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CHANGE OF RING AND TORSION-THEORETIC INJECTIVITY

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Let τ be a hereditary torsion theory in R-Mod. Then any ring homomorphism $\gamma : R \to S$ induces in S-Mod a torsion theory σ given by the condition that a left S-module is σ-torsion if and only if it is τ-torsion as a left R-module. We show that if $\gamma : R \to S$ is a ring epimorphism and A is a τ-injective left R-module, then $\operatorname{Ann}_A \operatorname{Ker}(\gamma)$ is σ-injective as a left S-module. As a consequence, we relate τ-injectivity and σ-injectivity, and we give some applications.

1. INTRODUCTION

A classical result states that if I is a two-sided ideal of a ring R and E is an injective left R-module, then $\operatorname{Ann}_E I$ is injective as a left R/I-module (for instance, see [**8**, Proposition 2.27]). In this note we generalise it for injectivity with respect to a hereditary torsion theory. As a consequence we are able to deal with the problem of the behaviour of torsion-theoretic injectivity under the action of a ring homomorphism $\gamma : R \to S$. Under certain conditions we pass from relative injectivity in R-Mod to relative injectivity in S-Mod and we deal with the converse. The latter was previously discussed by several authors, for instance, Golan [**5**], Izawa [**6**] and, in a more general framework, Teply and Torrecillas [**10**]. Some of the present results also generalise for an arbitrary hereditary torsion theory properties established in [**2**] in the case of the Dickson torsion theory [**4**].

Now let us give some basic notation and terminology. Throughout the paper we denote by R and S associative rings with non-zero identity and all modules are left unital R-modules. If B is a non-empty subset of a module A and I is a non-empty subset of R, we denote annihilators as follows:

$$\operatorname{Ann}_R B = \{r \in R \mid rb = 0\,, \forall b \in B\} \quad \text{and}$$
$$\operatorname{Ann}_A I = \{a \in A \mid ra = 0\,, \forall r \in I\}.$$

Also, τ will always be a hereditary torsion theory in the category R-Mod of left R-modules. For a left R-module A and a submodule B of A, B is called τ-dense (respectively τ-closed) in A if A/B is τ-torsion (respectively τ-torsionfree). A non-zero left R-module

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module A is said to be τ-cocritical if A is τ-torsionfree and each of its non-zero submodules is τ-dense in A. A left R-module A is said to be τ-injective if it is injective with respect to every monomorphism having a τ-torsion cokernel or, equivalently, if it is injective with respect to every monomorphism $I \to R$ with I a τ-dense left ideal of R. Any ring homomorphism $\gamma : R \to S$ allows us to induce in the category S-Mod a torsion theory σ given by the condition that a left S-module A is σ-torsion if and only if A is τ-torsion as a left R-module. Throughout σ will always be this induced torsion theory in the corresponding category S-Mod. In the above context, τ is called compatible with γ if the following condition holds: a left S-module A is σ-torsionfree if and only if A is τ-torsionfree as a left R-module.

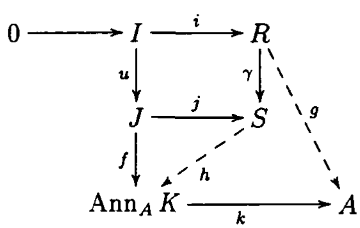
For additional information on torsion theories the reader is referred to [**5**].

2. From R to S

We begin with the torsion-theoretic generalisation of the result mentioned in the introduction.

Theorem 2.1. *Let $\gamma : R \to S$ be a ring epimorphism and let A be a τ-injective left R-module. Then $\mathrm{Ann}_A \mathrm{Ker}(\gamma)$ is σ-injective as a left S-module.*

Proof: Denote $K = \mathrm{Ker}(\gamma)$. Let J be a σ-dense left ideal of S and let $f : J \to \mathrm{Ann}_A K$ be an S-homomorphism. Since τ is compatible with γ by [**5**, Proposition 47.4], $I = \gamma^{-1}(J)$ is a τ-dense left ideal of R [**5**, Proposition 47.2]. Consider the following diagram with commutative square:



where i, j, k are inclusion homomorphisms and u is the natural R-epimorphism induced by γ. By the τ-injectivity of A as a left R-module, there exists an R-homomorphism $g : R \to A$ such that $gi = kfu$. Note that $K \subseteq I$ and $K \subseteq \mathrm{Ker}(g)$.

