J. Appl. Prob. **52,** 238–246 (2015)
Printed in England

ERGODIC INEQUALITY OF A TWO-PARAMETER INFINITELY-MANY-ALLELES DIFFUSION MODEL

YOUZHOU ZHOU,* *McMaster University*

Abstract

In this paper three models are considered. They are the infinitely-many-neutral-alleles model of Ethier and Kurtz (1981), the two-parameter infinitely-many-alleles diffusion model of Petrov (2009), and the infinitely-many-alleles model with symmetric dominance Ethier and Kurtz (1998). New representations of the transition densities are obtained for the first two models and the ergodic inequalities are provided for all three models.

Keywords: Two-parameter Poisson–Dirichlet distribution; transition density; ergodic inequality

2010 Mathematics Subject Classification: Primary 60J60
Secondary 37A30

1. Introduction

The Fleming–Viot process is a very important model in population genetics. It can include various evolutionary forces in a single model, such as mutations and selections. Let E be the type space, and $\mathcal{P}(E)$ be the set of probability measures on E. The Fleming–Viot process Z_t is a $\mathcal{P}(E)$-valued diffusion process with generator,

$$AF_f(\mu) = \sum_{1\le i<j\le m} (\langle \Phi_{ij}^{(m)} f, \mu^{m-1}\rangle - \langle f, \mu^m\rangle) + \langle B^{(m)} f, \mu^m\rangle \\ + 2\bar{\sigma}\sum_{i=1}^{m}(\langle K_i^{(m)} f, \mu^{m+2}\rangle - \langle f, \mu^m\rangle) + \bar{\sigma} m\langle f, \mu^m\rangle,$$

where $F_f(\mu) = \langle f, \mu^m\rangle$, $\mu \in \mathcal{P}(E)$, and $f \in \mathcal{B}(E^m)$. Let $\Phi_{ij}^{(m)}$ be the sampling operator, which replaces the jth variable of f by the ith variable. Let B be the mutation operator, generating a Feller semigroup $\{T_t, t \ge 0\}$ defined by a family of transition probabilities $P(t, x, \mathrm{d}y)(t > 0, x \in E)$, and $B^{(m)}$ is the generator of the semigroup

$$T_m(t)f = \int_E \cdots \int_E f(y_1, \dots, y_m) P(t, x_1, \mathrm{d}y_1) \cdots P(t, x_m, \mathrm{d}y_m).$$

Let $K_i^{(m)}$ be the selection operator and

$$K_i^{(m)} f = \frac{\bar{\sigma} + \sigma(x_i, x_{m+1}) - \sigma(x_{m+1}, x_{m+2})}{2\bar{\sigma}} f(x_1, \dots, x_m).$$

Received 5 July 2013; revision received 9 March 2014.
* Postal address: School of Statistics and Mathematics, Zhongnan University of Economics and Law, 182 South Lake Avenue, East Lake New Technology Development Zone, Wuhan, Hubei, China 430073. Email address: youzhouzhou1984@gmail.com

Let $\sigma(x, y)$ be a symmetric function called the relative fitness of genotype $\{x, y\}$. Define $\bar{\sigma}$ to be $\sup_{x,y,z} |\sigma(x, y) - \sigma(y, z)|$. For a more comprehensive introduction to the Fleming–Viot process; see [5].

If the mutation operator B of Fleming–Viot process Z_t is of the form

$$Bf(x) = \frac{\theta}{2} \int_E (f(y) - f(x))\nu_0(\mathrm{d}x), \qquad \theta > 0, \nu_0 \in \mathcal{P}(E),$$

then, for all $t > 0$, Z_t is almost surely of purely atomic measure. Denote the totality of purely atomic measures by $\mathcal{P}_\mathrm{a}$. For $\mu \in \mathcal{P}_\mathrm{a}$, if we consider the decreasing arrangement of the atomic mass of μ, then we will end up with $(x_1, x_2, \dots)$, which consists of a set

$$\bar{\nabla}_\infty = \left\{(x_1, x_2, \dots) \,\middle|\, x_1 \geq x_2 \geq \cdots \geq 0, \sum_{i=1}^{\infty} x_i \leq 1\right\}.$$

We can define an atomic mapping $\rho \colon \mathcal{P}(E) \to \bar{\nabla}_\infty$ by mapping μ to its decreasingly ordered atomic vector $(x_1, x_2, \dots)$. Therefore, $\rho(Z_t)$ is a $\bar{\nabla}_\infty$-valued process. The Fleming–Viot process is usually called a labeled model and its atomic process $\rho(Z_t)$ is called an unlabeled model.

