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Some definitions of finiteness

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Since Paul J. Cohen's 1963 result, it has been possible to investigate the consequences of the axioms of Zermelo-Fraenkel set theory without the Axiom of Choice. In this paper we examine the relative strengths of a number of finiteness criteria against the background of ZF set theory without AC .

1.

The study of finiteness criteria could be said to have been instigated by Tarski [3] in 1924; since then the properties and relative strengths of finiteness criteria have been investigated within various systems of set theory. To my knowledge, however, little work in this field has been done against the background of strict ZF set theory; Tarski's own paper was written within no particular system, but rather within what may be called "intuitive set theory".

The reason for this apparent lack of interest in ZF , at least up until recent years, can be readily explained. It is no good looking at different finiteness criteria in any reasonable set theory that contains AC , for in such a system all the well-known criteria are logically equivalent. Specifically, as will be shown in this paper, any "reasonable" finiteness criterion lies between two given criteria, namely *FN*-finiteness and *FDO*-finiteness (see §3); and it is well known that even a weakened form of AC is sufficient to enable us to prove these two conditions equivalent. But it was not until 1963 that we knew for certain that AC was independent of the ZF axioms; hence prior to that date any work on finiteness criteria that was done in ZF set theory had somewhat the appearance of an exercise in futility.

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Since, in my opinion, ZF is one of the most comfortable set theories in which to work, it seems reasonable to try to remedy this state of affairs, and the aim of this paper is to take a step towards such a remedy by stating and (where practicable) proving the relative strengths of a number of finiteness criteria within ZF .

2.

In order to alleviate some of the notational difficulties that appear to be inevitable in almost all branches of mathematics, I list here the main notational conventions that I shall try to stick to throughout this paper.

Logical:

1. "$\sim$" - negation;
2. "$\equiv$" - definitional equivalence;
3. "$\rightarrow$" - logical implication;
4. "$\leftrightarrow$" - logical equivalence;
5. "$\forall$" - universal quantification;
6. "$\exists$" - existential quantification;
7. "$\vee$" - disjunction;
8. "&" - conjunction.

Set theoretic:

1. "ω" - set of natural numbers;
2. "P" - power-set operator;
3. "$-$" - set theoretic difference;
4. "$\simeq$" - set theoretic equivalence;
5. "$+$" - union;
6. "$\cdot$" - intersection;
7. "$\sum$" - generalized union;
8. "$\prod$" - generalized intersection.

Additional:

1. x, y, z , and so forth, are variables that range over arbitrary sets;
2. f, g, h , and so forth, are variables that range over those sets

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that are functions;

3. k, m, n , and so forth, are variables that range over those sets that are natural numbers;

4. von Neumann's definition of natural number is adopted, that is, $0 = \emptyset$, $n + 1 = n + \{n\}$. [The symbol "+" is used in two senses here, but this slight ambiguity should cause no confusion.] Thus "$m < n$" and "$m \in n$" may be regarded as synonymous;

5. "In" is a unary predicate applicable to functions and denotes injectivity;

6. "$\uparrow$", "$\downarrow$" are unary predicates applicable to functions with domain ω :

$$\uparrow(f) \equiv \forall n\Big[f(n) \in P\big(f(n+1)\big)\Big] \ ; \quad \downarrow(f) \equiv \forall n\Big[f(n+1) \in P\big(f(n)\big)\Big] \ ;$$

7. occasionally the notation "$|\ |$" will be used to denote cardinality (of a set). Although not strictly legal in a paper of this type, its use in certain places circumvents a much clumsier notation;

8. all assertions and proofs are made within the language and framework of ZF .

3.

The formal definitions of the finiteness criteria to be studied are as follows.

DEFINITION 1. $FN(x) \equiv \exists n(x \simeq n)$.

DEFINITION 2. $FT(x) \equiv \forall y \exists z \forall w\Big[y \in P^2(x) - \{\emptyset\} \to z \in y \ \&$
$\big(w \in y \cdot P(z) \to w = z\big)\Big]$.

DEFINITION 3. $FS(x) \equiv \forall f\big(f \in P(x)^{\omega} \ \& \ \mathrm{In}(f) \to \sim\downarrow(f)\big)$.

DEFINITION 4. $FD0(x) \equiv \forall y\big(y \in P(x) \ \& \ y \simeq x \to y = x\big)$.

DEFINITION 5. $FD1(x) \equiv FD0\big(P(x)\big)$.

DEFINITION 6. $FD2(x) \equiv FD0\big(P^2(x)\big)$.

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Intuitively, a set x is

FN-finite if x contains n elements for some n ;

FT-finite if every non-empty subset of the poset $(P(x), \subset)$ has a minimal element;

FS-finite if the poset $(P(x), \subset)$ satisfies the descending chain condition;

$FD0$-finite if x is Dedekind finite;

$FD1$-finite if $P(x)$ is Dedekind finite;

$FD2$-finite if $P^2(x)$ is Dedekind finite.

