1. Introduction

For investigating the steady irrotational isentropic flow of a perfect gas in two dimensions, the hodograph method is to determine in the first instance the position coordinates $x, y$ and the stream function $\psi$ as functions of velocity components, conveniently taken as $q$ (the speed) and $\theta$ (direction angle). Inversion then gives $\psi, q, \theta$ as functions of $x, y$. The method has the great advantage that its field equations are linear, so that it is practicable to obtain exact solutions, and from any two solutions an infinity of others are obtainable by superposition. For problems of flow past fixed boundaries the linearity of the field equations is usually offset by non-linearity in the boundary conditions, but this objection does not arise in problems of trans-sonic nozzle design, where the rigid boundary is the end-point of the investigation.

Accordingly, this paper aims at showing something of the potentialities of the hodograph method for constructing nozzle flows.

It is well known that, for a trans-sonic nozzle, supersonic velocities which are near (in magnitude and direction) to the sonic velocity at the centre of the throat occur at three distinct points $(x, y)$; so on inversion $x, y$ must be three-valued functions of $q, \theta$. For applying the hodograph method to nozzle-flows, therefore, the essential problem is to find solutions of the hodograph equations that are three-valued in the neighbourhood of the sonic point. In [1] I have given a method for constructing such ‘trans-sonic’ solutions, with full numerical detail for one of them, here called ‘RIQ₂’. From the nozzle flow specified by this solution an infinity of others can be obtained by superposing other solutions (which in themselves may be devoid of physical interest). In this paper I survey the solutions which are readily available as the raw material for the superposition, and give specific examples, calculated for $\gamma$ (the adiabatic index) equal to 1.4. For application to conventional nozzle design the object of such constructions would be to obtain a nozzle flow-field for which the throat region is suitably
shaped and which is carried with adequate precision some distance into the supersonic region; the designer can then prolong the flow field through 'simple wave' regions so as to merge it into a uniform supersonic flow. In § 7, however, I show the possibility of avoiding the simple wave regions, and by hodograph procedures arriving at nozzle shapes for which the flow is everywhere analytic and ultimately uniform.

2. Field Equations

As fundamental dependent variable we take the Legendre potential $Q$, whose field equation is

$$(1)\quad q^2(1 - q^2)\Omega_q + (1 - q^2/q_e^2)(q\Omega_q + \Omega_{q\theta}) = 0;$$

here the unit for $q$ is so chosen that $q = 1$ gives the limiting or cavitation speed, while $q_s = \sqrt{(\gamma - 1)/(\gamma + 1)}$ gives the sonic speed. From any solution of (1) the position coordinates are obtained, in non-dimensional measure, by

$$(2)\quad x = X \cos \theta - Y \sin \theta, \quad y = X \sin \theta + Y \cos \theta,$$

where

$$(3)\quad X = Q q, \quad qY = \Omega_{q\theta},$$

and the stream function is determined from the consistent equations

$$(4)\quad \psi_q = (1 - q^2)^\beta(Y - X_{q\theta}), \quad \psi_{q\theta} = -q(1 - q^2)^\beta(X + Y_{q\theta}),$$

where $\beta = 1/(\gamma - 1)$. In any practical case the inversion to get $q, \theta$ as functions of $x, y$ has to be performed numerically; the first step is to determine loci $\psi = \text{constant}$ by interpolation in a table of $\psi(q, \theta)$, and then successions of points $(x, y)$ on such loci are found via tables of $X(q, \theta)$ and $Y(q, \theta)$. The arithmetic is elementary, and on modern standards not too burdensome. An overall check can be made from the fact that the slope of a streamline as found from its $x, y$ coordinates must equal $\theta$.

3. Standard Solutions

I. The Chaplygin set of potentials $C(\nu)$ of the form

$$(5)\quad \Omega = C(\nu) = q^\nu F_\nu(q^2) \cos \nu \theta \quad (\nu \text{ constant}),$$

where $F_\nu$ is in general a hypergeometric function*, is obtained from (1) by separating the variables. This represents a flow having the line $\theta = 0$ as an axis of symmetry, but it runs into a limit-line at the sonic point on

