Glasgow Math. J. **60** (2018) 225–230. © Glasgow Mathematical Journal Trust 2017.
doi:10.1017/S0017089517000064.

THE LOWER RANK OF DIRECT PRODUCTS OF HEREDITARILY JUST INFINITE GROUPS

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(Received 16 March 2016; accepted 13 January 2017; first published online 4 September 2017)

Abstract. We determine the lower rank of the direct product of finitely many hereditarily just infinite profinite groups of finite lower rank.

2010 *Mathematics Subject Classification.* Primary 20E18, Secondary 20F06.

1. Introduction. For primes p, the theory of p-adic analytic pro-p groups plays a central part in the study of pro-p groups and has interesting applications in infinite group theory; see [**2, 6**]. According to a well-known algebraic characterisation, a pro-p group G is p-adic analytic if and only if G has an open subgroup of finite rank. Here, the *rank* of G is defined as

$$\mathrm{rk}(G) = \sup\{\mathrm{d}(H) \mid H \leq_{\mathrm{o}} G\},$$

where $\mathrm{d}(H)$ denotes the minimal number of topological generators of the open subgroup $H \leq_{\mathrm{o}} G$. Indeed, Lubotzky and Mann [**7**] even established the following refinement: a pro-p group G is p-adic analytic if and only if the *upper rank* $\overline{L}_{\mathrm{d}}(G) = \limsup\{\mathrm{d}(H) \mid H \leq_{\mathrm{o}} G\}$ is finite, where the limit superior is taken over the net of open subgroups ordered by reverse inclusion; moreover, in this case, $\overline{L}_{\mathrm{d}}(G)$ is equal to the dimension $\dim(G)$ of G as a p-adic manifold.

Lubotzky and Mann also introduced, for a general profinite group G, the *lower rank*

$$\mathrm{lr}(G) = \underline{L}_{\mathrm{d}}(G) = \liminf\{\mathrm{d}(H) \mid H \leq_{\mathrm{o}} G\},$$

where again the limit inferior is taken over the net of open subgroups ordered by reverse inclusion. They proved that the lower rank of a compact p-adic analytic group coincides with the number of generators of its associated $\mathbb{Q}_p$-Lie algebra. By a classical theorem of Kuranishi [**5**], this implies that the lower rank of any compact p-adic analytic group with semi-simple $\mathbb{Q}_p$-Lie algebra is equal to 2.

Lubotzky and Shalev [**8**] continued the study of the lower rank and showed, for instance, that there exist non-analytic pro-p groups of finite lower rank. Refining their techniques, Barnea [**1**] established, for instance, that the lower rank of the $\mathbb{F}_p[[t]]$-analytic group $\mathrm{SL}_2(\mathbb{F}_p[[t]])$ is equal to 2. Computing the lower ranks of profinite groups is usually rather challenging and producing new families of groups of finite lower rank is of considerable interest.

In this paper, we are interested in the lower ranks of finite direct products of hereditarily just infinite profinite groups. We recall that a profinite group G is just

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infinite, if G is infinite and every non-trivial closed normal subgroup $N \trianglelefteq_c G$ is open in G. The group G is *hereditarily just infinite* if every open subgroup $H \leq_o G$ is just infinite.

There are many non-(virtually abelian) hereditarily just infinite profinite groups of finite lower rank, including those that are p-adic analytic; cf. [**4**]. However, Ershov and Jaikin-Zapirain proved that there exist also hereditarily just infinite pro-p groups of infinite lower rank; see [**3**, Corollary 8.10]. More recently, the second author of the present paper analysed the lower rank in a family of non-(virtually pro-p) hereditarily just infinite groups and conjectured that there exist such groups of any given lower rank in $\mathbb{N}_{\geq 2} \cup \{\infty\}$; see [**10**]. We establish the following theorem and corollary.

THEOREM A. *Let $G = \prod_{i=1}^{n} G_i$ be a non-trivial direct product of finitely many hereditarily just infinite profinite groups of finite lower rank. Set*

$$d = \max\{\mathrm{lr}(G_i) \mid 1 \leq i \leq n\} \quad \textit{and} \quad r = \max\{r_p \mid p \textit{ prime}\},$$

where r_p denotes the number of indices i such that G_i is virtually-$\mathbb{Z}_p$. Then the lower rank of G is $\mathrm{lr}(G) = \max\{d, r\}$.

COROLLARY B. *The non-trivial direct product $G = \prod_{i=1}^{n} G_i$ of finitely many pairwise non-commensurable hereditarily just infinite profinite groups of finite lower rank has lower rank*

$$\mathrm{lr}(G) = \max\{\mathrm{lr}(G_i) \mid 1 \leq i \leq n\}.$$

In the proof, we use basic facts about the structure of just infinite profinite groups, in particular, a result of Reid [**9**]. We emphasise that, in general, the lower rank of a direct product of profinite groups can be as large as the sum of the lower ranks of the factors. For instance, the lower rank of a free abelian pro-p group $\mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$, with n factors, is clearly n.