If there are only random sampling and mutations involved, then $\rho(Z_t)$ is the infinitely-many-neutral-alleles model [4], denoted by X_t. The generator of X_t is

$$G = \frac{1}{2} \sum_{i,j=1}^{\infty} x_i(\delta_{i,j} - x_j)\frac{\partial^2}{\partial x_i \partial x_j} - \frac{\theta}{2} \sum_{i=1}^{\infty} x_i \frac{\partial}{\partial x_i}, \qquad x \in \bar{\nabla}_\infty.$$

If we include selection as well, then the unlabeled model is usually non-Markovian. But if we consider selection of symmetric dominance introduced in [13], then the unlabeled model is a Markov process. We denote this unlabeled model by X_t^σ and call it infinitely-many-alleles diffusion with symmetric dominance; see [6]. The generator of X_t^σ is

$$G_\sigma = G + \sigma \sum_{i=1}^{\infty} x_i(x_i - \varphi_2(x))\frac{\partial}{\partial x_i}, \qquad x \in \bar{\nabla}_\infty,$$

where $\varphi_2(x) = \sum_{i=1}^{\infty} x_i^2$, and is called homozygosity in population genetics.

Both X_t and X_t^σ are reversible diffusions and have unique stationary distributions. The stationary distribution of X_t is the Poisson–Dirichlet distribution PD(θ), and the stationary distribution of X_t^σ is

$$\pi_\sigma(\mathrm{d}x) = C_\sigma \mathrm{Exp}\{\sigma\varphi_2(x)\}\mathrm{PD}(\theta)(\mathrm{d}x),$$

where C_σ is a normalized constant.

Moreover, there is a two-parameter generalization of the PD(θ). We call it a two-parameter Poisson–Dirichlet distribution (see [7]) PD(θ, α), $\theta + \alpha > 0, 0 < \alpha < 1$. Correspondingly, there is a two-parameter generalization [8], [11] of X_t, denoted by $X_t^{\theta,\alpha}$ and called a two-parameter infinitely-many-alleles diffusion model. The two-parameter Poisson–Dirichlet distribution PD(θ, α) is the associated stationary distribution. The generator of $X_t^{\theta,\alpha}$ is

$$G^{\theta,\alpha} = \frac{1}{2} \sum_{i,j=1}^{\infty} x_i(\delta_{i,j} - x_j)\frac{\partial^2}{\partial x_i \partial x_j} - \frac{1}{2} \sum_{i=1}^{\infty} (\theta x_i + \alpha)\frac{\partial}{\partial x_i}, \qquad x \in \bar{\nabla}_\infty.$$

However, $X_t^{\theta,\alpha}$ has no biological interpretation at all. Whether its labeled model exists is still open.

In [3], the transition probability of the neutral Fleming–Viot process Z_t is obtained. In [2], the transition density function of the unlabeled neutral process X_t is also obtained. Therefore, its explicit transition probability is available as well. We can actually obtain the transition probabilities of X_t through the transition probabilities of Z_t; see [7]. In [9], the transition density function of $X_t^{\theta,\alpha}$ is obtained as well. In this paper we reorganize the transition density functions of $X_t^{\theta,\alpha}$ and obtain a new representation of the transition density functions of $X_t^{\theta,\alpha}$. Interestingly, the associated transition probabilities resemble the structure of the transition probabilities for the neutral Fleming–Viot process. This can actually shed some light on the construction of the labeled model of $X_t^{\theta,\alpha}$.

Furthermore, the ergodic inequalities of Z_t and X_t are both available, but similar ergodic inequalities of $X_t^{\theta,\alpha}$ and X_t^{σ} are still missing. In this paper we will obtain the ergodic inequalities of $X_t^{\theta,\alpha}$ and X_t^{σ}. It turns out that $X_t^{\theta,\alpha}$ and X_t share the same ergodic inequality. Lastly, the ergodic inequality of X_t^{σ} is stronger than the ergodic theorem stated in [6].