* When $\nu$ is a negative integer $F_\nu$ is a logarithmically modified hypergeometric function. The case $\nu = -1$ is notable in that $F_{-1}$ is an elementary function (Ringleb [2]).
the axis so it is useful only in superposition. Tables (Ferguson and Light-
hill [3], Huckel [4], Petschacker [5]) are available whereby, for $\gamma = 1.4$
the $\Psi, X, Y$ corresponding to $C(\nu)$ can easily be found for a wide set of values
of $\nu, q$. In [3] are tabulated functions $\psi_\nu(\tau), \psi'_\nu(\tau)$, where $\tau = q^2$, and in
terms of these

$$
\begin{align*}
X_{C(\nu)} &= \frac{\nu}{\nu + 1} \frac{\psi_\nu(\tau) + 2\nu\psi'_\nu(\tau)}{q(1 - q^2)^{\beta}} \cos \nu \theta, \\
Y_{C(\nu)} &= -\frac{\nu^2 \psi_\nu(\tau) + 2\nu\psi'_\nu(\tau)}{q(1 - q^2)^{\beta}} \sin \nu \theta, \\
\psi_{C(\nu)} &= (\nu - 1)\psi_\nu(\tau) \sin \nu \theta.
\end{align*}
$$

[3] also gives the hypergeometric function $Y_\nu(\tau) = q^{-\nu} \psi_\nu(\tau)$ and its deriva-
tive, and these are the functions tabulated in [4]. [5] gives the $q^\nu F_\nu(\tau)$
and $F_\nu(\tau)$ of (5), but not their derivatives.

II. Radial flow. The solution

$$
\Omega = R = \int \frac{dq}{q(1 - q^2)^{\beta}}, \quad X_R = \frac{1}{q(1 - q^2)^{\beta}}, \quad Y_R = 0, \quad \Psi_R = -\theta
$$

represents purely radial flow, and is closely related to the vacuous case $\nu = 0$ of (5). Here also there is a limit line $q = q_s$, and the solution is useful
only in superposition.

III. A trans-sonic set*. To obtain a flow unencumbered by limit lines the
method of [1] is to change the variables from $q, \theta$ to $q, \phi$, where

$$
\theta = \phi - 2\alpha \arctan \frac{\sin \phi}{1 - q \cos \phi}, \quad 2\alpha(1 + \alpha) = \beta, \quad \alpha > 0.
$$

The singular locus $\partial \theta/\partial \phi = 0$ for this transformation is

$$
D = 1 - 2(1 + \alpha)q \cos \phi + (1 + 2\alpha)q^2 = 0,
$$
or in terms of $q, \theta$

$$
\theta = \pm \left[ \frac{1}{q_s} \arctan \sqrt{\frac{q^2 - q_s^2}{1 - q^2}} \right. - \arctan \sqrt{\frac{q^2/q_s^2 - 1}{1 - q^2}} \left. \right] = \pm \omega(q)
$$

The key fact is that this locus is characteristic for (1); hence it is that there
are potentials $\Omega(q, \phi)$ which are single-valued across the locus and which
in terms of $q, \theta$ are triple valued, as required for a nozzle-flow. The result
of transforming (1) by (8) is **

* Of the solutions $\Omega_s$ to be discussed only one represents a smooth trans-sonic nozzle-
flow, but I name the whole set from this one.

** It is to be noted that in the $q\phi$-context the suffix $q$ indicates a $q$-derivative for $\phi$ constant,
which is unequal to the $q$-derivative for $\theta$ constant implied by the suffix in (1), (3).
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\[ D \left( q^2 \Omega_{\theta \theta} + q \Omega_{\theta} + \Omega_{\phi \phi} - \frac{2 \beta q^3 \Omega_\phi}{1 - q^2} \right) + 4 \alpha q^2 \sin \phi \left( \Omega_{\phi \phi} - \frac{\beta q \Omega_\phi}{1 - q^2} \right) + \frac{4 \alpha q (\cos \phi - 2 q + q^2 \cos \phi)}{1 - q^2} \Omega_{\phi \phi} = 0, \]

and in [1] a set of solutions \( \Omega_0, \Omega_1, \cdots \) of this equation has been defined, with full numerical detail as regards \( \Omega_0, \Omega_1 \) and \( \Omega_2 \). They have the general form

\[ \Omega_\nu = H_\nu(q, \phi) + G_\nu(q^2, q e^{i \phi}), \]

where \( H_\nu \) is an explicit elementary function and \( G_\nu \) is a power series in \( q^2, q e^{i \phi} \) which converges for \( 0 \leq q^2 < 1 \) and \( |qe^{i \phi}| < 1 \); the convergence is fairly rapid for \( q \) as large as 0.75. It is hence practicable to use these solutions for \( \text{Im} \phi > 0 \) as well as for \( \phi \) real, and an illustration of this will be given in § 7. Regarding the elementary functions \( H_\nu \) it is sufficient to remark that \( H_1, H_2, \cdots \) are regular on the singular locus \( D = 0 \), and indeed for all \( q, \phi \) that come into question; but \( H_0 \) is singular on this locus:

\[ H_0 = \Omega_0 \text{ (princ. part)} = \frac{(1 - q^2)^{1 - \beta/2}}{(1 + \alpha)D}. \]

