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A CHARACTERIZATION AND A CLASS OF DISTRIBUTION FUNCTIONS FOR THE STIELTJES-WIGERT POLYNOMIALS

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1. In his classic memoir on the moment problem that bears his name, Stieltjes [2] exhibited

$$(1)\quad q^{-(n+1)^2/2} = \pi^{-1/2}\int_0^\infty x^n[1+A\sin(2\pi\log x)]\exp(-\log^2 x)\,dx$$
$$(q=\exp(-\tfrac{1}{2}),\quad |A|\le 1,\quad n=0,1,2,\ldots)$$

as an example of an indeterminate (Stieltjes) moment sequence.

Stieltjes also obtained the corresponding S-fraction and thus implicitly obtained the three-term recurrence formula satisfied by the corresponding orthogonal polynomials.

Later, Wigert [4] found an explicit formula for the orthonormal polynomials corresponding to the more general weight function

$$(2)\quad \begin{aligned} w(x) &= \gamma\pi^{-1/2}\exp(-\gamma^2\log^2 x),\quad x>0\\ (\gamma^2 &= -(2\log q)^{-1},\quad 0<q<1). \end{aligned}$$

Following Szegö [3], we call these polynomials the Stieltjes–Wigert polynomials and denote the orthonormal polynomial of degree n by $S_n(x)$.

Now the weight function (2) satisfies the identity

$$(3)\quad xw(x) = q^{-1/2}w(qx),\quad x>0.$$

This shows that $\{S_n(qx)\}$ is an orthogonal polynomial sequence (OPS) with respect to the weight function $xw(x)$ on $(0,\infty)$.

This raises the question of whether there are other OPS with this property but it is quickly seen that the answer is negative. More precisely, suppose $\{P_n(x)\}$ is an OPS with respect to a distribution, $d\psi(x)$, on $[0,\infty)$. Suppose also that $\{P_n(ax)\}$ is an OPS with respect to $x\,d\psi(x)$. Writing

$$(4)\quad \mu_n = \int_0^\infty x^n\,d\psi(x)\quad (n=0,1,2,\ldots)$$

it follows that $\{a^{-n}\mu_n\}_0^\infty$ is a moment sequence corresponding to $\{P_n(ax)\}$. But since $\{P_n(ax)\}$ is also an OPS with respect to $x\,d\psi(x)$, then $\{P_n(ax)\}$ is orthogonal with respect to $\{\mu_{n+1}\}_0^\infty$. Thus there exists $c>0$ such that

$$\mu_{n+1} = ca^{-n}\mu_n\quad (n\ge 0)$$

hence

$$\mu_n = c^n a^{-n(n-1)/2}\mu_0 = (ca^{3/2})^n a^{-(n+1)^2/2}(a^{1/2}\mu_0).$$

Now the condition

$$\Delta_2 \equiv \mu_2\mu_0-\mu_1^2 > 0$$

is necessary for (4) to be a Stieltjes moment sequence and this requires $0<a<1$. Hence imposing the latter condition and writing $a=q$, we see upon referring to (2), which has the moments, $q^{-(n+1)^2/2}(n\geq 0)$, that apart from a factor independent of x,

$$P_n(cq^{3/2}x) = S_n(x).$$

2. Although the generalization first suggested by (3) does not lead to any new OPS, it does lead to a rather interesting class of new distribution functions having the same moments as (2).

Consider the generalization of (3) to general distribution functions:

$$\text{(5)} \qquad \begin{aligned} \int_0^x t\, d\psi(t) &= c\int_0^x d\psi(qt), \quad x > 0 \\ \psi(t) &= 0 \quad \text{for } x \leq 0. \end{aligned}$$

Now if (5) has finite moments μ_n, then the preceding discussion still applies to yield

$$\text{(6)} \qquad \mu_n = q^{-(n+1)^2/2}$$

provided

$$\psi(\infty) \equiv \mu_0 = q^{-1/2}, \qquad c = q^{-3/2}.$$

Moreover, if ψ satisfies (5), then ψ is continuous at the origin and has jump J at τ if and only if it also has a jump $c^{-1}\tau J$ at $q\tau$. That is, if $\mathscr{D}$ denotes the set of discontinuities of a nondecreasing solution of (5), then

$$\tau \in \mathscr{D} \Leftrightarrow \tau > 0 \quad \text{and} \quad q^k\tau \in \mathscr{D} \quad (k = 0, \pm 1, \pm 2, \ldots).$$

Since $\mathscr{D}$ is at most denumerable, it follows that there is an at most denumerable set T such that

$$\text{(7)} \qquad \mathscr{D} = \{q^k\tau \mid \tau \in T, \quad k = 0, \pm 1, \pm 2, \ldots\}.$$