The remainder of this paper is organized as follows. In Section 2 we will consider the transition density functions of $X_t^{\theta,\alpha}$. In Section 3 we will discuss the ergodic inequalities of $X_t^{\theta,\alpha}$ and X_t^{σ}.

2. The transition density functions of $X_t^{\theta,\alpha}$

In [2] and [9], the explicit transition densities of X_t and $X_t^{\theta,\alpha}$ are obtained, respectively, through eigen expansion. By making use of these known transition densities, we obtain a new representation.

Theorem 2.1. *It holds that $X_t^{\theta,\alpha}$ has the following transition density:*

$$p^{\theta,\alpha}(t,x,y) = d_0^{\theta}(t) + d_1^{\theta}(t) + \sum_{n=2}^{\infty} d_n^{\theta}(t) p_n^{\theta,\alpha}(x,y),$$

where

$$d_n^{\theta}(t) = \sum_{m=n}^{\infty} \frac{2m+\theta-1}{m!}(-1)^{m-n}\binom{m}{n}(n+\theta)_{(m-1)} \mathrm{e}^{-\lambda_m t}, \qquad n \geq 1,$$

$$d_0^{\theta}(t) = 1 - \sum_{m=1}^{\infty} \frac{2m+\theta-1}{m!}(-1)^{m-1}\theta_{(m-1)} \mathrm{e}^{-\lambda_m t},$$

$$\lambda_1 = 0, \lambda_m = \frac{m(m-1+\theta)}{2}, \qquad m \geq 2.$$

Moreover,

$$p_n^{\theta,\alpha}(x,y) = \sum_{|\eta|=n} \frac{p_\eta(x) p_\eta(y)}{\int p_\eta d\mathrm{PD}(\theta,\alpha)},$$

$\eta = (\eta_1, \ldots, \eta_l)$ *is a partition of* n *and* $|\eta| = \sum_{i=1}^{l} \eta_i$.
Define $a_i(\eta) = \#\{j \mid \eta_j = i,\ 1 \leq j \leq l\}$. *Then* $p_\eta(x)$ *is the continuous extension of*

$$\frac{n!}{\eta_1! \ldots \eta_l! a_1(\eta)! \ldots a_n(\eta)!} \sum_{i_1,\ldots,i_l \text{ distinct}} x_{i_1}^{\eta_1} \ldots x_{i_l}^{\eta_l}.$$

Proof. Due to [9], the transition density of $X_t^{\theta,\alpha}$ is

$$p^{\theta,\alpha}(t,x,y) = 1 + \sum_{m=2}^{\infty} \mathrm{e}^{-\lambda_m t} Q_m^{\theta,\alpha}(x,y).$$

Moreover, for $\theta > -\alpha$, $0 < \alpha < 1$, there exist constants c and d, such that

$$Q_m^{\theta,\alpha}(x,y) \le (cm^d)^m.$$

Therefore, for $t_0 > 0$, and for all $t \in [t_0, \infty)$, we have

$$\begin{aligned}\sum_{m=2}^{\infty} \mathrm{Exp}\{-\lambda_m t\} Q_m^{\theta,\alpha}(x,y) &\le \sum_{m=2}^{\infty} \mathrm{Exp}\{-\lambda_m t\}(cm^d)^m \\ &= \sum_{m=2}^{\infty} \bigg(\frac{cm^d}{\exp(t(m+\theta-1)/2)}\bigg)^m \\ &\le \sum_{m=2}^{\infty} \bigg(\frac{cm^d}{\exp(t_0(m+\theta-1)/2)}\bigg)^m.\end{aligned}$$

Since $\lim_{m\to+\infty} cm^d/\exp(t_0(m+\theta-1)/2) = 0$, the exists $M > 0$ such that, for all $m > M$,

$$\frac{cm^d}{\exp(t_0(m+\theta-1)/2)} < \frac{1}{2}.$$

Because $\sum_{m\ge 1} 1/2^m$ is convergent; then, by Weierstrass's M-test, $p^{\theta,\alpha}(t,x,y)$ is uniformly convergent on $[t_0, +\infty) \times \bar{\nabla}_\infty \times \bar{\nabla}_\infty$, and, thus, is continuous. Next, by Fubini's theorem, we can rearrange $p^{\theta,\alpha}(t,x,y)$ by switching the order of summation. Then