Theorem A can be regarded as a generalisation of the aforementioned result, due to Kuranishi, Lubotzky and Mann, that every compact p-adic analytic group with semi-simple $\mathbb{Q}_p$-Lie algebra has lower rank 2. Finally, we remark that Theorem A can be applied to the family of hereditarily just infinite profinite groups of finite lower rank described in [**10**]. In this way, we obtain many new examples of profinite groups of finite lower rank.

2. Preliminaries. Clearly, profinite groups of finite lower rank are finitely generated and thus countably based. Restricting to the latter class of profinite groups, we can navigate around the general notion of the limit inferior of a net. The *lower rank* of a countably based profinite group G is

$$\mathrm{lr}(G) = \min\Big\{ \sup\big\{ \inf\{\mathrm{d}(H_i) \mid i \geq N\} \mid N \in \mathbb{N}\big\} \mid (H_i)_{i\in\mathbb{N}} \in \mathcal{C}(G)\Big\} \in \mathbb{N}_0 \cup \{\infty\},$$

where

$$\mathcal{C}(G) = \Big\{(H_i)_{i\in\mathbb{N}} \mid G = H_1 \,{}_o\!\geq H_2 \,{}_o\!\geq \ldots \text{ and } \bigcap\nolimits_{i\in\mathbb{N}} H_i = 1\Big\}$$

is the collection of all descending chains of open subgroups of G that form a neighbourhood base for the identity element.

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In other words, a countably based profinite group G has lower rank at most r if there exists a descending chain of r-generated open subgroups of G that form a neighbourhood base for the identity.

In preparation for the proof of Theorem A, we collect two basic lemmata. Recall that a profinite group G possesses *virtually* a group-theoretic property $\mathfrak{P}$ if G has an open subgroup H that has $\mathfrak{P}$. We abbreviate "virtually-(infinite procyclic pro-p)" to "virtually-$\mathbb{Z}_p$".

LEMMA 2.1. *Let G be a virtually abelian, hereditarily just infinite profinite group. Then G is virtually $\mathbb{Z}_p$ for a suitable prime p.*

Proof. Let A be an open abelian subgroup of G. As A is just infinite, it is a pro-p group for some prime p and infinite pro-cyclic. Thus, A is isomorphic to $\mathbb{Z}_p$. □

LEMMA 2.2. *Let K and L be open subgroups of a just infinite profinite group H that is not virtually abelian. Then there exists x in K such that L is not centralised by x.*

Proof. Assume that $L \subseteq \mathrm{C}_H(x)$ for all $x \in K$. Then $L \cap K$ is abelian and open in H, a contradiction. □

3. Proof of Theorem A. Before proving Theorem A, we establish another auxiliary result.

LEMMA 3.1. *Let $n \in \mathbb{N}$ and let $G = \prod_{i=1}^{n} G_i$ be a direct product of finitely many non-trivial hereditarily just infinite profinite groups of finite lower rank, where none of them is virtually abelian, and set $d = \max\{\mathrm{lr}(G_i) \mid 1 \le i \le n\}$. Then for every basic open neighbourhood $\prod_{i=1}^{n} U_i$ of the identity element in G, with $U_i \subseteq_{\mathrm{o}} G_i$, there exist open subgroups $H_i \le_{\mathrm{o}} G_i$, for $1 \le i \le n$, such that*

(1) *$H_i \subseteq U_i$ and $\mathrm{d}(H_i) \le d$ for $1 \le i \le n$,*
(2) *$H_i \not\cong H_j$ for $1 \le i < j \le n$.*

Proof. For each $i \in \{1, \dots, n\}$, the group G_i admits a descending chain of open d-generated subgroups $H_{i,1} \gneqq H_{i,2} \gneqq \dots$ satisfying $H_{i,k} \subseteq U_i$ for $k \in \mathbb{N}$. By [**9**, Theorem E], a non-(virtually abelian) just infinite profinite group does not contain any proper open subgroups isomorphic to the whole group. Hence, for $1 \le i \le n$, the groups $H_{i,k}, k \in \mathbb{N}$, are pairwise non-isomorphic. Consequently, there are $k_1, \dots, k_n \in \mathbb{N}$ such that $H_1 = H_{1,k_1}, \dots, H_n = H_{n,k_n}$ are pairwise non-isomorphic. □

Now, let $G = \prod_{i=1}^{n} G_i$ be a direct product of finitely many hereditarily just infinite profinite groups of finite lower rank. We set

$$\ell = \max\{d, r\},$$

where $d = \max\{\mathrm{lr}(G_i) \mid 1 \le i \le n\}$ and $r = \max\{r_p \mid p \text{ prime}\}$ are defined as in the statement of Theorem A; here, r_p denotes the number of $i \in \{1, \dots, n\}$ such that G_i is virtually-$\mathbb{Z}_p$.