The potential \( \Omega_\nu \) is \( i \)-times the \( \theta \)-derivative of \( \Omega_{\nu+1} \): in terms of \( q, \phi \)

\[ D \Omega_\nu = i(1 - 2q \cos \phi + q^2) (\Omega_{\nu+1})_\phi. \]

Since the coefficients in (1) are real, the real and unreal parts of \( \Omega_\nu \) (as functions of \( q, \theta \)) will separately satisfy (1), provided in the case where \( \phi \) is unreal that (8) gives a real \( \theta \). \( \text{Re} \Omega_2 \) gives a trans-sonic nozzle flow, plotted in figure 3.

For plotting the flow-field corresponding to the real or unreal part of \( \Omega_\nu \) we take \( \phi \) as an auxiliary parameter in the calculation of \( X, Y, \psi \). The values \( X_\nu, Y_\nu, \psi_\nu \) corresponding to \( \Omega_\nu \) are found, via (3), (4), (8) and (13), to be

\[ X_\nu = (\Omega_\nu)_\theta - \frac{2i \alpha \Omega_{\nu-1} \sin \phi}{1 - 2q \cos \phi + q^2}, \quad Y_\nu = - \frac{i \Omega_{\nu-1}}{q}, \]

\[ \psi_\nu = - q \left( 1 - q^2 \right)^\beta (iX_{\nu+1} + Y_\nu), \]

and corresponding to \( \text{Re} \Omega_\nu \) we take the real parts of these.

4. Symmetrization

To obtain a nozzle-flow having \( \theta = 0 \) as an axis of symmetry we require a potential \( \Omega \) which is an even function of \( \theta \); for then \( X, x \) are even functions and \( Y, y, \psi \) odd. Now if \( \Omega = f(q, \theta) \) is any solution of (1), \( f(q, \theta_0 + \theta) \) and
$f(q, \theta_0 - \theta)$ also are solutions, and by superposition we obtain $\Omega = f(q, \theta_0 + \theta) + f(q, \theta_0 - \theta)$ as an even solution.

This symmetrization process gives nothing new when applied to the potentials $C(\nu)$ or $R$, but as regards $\Omega_p$, it gives different flows for each value of $\theta_0$. Starting from the potential $\Omega_p(q, \phi)$, we obtain $f(q, \theta_0 + \theta) = f'(q, \theta)$ say, and $f(q, \theta_0 - \theta) = f''(q, \theta)$ by applying in place of (8) the transformations

$$\theta_0 \pm \theta = \phi - 2\pi \arctan \frac{q \sin \phi}{1 - q \cos \phi}. \tag{15}$$

For the upper sign in (15) the formulae (14) remain valid, but for the lower sign we have to reverse in (14) the signs of $Y_p, \psi_p$. Hence if, for a given $\theta$, we call $\phi', \phi''$ the values of $\phi$ that satisfy (15) with the upper and lower signs respectively; if $X_p', Y_p', \psi_p'$ are the values given by (14) for $\phi = \phi'$; and if $X_p'', Y_p'', \psi_p''$ are the values given again by (14) for $\phi = \phi''$: then the symmetrized potential $f' + f''$ leads to the values

$$X = X_p' + X_p'', \quad Y = Y_p' - Y_p'', \tag{16}$$

$$\psi = -q(1 - q^2)^\beta (iX_{p+1} - iX_{p+1}'' + Y_p' - Y_p''),$$

and thence the position coordinates $x, y$ follow from (2).

We denote by $S_p(\theta_0)$ the potential thus derived from $\Omega_p$.

### 5. The Axial $qx$-Relation

When the speed $q$ of a nozzle flow on the axis of symmetry $Ox$ is known or prescribed the whole flow pattern is determinate, and an approximation to it is furnished by the 'hydraulic' approximation that $yq(1 - q^2)^\beta$ is constant on any streamline. A rough comparison of two flow patterns may accordingly be inferred from their axial $qx$-relations. On the hodograph method, inversely, we consider $x$ as a function of $q$, and by (2) $x = X$ when $\theta = 0$, so by (3) the axial relation is $x = \partial \Omega / \partial q$.

This remark is significant in relation to the superposition of standard solutions; the functions $X$ attaching to them are primary numerical data, so the axial $qx$-relation for a given superposition can be tabulated by trivial labour.

