Banach algebras', *Bull. Amer. Math. Soc.* **74** (1968), 1020–1022.

[17] H. Leptin and D. Poguntke, 'Symmetry and nonsymmetry for locally compact groups', *J. Funct. Anal.* **33** (1979), 119–134.

[18] R. J. Loy, 'Identities in tensor products of Banach algebras', *Bull. Aust. Math. Soc.* **2** (1970), 253–260.

[19] J. Ludwig, 'A class of symmetric and a class of Wiener group algebras', *J. Funct. Anal.* **31** (1979), 187–194.

[20] T. W. Palmer, *Banach Algebras and the General Theory of $*$-Algebras II* (Cambridge University Press, Cambridge, 2001).

[21] R. R. Smith, 'Completely bounded module maps and the Haagerup tensor product', *J. Funct. Anal.* **102** (1991), 156–175.

[22] M. Takesaki, *Theory of Operator Algebras I*, 2nd Printing (Springer, Berlin–Heidelberg–New York, 2002).

[23] J. Tomiyama, 'Applications of Fubini type theorem to the tensor products of $C^*$-algebras', *Tohoku Math J.* **19** (1967), 213–226.

[24] S. Wassermann, 'The slice map problem for $C^*$-algebras', *Proc. Lond. Math. Soc.* **32** (1976), 537–559.

[25] S. Wassermann, 'On tensor products of certain group $C^*$-algebras', *J. Funct. Anal.* **23** (1976), 239–254.


https://doi.org/10.1017/S1446788711001479 Published online by Cambridge University Press

RANJANA JAIN, Department of Mathematics, Lady Shri Ram College for Women, New Delhi-110024, India
e-mail: ranjanaj_81@rediffmail.com

AJAY KUMAR, Department of Mathematics, University of Delhi, Delhi-110007, India
e-mail: akumar@maths.du.ac.in

https://doi.org/10.1017/S1446788711001479 Published online by Cambridge University Press