$$\begin{aligned}p^{\theta,\alpha}(t,x,y) = 1 + \sum_{m=2}^{\infty} \mathrm{e}^{-\lambda_m t}\bigg(&\sum_{n=2}^{m} \frac{2m+\theta-1}{m!}(-1)^{m-n}\binom{m}{n}(n+\theta)_{(m-1)} p_n^{\theta,\alpha}(x,y) \\ &+ \frac{2m+\theta-1}{m!}(-1)^{m-1}(\theta+1)_{(m-1)} m p_1^{\theta,\alpha}(x,y) \\ &+ \frac{2m+\theta-1}{m!}(-1)^m \theta_{(m-1)} p_0^{\theta,\alpha}(x,y)\bigg).\end{aligned}$$

Since $p_1^{\theta,\alpha}(x,y)$, $p_0^{\theta,\alpha}(x,y) = 1$, we have

$$\begin{aligned}p^{\theta,\alpha}(t,x,y) &= 1 + \sum_{m=2}^{\infty} \mathrm{e}^{-\lambda_m t} \sum_{n=2}^{m} \frac{2m+\theta-1}{m!}(-1)^{m-n}\binom{m}{n}(n+\theta)_{(m-1)} p_n^{\theta,\alpha}(x,y) \\ &\quad + \sum_{m=2}^{\infty} \mathrm{e}^{-\lambda_m t}\bigg(\frac{2m+\theta-1}{m!}(-1)^{m-1}(\theta+1)_{(m-1)} m \\ &\quad + \frac{2m+\theta-1}{m!}(-1)^m \theta_{(m-1)}\bigg) \\ &= 1 + \sum_{m=2}^{\infty} \mathrm{e}^{-\lambda_m t} \sum_{n=2}^{m} \frac{2m+\theta-1}{m!}(-1)^{m-n}\binom{m}{n}(n+\theta)_{(m-1)} p_n^{\theta,\alpha}(x,y) \\ &\quad + \sum_{m=2}^{\infty} \mathrm{e}^{-\lambda_m t} \frac{2m+\theta-1}{m!}(-1)^{m-1}\Big[m(\theta+1)_{(m-1)} - \theta_{(m-1)}\Big].\end{aligned}$$

https://doi.org/10.1239/jap/1429282618 Published online by Cambridge University Press

When $m=1$, $m(\theta+1)_{(m-1)}-\theta_{(m-1)}=0$. Then we have

$$\begin{aligned}
p^{\theta,\alpha}(t,x,y) &= 1-\sum_{m=1}^{\infty}\mathrm{e}^{-\lambda_m t}\frac{2m+\theta-1}{m!}(-1)^{m-1}\theta_{(m-1)} \\
&\quad +\sum_{m=1}^{\infty}\mathrm{e}^{-\lambda_m t}\frac{2m+\theta-1}{m!}(-1)^{m-1}m(\theta+1)_{(m-1)} \\
&\quad +\sum_{m=2}^{\infty}\mathrm{e}^{-\lambda_m t}\sum_{n=2}^{m}\frac{2m+\theta-1}{m!}(-1)^{m-n}\binom{m}{n}(n+\theta)_{(m-1)}p_n(x,y) \\
&= d_0^{\theta}(t)+d_1^{\theta}(t) \\
&\quad +\sum_{m=2}^{\infty}\mathrm{e}^{-\lambda_m t}\sum_{n=2}^{m}\frac{2m+\theta-1}{m!}(-1)^{m-n}\binom{m}{n}(n+\theta)_{(m-1)}p_n^{\theta,\alpha}(x,y).
\end{aligned}$$

Let us switch the order of summation. Then we have

$$p^{\theta,\alpha}(t,x,y)=d_0^{\theta}(t)+d_1^{\theta}(t)+\sum_{n=2}^{+\infty}d_n^{\theta}(t)p_n^{\theta,\alpha}(x,y).$$

Define $\nu_n^{\theta,\alpha}(x,\mathrm{d}y)=p_n^{\theta,\alpha}(x,y)\mathrm{PD}(\theta,\alpha)(\mathrm{d}y)$, then the transition probability of $X_t^{\theta,\alpha}$ is

$$P^{\theta,\alpha}(t,x,\mathrm{d}y)=(d_0^{\theta}(t)+d_1^{\theta}(t))\mathrm{PD}(\theta,\alpha)(\mathrm{d}y)+\sum_{n=2}^{\infty}d_n^{\theta}(t)\nu_n^{\theta,\alpha}(x,\mathrm{d}y). \tag{2.1}$$

Remark 2.1. The transition probability of X_t also has the same structure as (2.1). Moreover, since X_t has an entrance boundary $\bar{\nabla}_\infty-\nabla_\infty$, i.e. X_t immediately moves into ∇_∞ and never exits regardless of its starting point. S.N. Ethier informed the author that a similar result can also be obtained for $X_t^{\theta,\alpha}$.