Now define the S-homomorphism

$$h : S \to \mathrm{Ann}_A K\,, \quad h(s) = g(r),$$

where $r \in R$ is such that $\gamma(r) = s$. If $s = \gamma(r) = \gamma(r')$ for some $r, r' \in R$, then $r - r' \in K \subseteq \mathrm{Ker}(g)$, hence $g(r) = g(r')$. If $t \in K$ and $r \in R$, we have $tg(r) = g(tr) = 0$, hence $g(r) \in \mathrm{Ann}_A K$. Therefore h is well-defined.

For every $s \in J$, we have $s = u(r)$ for some $r \in I$ and

$$hj(s) = hju(r) = h\gamma i(r) = g\big(i(r)\big) = kfu(r) = kf(s) = f(s).$$

Therefore $\operatorname{Ann}_A K$ is σ-injective as a left S-module. ∎

Now we can transfer torsion-theoretic injectivity from R to S under the action of a ring epimorphism $\gamma : R \to S$.

THEOREM 2.2. *Let $\gamma : R \to S$ be a ring epimorphism and let A be a left R-module such that $\operatorname{Ker}(\gamma) \subseteq \operatorname{Ann}_R A$. If A is τ-injective as a left R-module, then A is σ-injective as a left S-module.*

PROOF: If $\operatorname{Ker}(\gamma) \subseteq \operatorname{Ann}_R A$, then A is a left S-module and $\operatorname{Ann}_A \operatorname{Ker}(\gamma) = A$. Now the result follows by Theorem 2.1. ∎

3. From S to R

We begin with a result in a more general setting, which slightly generalises [**10**, Proposition 2.7]. It has a similar proof, that we briefly give here for the sake of completeness.

THEOREM 3.1. *Let $\mathcal{C}$ and $\mathcal{C}'$ be two Abelian categories, $\mathcal{T}$ a torsion class in $\mathcal{C}$ and $\mathcal{T}'$ a torsion class in $\mathcal{C}'$. Also let $G : \mathcal{C} \to \mathcal{C}'$ be an additive functor and $F : \mathcal{C}' \to \mathcal{C}$ an additive functor which preserves the exactness of every short exact sequence $0 \to X \to Y \to Z \to 0$ with $Z \in \mathcal{T}'$. Furthermore, assume that G is a right adjoint to F and $F(\mathcal{T}') \subseteq \mathcal{T}$. Then $G(A)$ is a $\mathcal{T}'$-injective object in $\mathcal{C}'$ for every $\mathcal{T}$-injective object A in $\mathcal{C}$.*

PROOF: Let A be a $\mathcal{T}$-injective object in $\mathcal{C}$ and let $0 \to X \to Y \to Z \to 0$ be a short exact sequence of objects in $\mathcal{C}'$ with $Z \in \mathcal{T}'$. Then we have an induced short exact sequence

$$0 \to \operatorname{Hom}_{\mathcal{C}}(F(Z), A) \to \operatorname{Hom}_{\mathcal{C}}(F(Y), A) \to \operatorname{Hom}_{\mathcal{C}}(F(X), A) \to 0$$

and, by adjointness, a short exact sequence

$$0 \to \operatorname{Hom}_{\mathcal{C}'}\big(Z, G(A)\big) \to \operatorname{Hom}_{\mathcal{C}'}\big(Y, G(A)\big) \to \operatorname{Hom}_{\mathcal{C}'}\big(X, G(A)\big) \to 0$$

Then it follows that $G(A)$ is $\mathcal{T}'$-injective. ∎

Recall that a right R-module C is called τ-flat if the left R-module $\operatorname{Hom}_{\mathbb{Z}}(C, \mathbb{Q}/\mathbb{Z})$ is τ-injective or, equivalently, the functor $C \bigotimes_R -$ preserves the exactness of every short exact sequence $0 \to X \to Y \to Z \to 0$ of left R-modules with X a τ-dense submodule of Y (see [**5**, p.88] or [**7**]).