Assuming $\mathscr{D}\neq\phi$, write

$$T = \{\tau_1, \tau_2, \ldots\} = \{\tau_\nu\}_{\nu=1}^N, \quad 1 \leq N \leq \infty.$$

If j_ν denotes the jump of ψ at τ_ν, then the jump at $q^k\tau_\nu$ is given by

$$\text{(8)} \qquad J_{\nu k} = c^{-k}\tau_\nu^k q^{k(k-1)/2} j_\nu, \quad (k = 0, \pm 1, \pm 2, \ldots; \nu = 1, 2, \ldots, N).$$

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Conversely, consider now an arbitrary (finite or infinite) sequence $\{\tau_\nu\}_{\nu=1}^N$ of distinct positive numbers. Let

$$f(x) \equiv f(q;x) = \sum_{k=-\infty}^{\infty} q^{k^2/2}x^k, \quad 0 < q < 1, \quad x \neq 0,$$

and let $\{j_\nu\}_{\nu=1}^N$ be any sequence of positive numbers such that

$$\sum_{\nu=1}^{N} f(\tau_\nu)j_\nu < \infty.$$

It is easily verified that

$$f(qx) = q^{-1/2}x^{-1}f(x),$$

hence

$$(9) \qquad f(q^{n+1}\tau_\nu) = [q^{(n+1)^2/2}\tau_\nu^n]^{-1}f(\tau_\nu).$$

Now define

$$(10) \qquad \phi(x) = \sum_{q^k\tau_\nu \le x} J_{\nu k} \quad (\nu = 1, 2, 3, \ldots, N; \quad k = 0, \pm 1, \pm 2, \ldots)$$

where $\phi(x)=0$ for $x \le 0$ and $J_{\nu k}$ is given by (8) with $c=q^{-3/2}$. Then

$$\int_0^\infty x^n \, d\phi(x) = \sum_{\nu=1}^{\infty} \sum_{k=-\infty}^{\infty} (q^k\tau_\nu)^n c^{-n}\tau_\nu^k q^{k(k-1)/2} j_\nu$$

$$= \sum_{\nu=1}^{N} f(\tau_\nu q^{n+1})\tau_\nu^n j_\nu.$$

According to (9)

$$\int_0^\infty x^n \, d\phi(x) = q^{-(n+1)^2/2} \sum_{\nu=1}^{N} f(\tau_\nu)j_\nu.$$

Thus the jump function ψ defined by

$$(11) \qquad \psi(x) = K^{-1}\phi(x), \qquad K = \sum_{\nu=1}^{N} f(\tau_\nu)j_\nu,$$

has the moments (6).

We thus obtain a continuum of solutions to this Stieltjes moment problem, each of which is a jump function whose points of discontinuity form a set of the type (7) (and whose spectrum is thus the closure of $\mathscr{D}$). In particular, we can construct explicit solutions of the moment problem whose points of discontinuity form an everywhere dense subset of $[0, \infty)$. We remark that none of the above solutions are extremal solutions so by a theorem of M. Riesz [1], $\{S_n(x)\}$ is not complete in the corresponding L^2 spaces.

3. As a consequence of the above, we note a minor fact concerning the zeros of the Stieltjes–Wigert polynomials (about which very little seems to be known).

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It is well known that on any open interval in which a distribution function is constant, the corresponding orthogonal polynomials each have at most one zero (see Szegö [3, Theorem 3.41.2]). If we take $T=\{\tau\}$ ($\tau>0$) and construct the corresponding distribution function, it follows that $S_n(x)$ has at most one zero in every open interval of the form

$$(q^{k+1}\tau, q^k\tau) \quad (k = 0, \pm 1, \pm 2, \ldots)$$

for every $\tau>0$. (By a slightly more refined analysis, it is possible to extend this conclusion to the corresponding half-open intervals.)

References

1. M. Riesz, *Sur les problèmes des moments et le théorème de Parseval correspondant*, Acta Sci. Math. (Szeged) **1** (1922–23), 209–225.

2. T. J. Stieltjes, *Recherches sur les fractions continues*, Oeuvres, Tome II, Noordhoff, Groningen, 1918, 402–566.

3. G. Szegö, *Orthogonal Polynomials*, Colloq. Publ., Vol. **23**, Amer. Math. Soc., New York, 1939.

4. S. Wigert, *Sur les polynomes orthogonaux et l'approximation des fonctions continues*. Ark. Mat. Astr. Fys. (18) **17** (1923), 15 pp.

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