For both X_t and $X_t^{\theta,\alpha}$, the coefficients $d_n^{\theta}(t)$, $n\geq 0$, are the same. When $\theta\geq 0$, they are the distributions of the ancestral process discussed by Tavaré [12]. However, for $X_t^{\theta,\alpha}$, θ could be negative and $d_n^{\theta}(t)$ is not a probability distribution anymore. However, if we collapse the states 0 and 1, and relabel it as 1, then $d_1^{\theta}(t)+d_0^{\theta}(t)$, $d_n^{\theta}(t)$, $n\geq 2$, define a probability distribution. We can generalize this structure to the case where $\theta>-1$. The estimation of the tail probability obtained in [12] is still true when $\theta>-1$.

Proposition 2.1. *For $\theta>-1$, we have*

$$\mathrm{e}^{-\lambda_n t}\leq\sum_{k=n}^{\infty}d_k^{\theta}(t)\leq\frac{(n+\theta)_{(n)}}{n_{[n]}}\mathrm{e}^{-\lambda_n t}.$$

In particular, when $n=2$, we have

$$\sum_{k=2}^{\infty}d_k^{\theta}(t)\leq\frac{(2+\theta)(3+\theta)}{2}\mathrm{e}^{-(\theta+1)t}. \tag{2.2}$$

https://doi.org/10.1239/jap/1429282618 Published online by Cambridge University Press

Proof. Consider a pure-death Markov chain B_t in $\{1, 2, \dots, m\}$ with Q matrix,

$$Q = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ \lambda_2 & -\lambda_2 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_3 & -\lambda_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_m & -\lambda_m \end{pmatrix},$$

where $\lambda_k = \frac{1}{2}k(k+\theta-1), k \geq 2$. Following the similar arguments in Theorem 4.3 of [7], we will be able to find all the left eigenvectors and right eigenvectors of Q. Denote the matrix consisting of left eigenvectors by $U = (u_{ij})$ and the matrix consisting of right eigenvectors by $V = (v_{ij})$, where

$$u_{ij} = \begin{cases} \delta_{1j}, & i = 1, \\ 0, & j > i > 1, \\ (-1)^{i-j}\binom{i}{j}\dfrac{(j+\theta)_{(i-1)}}{(i+\theta)_{(i-1)}}, & j \leq i, i > 1, \end{cases}$$

and

$$v_{ij} = \begin{cases} 1, & j = 1, \\ 0, & j > i, \\ \binom{i}{j}\dfrac{(j+\theta)_{(j)}}{(i+\theta)_{(j)}}, & 1 < j \leq i. \end{cases}$$

Note that the row vectors of U are the left eigenvectors of Q and the column vectors of V are the right eigenvectors of Q. Similarly, we can also show that $UV = I$ and Q is diagonlized as $V\Lambda U$, where $\Lambda = \operatorname{diag}\{0, -\lambda_2, \dots, -\lambda_m\}$. Therefore, the transition matrix P_t is

$$P_t = \mathrm{e}^{tQ} = V\mathrm{e}^{\Lambda t}U.$$

By direct computation, we know that, for $2 \leq n \leq m$,

$$P_{mn}(t) = \sum_{k=n}^{m}(-1)^{k-n}\binom{m}{k}\binom{k}{n}\frac{(\theta+k)_{(k)}}{(\theta+m)_{(k)}}\frac{(\theta+n)_{(k-1)}}{(\theta+k)_{(k-1)}}\mathrm{e}^{-\lambda_k t}.$$

Letting $m \to +\infty$, we have $d_n^\theta(t) = \lim_{m\to\infty} P_{mn}(t)$.

The remaining arguments are essentially due to Tavaré.