COROLLARY 3.2. *Let $\gamma : R \to S$ be a ring homomorphism such that S is τ-flat as a right R-module. Also, assume that τ is compatible with γ. Then every σ-injective left S-module is τ-injective as a left R-module.*

PROOF: In Theorem 3.1 take $G : S - \text{Mod} \to R - \text{Mod}$ to be the forgetful functor, $F = S \bigotimes_R - : R - \text{Mod} \to S - \text{Mod}$, $\mathcal{T}'$ the torsion class in R-Mod associated to the torsion theory τ and $\mathcal{T}$ the torsion class in S-Mod associated to the induced torsion theory σ. Then G is a right adjoint to F. By [**5**, Proposition 47.2], the compatibility of τ with γ assures that for every τ-torsion left R-module Z, $F(Z) = S \bigotimes_R Z$ is τ-torsion as a left R-module, hence σ-torsion as a left S-module. Now the conclusion follows by Theorem 3.1. $\square$

In order to establish a characterisation of the condition that every σ-injective left S-module is τ-injective as a left R-module, we use some generalisation of purity. To recall some terminology, let Ω be a class of left R-modules and let

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \tag{1}$$

be a short exact sequence of right R-modules. If the map $f \bigotimes_R 1_D : A \bigotimes_R D \to B \bigotimes_R D$ is a monomorphism for every $D \in \Omega$, then the sequence (1) is called Ω-pure [**9**]. If A is a submodule of B, f is the inclusion monomorphism and the sequence (1) is Ω-pure, then A is said to be an Ω-pure submodule of B. The following characterisation will be useful.

PROPOSITION 3.3. ([**9**, p. 170]) *Let Ω be a class of cyclic left R-modules. The following are equivalent:*

(i) *The sequence (1) is Ω-pure;*

(ii) *$AJ = A \cap BJ$ for every left ideal J of R such that $R/J \in \Omega$.*

Now let Ω' be the set of all left R-modules R/J with J a τ-dense left ideal of R.

THEOREM 3.4. *Let $\gamma : R \to S$ be a ring epimorphism and let $K = \text{Ker}(\gamma)$. The following are equivalent:*

(i) *K is an Ω'-pure right ideal of R;*

(ii) *S is τ-flat as a right R-module;*

(iii) *Every σ-injective left S-module is τ-injective as a left R-module.*

PROOF: (i) $\Longleftrightarrow$ (ii) By Proposition 3.3 and [**5**, p.89] both conditions are equivalent to $KI = K \cap I$ for every τ-dense left ideal of R.

(ii) $\Longrightarrow$ (iii) By Corollary 3.2.

(iii) $\Longrightarrow$ (ii) Use the injective left S-module $\text{Hom}_{\mathbb{Z}}(S, \mathbb{Q}/\mathbb{Z})$ and the definition of τ-flatness. $\square$

Now we can put together Theorems 2.2 and 3.4 to get the following result.

THEOREM 3.5. *Let $\gamma : R \to S$ be a ring epimorphism and let A be a left R-module such that $\text{Ker}(\gamma) \subseteq \text{Ann}_R A$ is an Ω'-pure right ideal of R. Then A is τ-injective as a left R-module if and only if it is σ-injective as a left S-module.*

4. Applications

For a commutative ring R, denote by $\mathrm{Spec}(R)$ the set of all prime ideals of R and let $p \in \mathrm{Spec}(R)$. Following [**12**, p.83], for each integer $m \geqslant 1$ denote $A_m = \mathrm{Ann}_{E(R/p)}\, p^m$, where $E(R/p)$ is the injective hull of R/p. Note that $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_m \subseteq A_{m+1} \subseteq \dots$ and, if R is Noetherian, $E(R/p) = \bigcup_{m=1}^{\infty} A_m$ [**12**, p.83].

Corollary 4.1. *Let R be commutative and let $p \in \mathrm{Spec}(R)$ be such that R/p is τ-cocritical. Then $A_1 = E_\tau(R/p)$ is σ-injective as an R/p-module, where $E_\tau(R/p)$ is the τ-injective hull of R/p and σ is the torsion theory induced by τ in R/p-Mod via the natural ring epimorphism $R \to R/p$.*

Proof: By [**3**, Theorem 2.5], $A_1 = E_\tau(R/p)$. Since $R/p \subseteq \mathrm{Ann}_{E(R/p)}\, p$, we have $E_\tau(R/p) \subseteq \mathrm{Ann}_{E(R/p)}\, p$, hence $p \subseteq \mathrm{Ann}_R\, E_\tau(R/p)$. Now use Theorem 2.2. ∎

Corollary 4.2. *Let R be commutative Noetherian and let $p \in \mathrm{Spec}(R)$ be τ-closed in R. Then:*

(i) *Each A_m is σ_m-injective as an R/p^m-module, where σ_m is the torsion theory induced by τ in R/p^m-Mod via the natural ring epimorphism $R \to R/p^m$.*

(ii) *The τ-injective hull $E_\tau(R/p)$ of R/p is σ-injective as an R/p-module, where σ is the torsion theory induced by τ in R/p-Mod via the natural ring epimorphism $R \to R/p$.*

Proof: (i) Since p is τ-closed in R, A_m is τ-injective as an R-module [**3**, Theorem 2.10]. Now use Theorem 2.1.