In figure 1 are shown the axial $qx$-relations for the potentials $R1, \Omega_2$ and $R$. The former shows a trans-sonic flow originating at a stagnation point ($x$ finite for $q = 0$). For the latter, the minimum at $q = q_s$ indicates a flow with the limit-line mentioned in § 3 II. The figure shows also the axial relations for the potential

$$R1 \Omega_2 - AR \tag{17}$$
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for certain values of $A$. The leading feature here is that for $q \sim 0$ the term $-AR$ is dominant, so on the subsonic side we get a flow coming from infinity, approximately radially. Another feature, which however, for practical purposes is likely to be less important, is that for $q \sim 1$ the $R$-curve is steeper than that for $\text{Rl } \Omega$, so (17) must give $dx/dq = 0$ for some supersonic $q$ depending on $A$, and specifies a nozzle-flow terminating in a limit line in the far supersonic region*.

For the potential $C(v)$ the axial relation is $x = d(q^v F_v)/dq$. For $0 < v < 1$ its graph is qualitatively like that for $R$, the significant distinction being that for $q \sim 0$, $x \sim q^{v-1}$. For a certain negative range $v_0 < v < 0$ also the

* The behaviour for $q \sim 1$ is

for $\text{Rl } \Omega$, $x \sim A_p(1 - q)^{1-\beta} / $2

for $R$, $x \sim (1 - q)^{-\beta}$

for $C(v)$, $x \sim A_v(1 - q)^{-\beta}$

where $A_p$ changes sign at transitional values of $v$. 

---

Fig. 1. Axial $xq$-relations for the potentials: Curve 1: $R$: ($x$-scale on right). Curves 2, 3, 4, 5: $\text{Rl } \Omega - AR$, for $A = 0, 1, 2, 3$, ($x$-scale on left).
graph is like that for $R$, but the other way up; $v_0 = (13 - \sqrt{889})/10 = -1.68$ for $\gamma = 1.4$. As $v$ decreases from $v_0$ the graph develops turning points additional to that at $q_s$, while in transitional cases $x$ remains finite at $q = 1$; for example $F_v$ is a quadratic in $q^2$ when $v = v_0$.

From these facts the general march of the axial $xq$-relations for $C(v)$ superposed on $R \Omega_2$ can be inferred; for example for $v = v_0$ the relation is monotonic over the whole range $0 < q < 1$ and the superposition gives a nozzle unencumbered by limit lines anywhere near its axis. In general, for $q \sim 0$ the flow is dominated by the component $C(v)$ and the streamlines are approximately $x = (\text{const.}) y^{1-v}$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig2.png}
\caption{Axial $xq$-relations derived from $R \Omega_1$ by symmetrization, for $\theta_0 = 0^\circ$, 5$^\circ$, 10$^\circ$, 20$^\circ$, 30$^\circ$, 40$^\circ$, 80$^\circ$, 120$^\circ$, 180$^\circ$. (For compactness the curves for 80$^\circ$, 120$^\circ$, 180$^\circ$ have been displaced upwards 2, 3, 3.5 units respectively).}
\end{figure}

In fig. 2 are shown the axial $xq$-relations derived from the potential $R \Omega_2$ by symmetrization ($\S$ 4) for various values of the parameter $\theta_0$. For $\theta_0 = 0$ the curve branches into two at $q = q_s$; this is because the inversion of (8) gives $\phi$ as a multiple valued function of $\theta$. The smooth branch of this curve belongs to a smooth trans-sonic flow, but all the others have $dz/dq = 0$ for $q = q_s$. For $\theta_0$ around 120$^\circ$ the relation is remarkable in that $x$ is almost...
constant over a long range of $q$.

An arbitrary $xz$-relation can be fitted in the Fourier manner by a series of the $X$'s derived from a suitably chosen sequence of potentials, at least provided the relation is analytic. I shall not here enter into any general theory on this matter, but I give an example of a fitting arrived at by judicious trial. By finite difference methods Emmons [6] approximated to the trans-sonic solution of the non-linear physical flow equations (for $\gamma = 1.4$) for the region bounded by a hyperbola and its minor axis. His paper gives a graph of the axial $xz$-relation which emerged from this solution. An engineering friend converted the graph into a table, by machine readings recorded (rather speculatively) 'correct' to 0.001 inch, and I fitted this table by superposing four of the standard solutions, viz. (18)

$$1.27 x(\text{Emmons}) \sim R_l X_2 - 1.091 X_R - 0.331 X_{S(22^\circ)} - 8.35 X_{C(10)} .$$

The fitting was made at 19 points ranging from $q = 0.176 x = -3.995$ to $q = 0.648, x = 3.108$, (specimens are shown in the table below) and the maximum residual was 0.004, a proportional accuracy of about 1 in 1000.