By the martingale argument in Chapter 6 of [10], we know that

$$Z_n(t) = \frac{\mathrm{e}^{\lambda_n t}(B_t)_{[n]}}{(B_t+\theta)_{(n)}},$$

because $\mathrm{e}^{-\lambda_n t}$ is one eigenvalue of P_t and

$$\left(0, 0, \dots, 0, \frac{n_{[n]}}{(n+\theta)_{(n)}}, \dots, \frac{k_{[n]}}{(k+\theta)_{(n)}}, \dots, \frac{m_{[n]}}{(m+\theta)_{(n)}}\right)^\top$$

is the corresponding eigenvector. So,

$$EZ_n(t) = Z_n(0) = \frac{m_{[n]}}{(m+\theta)_{(n)}}.$$

Since, for $n \le k \le m$,

$$\frac{n_{[n]}}{(n+\theta)_{(n)}} \le \frac{k_{[n]}}{(k+\theta)_{(n)}} \le \frac{m_{[n]}}{(m+\theta)_{(n)}},$$

and

$$\frac{\mathrm{e}^{-\lambda_n t} m_{[n]}}{(m+\theta)_{(n)}} = \mathrm{e}^{-\lambda_n t} E Z_n(t) = \sum_{k=n}^{m} \frac{k_{[n]}}{(k+\theta)_{(n)}} P_{mk}(t),$$

we have

$$\frac{n_{[n]}}{(n+\theta)_{(n)}} P(B_t \ge n \mid B_0 = m) \le \frac{\mathrm{e}^{-\lambda_n t} m_{[n]}}{(m+\theta)_{(n)}} \le \frac{m_{[n]}}{(m+\theta)_{(n)}} P(B_t \ge n \mid B_0 = m).$$

Thus, we have

$$\mathrm{e}^{-\lambda_n t} \le P(B_t \ge n \mid B_0 = m) \le \frac{(n+\theta)_{(n)}}{n_{[n]}} \mathrm{e}^{-\lambda_n t}.$$

Letting $m \to \infty$, we have

$$\mathrm{e}^{-\lambda_n t} \le \sum_{k=n}^{\infty} d_k^{\theta}(t) \le \frac{(n+\theta)_{(n)}}{n_{[n]}} \mathrm{e}^{-\lambda_n t}.$$

3. Ergodic inequalities

By making use of the transition probability (2.1) and the estimation of the tail probability (2.2), we can easily obtain the following ergodic inequality of $X_t^{\theta,\alpha}$.

Theorem 3.1. *For $X_t^{\theta,\alpha}$, we have the ergodic inequality*

$$\sup_{x \in \bar{\nabla}_\infty} ||P^{\theta,\alpha}(t,x,\cdot) - \mathrm{PD}(\theta,\alpha)(\cdot)||_{\mathrm{var}} \le \frac{(2+\theta)(3+\theta)}{2} \mathrm{Exp}\{-(\theta+1)t\}, \qquad t \ge 0.$$

Proof. Denote $\mathcal{B}$ to be the totality of Borel subsets of $\bar{\nabla}_\infty$. Then

$$\begin{aligned}
||P^{\theta,\alpha}(t,x,\cdot) - \mathrm{PD}(\theta,\alpha)(\cdot)||_{\mathrm{var}} &= \sup_{A \in \mathcal{B}} |P^{\theta,\alpha}(t,x,A) - \mathrm{PD}(\theta,\alpha)(A)| \\
&= \sup_{A \in \mathcal{B}} |(d_0^{\theta}(t) + d_1^{\theta}(t))\mathrm{PD}(\theta,\alpha)(A) \\
&\quad + \sum_{n=2}^{\infty} d_n^{\theta}(t) \nu_n^{\theta,\alpha}(A) - \mathrm{PD}(\theta,\alpha)(A)| \\
&= \sup_{A \in \mathcal{B}} \left| \sum_{n=2}^{\infty} d_n^{\theta}(t)(\nu_n^{\theta,\alpha}(A) - \mathrm{PD}(\theta,\alpha)(A)) \right| \\
&\le \sum_{n=2}^{\infty} d_n^{\theta}(t) \sup_{A \in \mathcal{B}} |\nu_n^{\theta,\alpha}(A) - \mathrm{PD}(\theta,\alpha)(A)| \\
&\le \sum_{n=2}^{\infty} d_n^{\theta}(t) \le \frac{(\theta+2)(\theta+3)}{2} \mathrm{e}^{-(\theta+1)t}.
\end{aligned}$$

Remark 3.1. As can be seen, the ergodic inequality of $X_t^{\theta,\alpha}$ is the same as the ergodic inequality of X_t obtained in [2]. But for $X_t^{\theta,\alpha}$, θ could be negative.