(ii) By [**3**, Theorem 2.10], $A_1 = \mathrm{Ann}_{E(R/p)}\, p$ is a τ-injective R-module. As in the proof of Corollary 4.1, we have $p \subseteq \mathrm{Ann}_R\, E_\tau(R/p)$. Now use Theorem 2.2. ∎

Recall that a ring R is called left fully idempotent if every left ideal of R is idempotent. In particular, any von Neumann regular ring is left fully idempotent.

Corollary 4.3. *Let R be a left fully idempotent ring and let $\gamma : R \to S$ be a ring epimorphism. Then every σ-injective left S-module is τ-injective as a left R-module.*

Proof: By [**11**, p.320], S is flat as a right R-module. Now use Corollary 3.2. ∎

Let us particularise Theorem 3.4 in the case of the Dickson torsion theory τ_D, which is generated by all simple left R-modules [**4**]. Then we may restrict Ω' to all simple left R-modules and Ω'-purity becomes s-purity. Recall that a short exact sequence of right R-modules is called s-pure if it stays exact when tensored by any simple left R-module [**1**]. We denote by σ_D the torsion theory induced by τ_D in S-Mod under the action of a ring homomorphism $\gamma : R \to S$. Thus we obtain the following result.

Corollary 4.4. *Let $\gamma : R \to S$ be a ring epimorphism and let $K = \mathrm{Ker}(\gamma)$. The following are equivalent:*

(i) *K is an s-pure right ideal of R;*

(ii) *S is τ_D-flat as a right R-module;*

(iii) *Every σ_D-injective left S-module is τ_D-injective as a left R-module.*

Corollary 4.5. *Let $\gamma : R \to S$ be a ring epimorphism such that $K = \mathrm{Ker}(\gamma)$ is an idempotent ideal. Suppose that either R is commutative or K is contained in the Jacobson radical of R. Then every σ_D-injective left S-module is τ_D-injective as a left R-module.*

Proof: If either R is commutative or K is contained in the Jacobson radical of R, then K is an s-pure ideal of R by [**1**, Proposition 3.6] and [**1**, Corollary 2.8]. Now the result follows by Corollary 4.4. ∎

Now assume that τ is a Jansian torsion theory, that is, it has a Gabriel filter consisting of all left ideals J of R containing an idempotent two-sided ideal K of R. Hence $\Omega' = \Omega_0$ consists of all left R-modules R/I with I a left ideal of R such that $K \subseteq I$. We may characterise idempotent ideals in terms of some purity.

Lemma 4.6. *Let K be a two-sided ideal of R. Then K is idempotent if and only if K is Ω_0-pure as a right ideal of R.*

Proof: Assume first that K is idempotent. Let I be a left ideal of R such that $R/I \in \Omega_0$. Then $K = K^2 \subseteq KI \subseteq K \cap I = K$. Hence $KI = K \cap I$, so that K is Ω_0-pure in R.

Conversely, if K is Ω_0-pure in R, then we have $KI = K \cap I = K$ for every left ideal I of R such that $R/I \in \Omega_0$. In particular, $K^2 = K$. ∎

Corollary 4.7. *Let $\gamma : R \to S$ be a ring epimorphism and suppose that $K = \mathrm{Ker}(\gamma)$ is an idempotent ideal. Let τ be the Jansian torsion theory having the Gabriel filter given by K. Then every σ-injective left S-module is τ-injective as a left R-module.*

Proof: From the proof of Lemma 4.6, we see that the property that K is idempotent is characterised by the condition $KI = K$ for every left ideal I of R such that $R/I \in \Omega_0$, that is, for every τ-dense left ideal I of R. Also note that one always has $K \subseteq I$, hence $KI = K \cap I$, for any τ-dense left ideal I of R. Now use Theorem 3.4. ∎

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