Clearly, $\mathrm{lr}(G) \ge \ell$. Let $U = \prod_{i=1}^{n} U_i$ be a basic open neighbourhood of the identity element in G, with $U_i \subseteq_{\mathrm{o}} G_i$. We need to find an ℓ-generated open subgroup $K \le_{\mathrm{o}} G$ with $K \subseteq U$.

Without loss of generality, the first m factors $G_1, \dots, G_m$ are not virtually abelian, while the remaining $n - m$ factors $G_{m+1}, \dots, G_n$ are virtually abelian. By

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Lemma 2.1, there exists, for each $i \in \{m+1, \dots, n\}$, a prime q_i such that G_i is virtually-$\mathbb{Z}_{q_i}$. Reordering the factors and descending to an appropriate open subgroup $C \le_o \prod_{i=m+1}^n G_i$ we can arrange that $C \subseteq \prod_{i=m+1}^n U_i$ and

$$C = C_1 \times \cdots \times C_s, \quad \text{with } C_i = \overline{\langle y_{i1}, \dots, y_{it_i} \rangle} \cong \mathbb{Z}_{p_i}^{t_i} \text{ for } 1 \le i \le s,$$

where $s \in \mathbb{N} \cup \{0\}$ with $s \le n - m$, the positive integers $t_i = r_{p_i}$ satisfy $\sum_{i=1}^s t_i = n - m$ and $p_1, \dots, p_s$ denote distinct primes. It is convenient to set $y_{ij} = 1$ for $1 \le i \le s$ and $t_i + 1 \le j \le \ell$ as well as for $s + 1 \le i \le n$ and $1 \le j \le \ell$. We now work in the open subgroup $\prod_{i=1}^m G_i \times C \le_o G$.

By Lemma 3.1, we can choose subgroups $H_i \le_o G_i$ with generators $h_{i1}, \dots, h_{id}$, for $1 \le i \le m$, such that

$$H_i = \overline{\langle h_{i1}, \dots, h_{id} \rangle} \subseteq U_i \qquad \text{and} \qquad H_i \not\cong H_j \quad \text{for } 1 \le i < j \le m.$$

Again, it is convenient to set $h_{ij} = 1$ for $1 \le i \le m$ and $d + 1 \le j \le \ell$ as well as for $m + 1 \le i \le n$ and $1 \le j \le \ell$. We write $H = \prod_{i=1}^m H_i$ for the internal direct product of $H_1, \dots, H_m$.

To conclude the proof, it suffices to produce an ℓ-generated open subgroup $K \le_o H \times C \le_o G$. We consider

$$K = \overline{\langle g_1, g_2, \dots, g_\ell \rangle} \le_c H_1 \times \cdots \times H_m \times C = H \times C,$$

where $g_i = h_i y_i$ with

$$h_i = h_{1i} h_{2i} \cdots h_{ni} \in H \quad \text{and} \quad y_i = y_{1i} y_{2i} \cdots y_{ni} \in C \quad \text{for } 1 \le i \le \ell.$$

Clearly, K is ℓ-generated. Furthermore, K is a sub-direct product of $H_1, \dots, H_m$ and C, i.e., it is a closed subgroup of $H_1 \times \cdots \times H_m \times C$ that projects onto each of the $m + 1$ direct factors. The proof of Theorem A can therefore be completed by appealing to the next proposition, which is also of independent interest.

PROPOSITION 3.2. *Let $m \in \mathbb{N}_0$, and let*

$$K \le_c H_1 \times \cdots \times H_m \times C$$

be a sub-direct product of m pairwise non-isomorphic finitely generated just infinite profinite groups $H_1, \dots, H_m$ that are not virtually abelian and a finitely generated abelian profinite group C.

Then K is an open subgroup of $\prod_{i=1}^m H_i \times C$.

Proof. We may assume that $m \ge 1$. Put $H = \prod_{i=1}^m H_i$. For each $j \in \{1, \dots, m\}$, let $\pi_j \colon H \times C \to H_j$ denote the projection onto H_j; and let $\pi_C \colon H \times C \to C$ denote the projection onto C. Fix finitely many generators $h_1, \dots, h_d$ for H_1, where $d \ge \max\{\mathrm{d}(H_i) | 1 \le i \le m\}$, and $y_1, \dots, y_s$ for C so that

$$H_1 = \overline{\langle h_1, \dots, h_d \rangle} \quad \text{and} \quad C = \overline{\langle y_1, \dots, y_s \rangle}.$$

Since K is a sub-direct product, we find $h_1^*, \dots, h_d^*, y_1^*, \dots, y_s^* \in K$ such that $h_i^* \pi_1 = h_i$ for $1 \le i \le d$ and $y_j^* \pi_C = y_j$ for $1 \le j \le s$.