<table>
<thead>
<tr>
<th>$q$</th>
<th>$R_l X_2 - 1.091 X_R - 0.331 X_{S(22^\circ)} - 8.35 X_{C(10)}$ sum</th>
<th>sum $+ 1.270$ $x_E$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.176</td>
<td>3.737 - 6.703 - 2.110</td>
<td>0.000 - 5.076 - 3.997 - 3.995</td>
</tr>
<tr>
<td>0.400</td>
<td>5.756 - 4.218 - 2.138</td>
<td>-0.015 - 0.615 - 0.484 - 0.483</td>
</tr>
<tr>
<td>0.566</td>
<td>9.594 - 5.058 - 2.512</td>
<td>+0.040 2.064 1.625 1.627</td>
</tr>
<tr>
<td>0.648</td>
<td>13.828 - 6.571 - 3.296</td>
<td>-0.010 3.951 3.111 3.108</td>
</tr>
</tbody>
</table>

The streamline calculated thence from the hodograph formulae, with a view to fitting Emmons's hyperbolic boundary, is however discrepant by about 1 in 30 at its subsonic end. The explanation, in view of the close axial fitting, is perhaps that Emmons's numerical solution was more accurate — as a solution of the flow equations — near the axis than at remote points.

### 6. Flow-Fields Calculated by Superposition

For the potential $R_l \Omega_2 - AR$ the flow-field is shown in figure 3 for $A = 0$, and in figure 4 for $A = 1.012, 1.840$. In figure 3 the whole flow-field, up to the bounding limit-lines, is indicated. In figure 4 the fields are carried far enough transversely to show the breakdown of regular flow by the appearance of limit-lines in the throat region; the outermost regular streamline has infinite curvature at the point of appearance (cusp) of the limit-lines, and this point is upstream from the geometrical throat for $A = 1.012$, but about at the throat for $A = 1.84$.

Figure 5 shows (i) one streamline for the superposition of four standard potentials used to imitate Emmons's solution as in (18), in comparison with
Fig. 3. Flow-field for potential $\text{Re} \Omega_2$. Streamlines: thick. Isovels: thin. Limit lines: chain dot.

Fig. 4. Flow-fields for potentials $\text{Re} \Omega_2 - AR$. $A = 1.012$: lower half. $A = 1.840$: upper half. In each case the isovels are, from the left, $q/q_* = 0.34, 0.69, 0.98, 1.20, 1.39, 1.53, 1.70$ i.e. $M = 0.32, 0.66, 0.98, 1.26, 1.53, 1.83, 2.15$. Enlargements show incipient limit lines (dotted).
Fig. 5. Flows close to that of Emmons (see text). Isovels for \( M = 0.46, 0.75, 1.0, 1.32, 1.68, 1.90 \) are shown for solution (i) above and solution (iii) below.

(ii) Emmons's hyperbolic boundary and (iii) a streamline for a superposition of three standard potentials \( R \Omega_2 - 1.012R + 4.86C(5) \). The comparison between (i) and (iii) is of interest because their axial \( xq \)-relations agree to about 1 in 1000 over the range \( 0.3 \leq q \leq 0.52 \).

7. Nozzles in Which the Supersonic Flow is Ultimately Uniform

A supersonic wind tunnel has to have a test-section in which the velocity is uniform. In practice this section must be not too far downstream from the throat, and the transition to uniform flow from the non-uniform flow in the throat requires that there be at least two surfaces which in the analytical sense are singular for the flow, in that certain velocity-derivatives are there discontinuous. However, there are flows without singularities for which the velocity *tends* to uniformity at infinity downstream, and such a flow might be a good enough approximation to what is practically required.

Such asymptotically uniform nozzle flows may be constructed by the
hodograph method. The ideas underlying what follows are (1) that because of the possibility of symmetrization (§ 4) we can in the first instance disregard the requirement of symmetry; (2) that to require \( q \to q_0 \) as \( x \to \infty \) is in hodograph terms to require \( x \to \infty \) as \( q \to q_0 \), and by (12), (14) we get this infinity on the singular locus \( D = 0 \) for the flow determined by the potential \( \Omega_4 \). By reference to (10) we see then that, for a given limiting \( q_0 \), the \( \theta_0 \) to use in the symmetrization process will be \( \theta_0 = \omega(q_0) \).