Since X_t^σ is absolutely continuous with respect to X_t, $\bar{\nabla}_\infty - \nabla_\infty$ should also serve as an entrance boundary of X_t^σ. Hence, we can change the value of the density function $p_\sigma(t, x, y)$ when x or y is in $\bar{\nabla}_\infty - \nabla_\infty$. Therefore, $p_\sigma(t, x, y)$ can be chosen to be the continuous extension of $p_\sigma(t, x, y)\big|_{\nabla_\infty \times \nabla_\infty}$. Moreover, $p_\sigma(t, x, y)$ is symmetric because X_t^σ is reversible. As stated in [9], the Poincaré inequality of X_t^σ also holds. Therefore, it guarantees the L_2-exponential convergence of X_t^σ. By running the argument in Theorem 8.8 of [1], we can also obtain the following ergodic inequality.

Theorem 3.2. *For X_t^σ, there exists $K(\theta, \sigma)$, such that*

$$\sup_{x \in \bar{\nabla}_\infty} \|P^\sigma(t, x, \cdot) - \pi_\sigma(\cdot)\|_{\text{var}} \le K(\theta, \sigma)\text{Exp}\{-\text{gap}(G_\sigma)t\}, \qquad t \ge 0.$$

Proof. We will follow the argument in Theorem 8.8 of [1]. Define $\mu^x(\cdot) = P_x(X_s^\sigma \in \cdot)$. Since

$$P^\sigma(t, x, \cdot) = \int_{\bar{\nabla}_\infty} P^\sigma(t-s, z, \cdot) P^\sigma(s, x, \mathrm{d}z),$$

we have

$$P^\sigma(t, x, \cdot) = \mu^x P_{t-s}^\sigma(\cdot).$$

Therefore,

$$\|P^\sigma(t, x, \cdot) - \pi(\cdot)\|_{\text{var}} = \|\mu^x P_{t-s}^\sigma(\cdot) - \pi(\cdot)\|_{\text{var}}.$$

By part (1) in Theorem 8.8 of [1], we have, for all $t \ge s$,

$$\begin{aligned}\|P^\sigma(t, x, \cdot) - \pi(\cdot)\|_{\text{var}} &\le \left\| \frac{\mathrm{d}\mu^x}{\mathrm{d}\pi_\sigma} - 1 \right\|_2 \mathrm{e}^{-(t-s)\text{gap}(G_\sigma)} \\ &= \sqrt{\int p_\sigma(s, x, y)^2 \pi_\sigma(\mathrm{d}y) - 1}\,\mathrm{e}^{-(t-s)\text{gap}(G_\sigma)}.\end{aligned}$$

Therefore, for $t \ge s$, we have

$$\|P^\sigma(t, x, \cdot) - \pi(\cdot)\|_{\text{var}} \le \sqrt{\int p_\sigma(s, x, y)^2 \pi_\sigma(\mathrm{d}y) - 1}\,\mathrm{e}^{s\cdot\text{gap}(G_\sigma)}\text{Exp}\{-\text{gap}(G_\sigma)t\}.$$

Due to (4.17) of [6] and Theorem 3.3 of [9], we can conclude that there exists a constant $D(\sigma, t) > 0$, such that

$$p_\sigma(t, x, y) \le D(\sigma, t).$$

If we choose $s = \frac{1}{2}$, the constant

$$K'(\theta, \sigma) = \sqrt{D^2(\sigma, t) + 1}\,\mathrm{e}^{\text{gap}(G_\sigma)/2} \ge \sqrt{\int p_\sigma(s, x, y)^2 \pi_\sigma(\mathrm{d}y) - 1}\,\mathrm{e}^{\text{gap}(G_\sigma)/2}.$$

Then we have

$$\sup_{x \in \bar{\nabla}_\infty} \left\| P^\sigma(t, x, \cdot) - \pi_\sigma(\cdot) \right\|_{\text{var}} \le K'(\theta, \sigma)\text{Exp}\{-\text{gap}(G_\sigma)t\}, \quad \text{for all } t \ge \tfrac{1}{2}.$$