We observe that it suffices to show that, for each $j \in \{1, \dots, m\}$, there exist an open subgroup $K_j \le_o H_j$ with $K_j \le K$. For then, we get $\widetilde{K} = K_1 \times \cdots \times K_m \le_o H$

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and, setting $N = |H : \widetilde{K}|!$, we obtain $h^N \in \bigcap\{\widetilde{K}^g \mid g \in H\} \subseteq \widetilde{K}$ for every $h \in H$. As C is central, this implies $y_i^N = (y_i^*)^N((y_i^*)^{-1}y_i)^N \in K$ for $1 \leq i \leq s$, and we deduce from $\widetilde{K} \times \overline{\langle y_1^N, \ldots, y_s^N \rangle} \leq_{\mathrm{o}} H \times C$ that $K \leq_{\mathrm{o}} H \times C$.

It remains to construct the aforementioned subgroups $K_j \leq_{\mathrm{o}} H_j$ with $K_j \leq K$ for $1 \leq j \leq m$. By symmetry, it is enough to manufacture K_1. Indeed, we construct recursively, for each $1 \leq i \leq m$, a subgroup $K^{(i)} \leq_{\mathrm{c}} K$ such that

$$K^{(i)} \leq H_1 \times H_{i+1} \times H_{i+2} \times \cdots \times H_m \qquad \text{and} \qquad K^{(i)}\pi_1 \leq_{\mathrm{o}} H_1.$$

Then we take $K^{(m)}$ for K_1 and the proof is complete.

Note that $K^{(1)} = [K, K]$ satisfies the relevant conditions, because $H_1 = K\pi_1$ is just infinite and non-abelian. Now suppose that for $i \in \{2, \ldots, m\}$ the group $K^{(i-1)}$ is already available and build $K^{(i)}$ as follows. Let $F = F_d$ denote the free profinite group on d generators $a_1, \ldots, a_d$ and define profinite presentations

$$1 \to R_i \to F \xrightarrow{\varphi_i} H_i \to 1, \qquad 1 \leq i \leq m,$$

with $a_j\varphi_1 = h_j$ for $1 \leq j \leq d$.

Recall that $F/R_1 \cong H_1$ and $F/R_i \cong H_i$ are non-isomorphic just infinite groups. This gives $R_i \not\subseteq R_1$, and R_iR_1/R_1 is a non-trivial closed normal subgroup of $F/R_1 \cong H_1$. This implies $R_i\varphi_1 \trianglelefteq_{\mathrm{o}} H_1$, and we obtain $R_i\varphi_1 \cap K^{(i-1)}\pi_1 \leq_{\mathrm{o}} K^{(i-1)}\pi_1$. By Lemma 2.2, there exists $x_{i-1} \in K^{(i-1)}$ such that $R_i\varphi_1 \cap K^{(i-1)}\pi_1 \not\leq \mathrm{C}_{K^{(i-1)}\pi_1}(x_{i-1}\pi_1)$.

Consequently, we find a word $w_i \in R_i$ such that $w_i(h_1, \ldots, h_d) \notin \mathrm{C}_{K^{(i-1)}\pi_1}(x_{i-1}\pi_1)$. Using the properties of $K^{(i-1)}$ and $w_i \in \ker(\pi_i)$, it follows that $z_i = [w_i(h_1^*, \ldots, h_d^*), x_{i-1}] \in K$ satisfies

$$z_i \equiv b \pmod{H_{i+1} \times \cdots \times H_m}, \quad \text{where } 1 \neq b \in H_1. \tag{3.1}$$

Set $K^{(i)} = \overline{\langle z_i \rangle^K} \leq_{\mathrm{c}} K$. Visibly $z_i \in H_1 \times H_{i+1} \times \cdots \times H_m$, hence $K^{(i)} \leq H_1 \times H_{i+1} \times \cdots \times H_m$. Moreover, $K^{(i)}\pi_1 = \overline{\langle b \rangle^{H_1}}$ is a non-trivial closed normal subgroup of the just infinite group H_1, so $K^{(i)}\pi_1 \leq_{\mathrm{o}} H_1$. □

Acknowledgements. A weak version of Theorem A formed part of the second author's PhD thesis, Royal Holloway, University of London, 2015. We thank the referee for encouraging us to state Proposition 3.2 as a separate result.

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