Working now with the variables \( q, \phi \), (15) gives on the axis of the proposed nozzle

\[
\theta_0 = \phi - 2\alpha \arctan \frac{q \sin \phi}{1 - q \cos \phi} = f(q, \phi), \text{ say,}
\]

which is to be satisfied by a point \((q_0, \phi_0)\) on the singular locus \( D = 0 \).

For neighbouring points \((q_0 + \delta q, \phi_0 + \delta \phi)\) on the axis the relation between \( \delta q, \delta \phi \) is found from a Taylor expansion. We have

\[
\frac{\partial f}{\partial \phi} = \frac{D}{1 - 2q \cos \phi + q^2}, \quad \frac{\partial f}{\partial q} = -\frac{2\alpha \sin \phi}{1 - 2q \cos \phi + q^2},
\]

and when \( D = 0 \),

\[
\frac{\partial^2 f}{\partial \phi^2} = \frac{2(1 + \alpha)q \sin \phi}{1 - 2q \cos \phi + q^2}.
\]

Hence the Taylor series for points on the axis is

\[
0 = -\frac{2\alpha \sin \phi}{1 - 2q \cos \phi + q^2} \delta q + \frac{(1 + \alpha)q \sin \phi}{1 - 2q \cos \phi + q^2} \delta \phi + \cdots,
\]

and since for a trans-sonic nozzle the relevant values of \( \delta q \) are negative, \( \delta \phi \) must be to a first approximation pure-imaginary. Hence, for real \( \theta_0, q \), it is an unreal solution \( \phi \) of (19) that is relevant; and since the non-elementary parts of the potentials \( \Omega_0, \Omega_1, \Omega_2 \) are power series in \( q^2, qe^{i\phi} \), computation will be easier if we take \( \phi \) to have a positive imaginary part.

We have now to show that the complete ranges of \( q, \theta \) that can occur in a nozzle such as is proposed are furnished by unreal solutions of (15).

If \( \phi = \sigma + i\xi \), the condition that (15) gives \( \theta_0 \pm \theta \) real is

\[
\cos \sigma = \frac{\sinh (\xi/\alpha) + q^2 \sinh (2\xi + \xi/\alpha)}{2q \sinh (\xi + \xi/\alpha)},
\]

and then

\[
\theta_0 \pm \theta = \sigma - \alpha \arctan \frac{2q \cosh \xi \sin \sigma - q^2 \sin 2\sigma}{1 - 2q \cosh \xi \cos \sigma + q^2 \cos 2\sigma}.
\]

For a given \( \xi \), (21) gives a real \( \sigma \) between \( \pm \frac{1}{2} \pi \) if
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and it is easily shown that as \( q \) runs over this range (so that \( \sigma \) makes an excursion from 0 towards \( \pm \frac{1}{2}\pi \) and then back to 0), the inverse tangent in (22) runs from 0 to \( \pm \pi \), so that \( \theta_0 \pm \theta \) runs from 0 to \( \mp \pi \). By continuation, values of \( \theta_0 \pm \theta \) outside \( (\mp \pi, \pi) \) are obtained by further excursions of \( q \) over the range (23), so all values of \( \theta \) are attainable. Finally, in the limit \( \zeta = 0 \), (21) gives \( \cos \phi = \cos \sigma = \frac{1 + (1 + 2\alpha)^2}{2(1 + \alpha)}q \), so that \( (q, \phi) \) is on the singular locus \( D = 0 \); and as \( \zeta \) increases to \( \infty \) the range (23) contracts continuously towards \( q = 0 \). Hence all relevant values of \( q, \theta \) are covered; it may be noted that for \( \theta = \alpha \pi \), (10) gives \( q = 1 \), so any supersonic speed in the ultimate test-section can be secured by suitably choosing \( \theta_0 \) between 0 and \( \alpha \pi \).

The potential \( \Omega_1 \) is unreal, and we shall show that a flow having the desired character when \( (q, \phi) \) is near the singular locus is obtained by symmetrizing its real part. In the desired neighbourhood \( \zeta \) is small, and (2) gives (suppressing till the end the zero suffix from \( q_0, \sigma_0 \))

\[
\cos \sigma = \frac{1 + (1 + 2\alpha)^2}{2(1 + \alpha)}q + O(\zeta^2),
\]

and thence

\[
D = 1 + (1 + 2\alpha)^2q^2 - 2(1 + \alpha)q \cos \sigma \cosh \zeta + 2i(1 + \alpha)q \sin \sigma \sinh \zeta
\]

\[
= 2i(1 + \alpha)q \zeta \sin \sigma + O(\zeta^2).
\]

Hence (11) and (12) give

\[
i\Omega_0 = \frac{(1 - q^2)^{1/2}}{2(1 + \alpha)^2q \zeta \sin \sigma} + O(1).
\]