Moreover,

$$\sup_{x\in\bar{\nabla}_\infty}\left\|P^\sigma(t,x,\cdot)-\pi_\sigma(\cdot)\right\|_{\text{var}}\le 1,\qquad \text{for all } t\ge 0.$$

Thus, for all $t\in[0,\frac{1}{2}]$, if we choose $K^{''}(\theta,\sigma)$ such that

$$K^{''}(\theta,\sigma)\mathrm{e}^{-\mathrm{gap}(G_\sigma)/2}\ge 1,$$

then, for all $t\in[0,\frac{1}{2}]$,

$$\sup_{x\in\bar{\nabla}_\infty}\left\|P^\sigma(t,x,\cdot)-\pi_\sigma(\cdot)\right\|_{\text{var}}\le 1\le K^{''}(\theta,\sigma)\mathrm{e}^{-\mathrm{gap}(G_\sigma)/2}\le K^{''}(\theta,\sigma)\mathrm{Exp}\{-\mathrm{gap}(G_\sigma)t\}.$$

Therefore, choosing $K(\theta,\sigma)=\max\{K^{'}(\theta,\sigma),K^{''}(\theta,\sigma)\}$, we have

$$\sup_{x\in\bar{\nabla}_\infty}\left\|P^\sigma(t,x,\cdot)-\pi_\sigma(\cdot)\right\|_{\text{var}}\le K(\theta,\sigma)\mathrm{Exp}\{-\mathrm{gap}(G_\sigma)t\}.$$

Remark 3.2. Presumably, we could have an ergodic inequality of X_t^σ if we apply Theorem 8.8 of [1]. But due to the special property of X_t^σ, this theorem is actually a refinement of the ergodic inequality deduced from Theorem 8.8 of [1]. Furthermore, this theorem is stronger than the ergodic theorem of [6].

Acknowledgements

This paper is part of my PhD work. I am so grateful for the support of my PhD advisor, Shui Feng. Moreover, I would also like to thank Stewart N. Ethier for his suggestion in the proof of proposition 2.1.

References

[1] Chen, M.-F. (2005). *Eigenvalues, Inequalities, and Ergodic Theory*. Springer, London.
[2] Ethier, S. N. (1992). Eigenstructure of the infinitely-many-neutral-alleles diffusion model. *J. Appl. Prob.* **29,** 487–498.
[3] Ethier, S. N. and Griffiths, R. C. (1993). The transition function of a Fleming–Viot process. *Ann. Prob.* **21,** 1571–1590.
[4] Ethier, S. N. and Kurtz, T. G. (1981). The infinitely-many-neutral-alleles diffusion model. *Adv. Appl. Prob.* **13,** 429–452.
[5] Ethier, S. N. and Kurtz, T. G. (1993). Fleming–Viot processes in population genetics. *SIAM J. Control Optimization* **31,** 345–386.
[6] Ethier, S. N. and Kurtz, T. G. (1998). Coupling and ergodic theorems for Fleming–Viot processes. *Ann. Prob.* **26,** 533–561.
[7] Feng, S. (2010). *The Poisson–Dirichlet Distribution and Related Topics. Models and Asymptotic Behaviors.* Springer, Heidelberg.
[8] Feng, S. and Sun, W. (2010). Some diffusion processes associated with two parameter Poisson–Dirichlet distribution and Dirichlet process. *Prob. Theory Relat. Fields* **148,** 501–525.
[9] Feng, S., Sun, W., Wang, F.-Y. and Xu, F. (2011). Functional inequalities for the two-parameter extension of the infinitely-many-neutral-alleles diffusion. *J. Funct. Anal.* **260,** 399–413.
[10] Karlin, S. and Taylor, H. M. (1981). *A Second Course in Stochastic Processes*. Academic Press, New York.
[11] Petrov, L. A. (2009). A two-parameter family of infinite-dimensional diffusions on the Kingman simplex. *Funct. Anal. Appl.* **43,** 279–296.
[12] Tavaré, S. (1984). Line-of-descent and genealogical processes, and their applications in population genetics models. *Theoret. Pop. Biol.* **26,** 119–164.
[13] Watterson, G. A. (1977). Heterosis or neutrality? *Genetics* **85,** 789–814.