A CONCAVITY PROBLEM IN NUMBER THEORY

by IAN ANDERSON

(Received 10 February, 1970)

For any fixed value of x, let $\{a_i^{(k)}\}$ denote the set of all positive integers with exactly k prime factors counted according to multiplicity, each prime factor being $\leqq x$. In an earlier paper [**1**] in this journal we posed the following problem. Let

$$A_k = \sum_i \frac{1}{a_i^{(k)}}. \tag{1}$$

Show the existence or non-existence of an integer K such that, if

$$k_1 < k_2 < K < k_3 < k_4,$$

then

$$A_{k_1} \leqq A_{k_2} \leqq A_K, \qquad A_K \geqq A_{k_3} \geqq A_{k_4}. \tag{2}$$

We now show that such a K exists, and that in (2) there is strict inequality in each case.

A sequence $\{A_k: 1 \leqq k \leqq n\}$ of positive real numbers is said to have the *logarithmic concavity* (L.C.) property if, for $2 \leqq k \leqq n-1$,

$$A_{k-1}A_{k+1} \leqq A_k^2.$$

Then $B_k = \log A_k$ is a concave function of k, and so the numbers A_k are either monotonic (increasing or decreasing) or *unimodal*, first increasing with k, and then eventually decreasing after perhaps remaining unchanged for several values of k. For the numbers A_k defined by (1), we have a monotonic decreasing sequence for $2 \leqq x \leqq 4$, where $K = 1$, but we cannot have a monotonic increasing sequence, since, for any x, $A_k \to 0$ as $k \to \infty$. To see this, observe that the number of terms contributing to A_k is

$$\frac{(l+k-1)!}{(l-1)!\,k!},$$

where l denotes the number of primes $\leqq x$. Since each term is $\leqq 1/2^k$, we therefore have

$$A_k \leqq \frac{(l+k-1)!}{(l-1)!\,k!\,2^k} \to 0 \qquad \text{as} \qquad k \to \infty.$$

Thus the A_k are unimodal. We shall show that the plateau between the increasing and the decreasing phases is non-existent. Examples of numbers with L.C., which are hence unimodal, are, for fixed n, the binomial coefficients $\binom{n}{k}$, the Stirling numbers S_n^k of both the first and the second kind [3], and, for a given integer m, $N_k(m)$, the number of divisors of m with k prime factors [**2**].

https://doi.org/10.1017/S0017089500001075 Published online by Cambridge University Press

The sequence $\{A_k\}$ given by (1) has L.C. For

$$A_{k-1}A_{k+1} = \sum_M \frac{c(M)}{M},$$

where $c(M)$ denotes the number of representations of M in the form $M = a_i^{(k-1)}a_j^{(k+1)}$, and where summation is over all such M with $2k$ prime factors. But it is shown in the proof of Theorem 1 of [2] that, for each M,

$$c(M) \leqq d(M),$$

where $d(M)$ denotes the number of representations of M in the form $M = a_i^{(k)}a_j^{(k)}$. Thus

$$A_{k-1}A_{k+1} \leqq \sum_M \frac{d(M)}{M} = A_k^2.$$

Thus we have unimodality. Finally, there is strict inequality everywhere in (2). For if $A_{k-1} = A_k$ for some k, then since

$$A_{k-1}\left(\sum_{p \leqq x} \frac{1}{p}\right) = \sum_i \frac{\omega(a_i^{(k)})}{a_i^{(k)}},$$

where $\omega(a)$ denotes the number of distinct prime factors of a, it follows that

$$\left(\sum_i \frac{1}{a_i^{(k)}}\right)\left(\sum_{p \leqq x} \frac{1}{p}\right) = \sum_i \frac{\omega(a_i^{(k)})}{a_i^{(k)}}. \tag{3}$$

Let $P = \prod_{p \leqq x} p$. Then the right-hand side of (3) can be expressed as a rational with denominator P^k. However, since for each prime $p \leqq x$ there is exactly one $a_i^{(k)}$ divisible by p^k, namely p^k itself, A_k, when expressed as a rational in lowest terms, has denominator P^k; and $\sum_{p \leqq x} 1/p$ has denominator P. Thus the left hand side of (3) has denominator P^{k+1}. Thus (3) is in fact impossible.

REFERENCES

1. I. Anderson, Primitive sequences whose elements have no large prime factors, *Glasgow Math. J.* **10** (1969), 10–15.

2. I. Anderson, On the divisors of a number, *J. London Math. Soc.* **43** (1968), 410–418.

3. E. H. Lieb, Concavity properties and a generating function for Stirling numbers, *J. Combinatorial Theory*, **5** (1968), 203–206.

UNIVERSITY OF GLASGOW