The principal parts of \( X_1, Y_1, \psi_1 \) now follow from (14), viz.

\[
\begin{align*}
X_1 &= -\frac{\alpha(1 - q^2)^{1-\beta/2}}{(1 + \alpha)^2q(1 - 2q \cos \sigma + q^2)\zeta} + O(1), \\
Y_1 &= -\frac{(1 - q^2)^{1-\beta/2}}{2(1 + \alpha)^2q^3 \sin \sigma \cdot \zeta} + O(1) \\
\psi_1 &= -q(1 - q^2)^\beta Y_1 + O(1),
\end{align*}
\]

and taking real parts leaves these unaffected.

For the symmetrization we must form combinations, as in (16), of the values (24) that belong to the two transformations (22). At the point \( (q, \sigma) \) on the singular locus with which we are concerned the right hand side of (22) (with \( \zeta = 0 \)) reduces to \( \theta_0 \). For a neighbouring point \( (q - Q, \sigma + \delta\phi) \), we obtain as a generalization of (20)
\[ \pm \theta = \frac{2x \sin \sigma \cdot Q}{1 - 2q \cos \sigma + q^2} + \frac{(1 + \alpha) Q \sin \sigma}{1 - 2q \cos \sigma + q^2} (\delta \phi)^2 + \cdots, \]

and for \( \theta \) small this gives

\[ \zeta = -i \delta \phi = \sqrt{\frac{2xQ}{(1 + \alpha)q}} \sqrt{1 - \frac{2q \cos \sigma + q^2 \theta}{2x \sin \sigma \cdot Q}}. \]

Distinguishing as \( \zeta', \zeta'' \) the values of \( \zeta \) belonging to the two signs, the principal part of \( \psi \) becomes

\[ -q(1 - q^2)^{1/2} (Y_1' - Y_1'') = \frac{q(1 - q^2)^{1+\beta/2}}{2(1 + \alpha)^2 q^2 \sin \sigma} \left( \frac{1}{\zeta'} - \frac{1}{\zeta''} \right). \]

Now near the throat of the proposed nozzle all the formulae are regular and it will be sufficient to restrict \( \psi \) to be bounded. Since \( \psi \) is constant along a streamline we may therefore restrict \( \theta \) so that in (27) \( 1/\zeta' - 1/\zeta'' \) is bounded, and by (26) the condition for this is

\[ \theta = O(Q^{\mu}). \]

Hence the values of \( 1/\zeta', 1/\zeta'' \) derived from (26) may be expressed as binomial series in \( \theta/Q \), and for the symmetrized solution we obtain from (24) the principal parts

\[ \begin{align*}
X &= -\frac{\alpha(1 - q_0^2)^{1-\beta/2}}{(1 + \alpha)^2 q_0(1 - 2q_0 \cos \sigma_0 + q_0^2)} \left( \frac{1}{\zeta'} + \frac{1}{\zeta''} \right) = -\frac{A}{\sqrt{Q}}, \\
Y &= -\frac{(1 - q_0^2)^{1-\beta/2}}{2(1 + \alpha)^2 q_0^2 \sin \sigma_0} \left( \frac{1}{\zeta'} - \frac{1}{\zeta''} \right) = -\frac{B \theta}{q_0 Q^\mu},
\end{align*} \]

where

\[ A = \frac{2(1 - q_0^2)^{-\beta/2}}{\sqrt{2x(1 + \alpha)q_0}}, \quad B = \frac{1}{2} A \left( \frac{1 - q_0^2}{2(1 + \alpha) \sin \sigma_0} \right)^2; \]

in arriving at these values of \( A, B \) we have used the relation

\[(1 + \alpha)(1 - 2q_0 \cos \sigma_0 + q_0^2) = \alpha(1 - q_0^2),\]

which says that \((q_0, \sigma_0)\) is on the singular locus \( D = 0 \). By (2) the relations between position and velocity coordinates are accordingly, to leading order,

\[ x = -\frac{A}{\sqrt{(q_0 - q)}}, \quad y = -\frac{B \theta}{q_0(q_0 - q)^{\mu/2}}. \]

For the leading approximation to \( \psi \) we require by (16) the principal part of \( i(X_2' - X_2'') \), which like \( Y_1' - Y_1'' \) is in general \( O(1) \). This is

\[ iX_2(q, \phi + \delta \phi') - iX_2(q, \phi + \delta \phi'') \sim i(\delta \phi' - \delta \phi'') X_2 \left( q, \phi + \frac{\delta \phi' + \delta \phi''}{2} \right), \]
where \( i(\delta \phi' - \delta \phi'') = \zeta'' - \zeta' \) and by (26), \( \frac{1}{2}(\delta \phi' + \delta \phi'') = i\{2\alpha q/(1 + \alpha)q\}^{1/2} \).

Now from (13) and (14) we find

\[
(33) \quad iX_{2\phi} = \frac{DX_1}{1 - 2q \cos \phi + q^2},
\]

and thence (32) is found to reduce to

\[
(34) \quad i(X'_2 - X''_2) = \theta X,
\]

to leading order. Thus by (2), (16) becomes

\[
(34) \quad \psi = -q(1 - q^2)^{\beta}(\theta X + Y) = -q(1 - q^2)^{\beta} y,
\]

which could indeed have been asserted ab initio as the hydraulic approximation. However, by suitably judged work with expansions the approximations such as (32) can be refined; the result is

\[
(35) \quad \psi = -q(1 - q^2)^{\beta} \text{Re} \{iX'_2 - iX''_2 + Y'_2 - Y''_2\}
\]

\[
= -q(1 - q^2)^{\beta}(Y + \theta X - \frac{1}{2} Yq^2/Q + O(\theta^3 Q^{-1/4})),
\]

where \( X, Y \) are given by (29) with errors \( O(1), O(\theta Q^{-1/4}) \) respectively. Also by (2), to the same approximation as (35), \( Y + \theta X = y \), and

\[
q(1 - q^2)^{\beta} = q_0(1 - q_0^2)^{\beta}(1 + C Q),
\]

where \( C = (q_0^2 - q_x^2)/q_0^2 q_x^2(1 - q_0^2) \). Thus finally we find, on substituting for \( Q \) and \( \theta \) from (29)

\[
(36) \quad \psi = -q_0(1 - q_0^2)^{\beta} y \left( 1 + \frac{a^2}{x^2} \left( 1 - \frac{\lambda y^2}{x^2} \right) \right) + O \left( \frac{y^3}{x^6} \right),
\]

with

\[
(37) \quad a^2 = \frac{4(q_0^2/q_x^2 - 1)}{\beta q_0^2(1 - q_0^{2+1+\beta})}, \quad \lambda = \frac{2(q_0^2/q_x^2 - 1)}{q_0(1 - q_0^2)}. \]

The approximation (36) is of course valid only when \( x \) is large, but it suggests a nozzle with a throat, as in figure 6; the flow is in the direction of \( x \) decreasing because of the negative sign in (29).

![Fig. 6. Nozzle with ultimately uniform section.](https://www.cambridge.org/core/terms).
Now in (11) \( H_1 = -\log(1 - q e^{-i\phi}) \) and \( G_0 = (1 + \alpha)^{-1} + O(q e^{i\phi}) \). Hence

\[ \Omega_0 \sim -\frac{1}{\alpha} - \frac{(qe^{-i\phi})^{1/\alpha}}{(1 + \alpha)^2}, \]

and by (14)

\[ X_1 \sim \frac{e^{-i\phi}}{1 - q e^{-i\phi}} + \frac{2i \sin \phi (1 + \alpha(q e^{-i\phi})^{1/\alpha} (1 + \alpha)^2)}{(1 - q e^{-i\phi})(1 - q e^{i\phi})} \]

\[ \sim -\frac{\alpha q^{1/\alpha - 1} e^{-i\phi/\alpha}}{(1 + \alpha)^2(q e^{-i\phi})^{1/\alpha}}, \]

provided \( 1/\alpha < 2 \), which is so if \( \beta > \frac{3}{2}, \gamma < \frac{5}{3} \). Hence (38)

\[ X \sim -2\alpha/(1 + \alpha)^2 q \sim -\infty. \]

Since also by (29) \( X \sim -\infty \) for \( q \sim q_0 \), the axial \( xq \)-relation for the symmetrized flow \( R_1 (\Omega_1(q, \theta_0 + \theta) + \Omega_1(q, \theta_0 - \theta)) \) must have a maximum — which in fact occurs for \( q = q_s \) — and the flow has a limit-line. To arrive at a smooth trans-sonic nozzle flow we therefore superpose the potential \(-A R_1 \Omega_2\), with \( A > 0 \). The maximum of the axial relation is thereby displaced to a subsonic \( q \), say at \((x_1, q_1)\), and the corresponding singularity of the flow ([1], Appendix 1) is that the axial streamline is prolonged by two non-axial branches meeting at a cusp at \( x_1 \). Finally this singularity can be removed, if desired, by superposing also a suitable multiple of the potential \( R \).

**References**


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