Clearly FT and FS have duals, FT^* and FS^* , for example $FS^*(x) \equiv \forall f(f \in P(x)^\omega \;\&\; \mathrm{In}(f) \rightarrow \sim\uparrow(f))$. It is not at all difficult to show that $FT \leftrightarrow FT^*$ and $FS \leftrightarrow FS^*$: the proof of the latter is given here, since the equivalence will be required later on in this paper.

LEMMA 1. $FS \leftrightarrow FS^*$.

Proof. Assume $\sim FS^*(x)$, that is, $\exists f(f \in P(x)^\omega \;\&\; \mathrm{In}(f) \;\&\; \uparrow(f))$. For any such f , define $g = g_f \in P(x)^\omega$ by $g(n) = x - f(n)$. Then clearly $\mathrm{In}(g) \;\&\; \downarrow(g)$, and so $\sim FS(x)$. Hence $FS \rightarrow FS^*$; and the converse is exactly analogous.

The relative strengths of the six criteria may be diagrammed as follows:-

$$\begin{array}{c} FN \leftrightarrow FD2 \leftrightarrow FT \\ \downarrow \\ FS \leftrightarrow FD1 \\ \downarrow \\ FD0 \end{array}$$

The implications indicated by single arrows are in fact strict implications; however, the proofs of their strictness require the construction of non-standard ZF-models, and so, because of the complexity of even the simplest of these, cannot be presented here. Suffice it to say that these models can be constructed by the boolean technique described in Rosser [2], but there would appear to be no reason why forcing methods as described in

Cohen [1] could not be used.

The following lemma is used repeatedly in almost all work on finiteness:

LEMMA 2. $\forall x\left[FD0(x) \leftrightarrow \forall f\left(f \in x^{\omega} \rightarrow \sim\mathrm{In}(f)\right)\right]$.

Proof. Assume $\sim FD0(x)$, that is, $\exists y\left(y \in P(x)-\{x\} \ \& \ y \simeq x\right)$. Then it is routine to prove the existence of a triple (y, f, z) , where $y \in P(x)-\{x\}$, $f \in y^{x}$ is bijective, and $z \in x-y$. For any such triple, define $g = g_{y,f,z} \in x^{\omega}$ by $g(0) = z$, $g(n+1) = f\left(g(n)\right)$. Then clearly $\mathrm{In}(g)$. Conversely, assume $\exists f\left(f \in x^{\omega} \ \& \ \mathrm{In}(f)\right)$, and for any such f , $y = y_f \in P(x)-\{x\}$ by $y = x - \{f(0)\}$, and define $g = g_f \in y^{x}$ by $g\left(f(n)\right) = f(n+1)$ and $g(z) = z$ for $z \in x-f''\omega$. Then clearly g is bijective.

4.

The most difficult of all the implications to prove is $FS \rightarrow FD1$; this requires a couple of preliminary lemmas, and will be deferred to §5. In this section the "lightweights" are dispatched.

Three of these are really trivial, and are dealt with at once.

THEOREM 1.

(i) $FD1 \rightarrow FD0$;

(ii) $FD2 \rightarrow FD1$;

(iii) $FD1 \rightarrow FS$.

Proof *(i)*. Assume $\sim FD0(x)$, that is by Lemma 2, $\exists f\left(f \in x^{\omega} \ \& \ \mathrm{In}(f)\right)$. For any such f , define $g = g_f \in P(x)^{\omega}$ by $g(n) = \{f(n)\}$. Clearly $\mathrm{In}(g)$, and so by Lemma 2 it follows that $\sim FD1(x)$. Hence $FD1 \rightarrow FD0$.

(ii). This follows immediately from *(i)* upon the replacement of x by $P(x)$.

(iii). Assume $\sim FS(x)$, that is, $\exists f\left(f \in P(x)^{\omega} \ \& \ \mathrm{In}(f) \ \& \ \downarrow(f)\right)$:

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$\sim FD1(x)$ follows at once by Lemma 2.

Because of these implications, there only remain the above-mentioned implication $FS \to FD1$, and the two equivalences $FN \leftrightarrow FT$, $FN \leftrightarrow FD2$, to prove.

THEOREM 2. $FN \leftrightarrow FT$.

Proof. Assume $FN(x)$; if $x = \emptyset$, then $FT(x)$ holds vacuously; hence assume $x \neq \emptyset$. It is routine to prove (by induction on $|x|$) that $z \in P(x) \to FN(z)$. For $y \in P^2(x)-\{\emptyset\}$, define $S_y \in P(\omega)$ by $S_y = \{m ; \exists z(z \in y \ \&\ z \simeq m)\}$. Then $S_y \neq \emptyset$, and so $\min S_y$ exists. Thus $z \simeq \min S_y$ for some $z \in y$, and clearly any such z is minimal in y. Hence $FT(x)$, and so $FN \to FT$. Now assume $\sim FN(x)$; then it is routine to prove, by induction on n, $\forall n \exists y \big(y \in P(x) \ \&\ y \simeq n\big)$. Hence the following set $z = \{y \in P(x); FN(x-y)\} \in P^2(x)-\{\emptyset\}$, and has no minimal element. Thus $\sim FT(x)$, and so $FT \to FN$.

THEOREM 3. $FN \leftrightarrow FD2$.

Proof. From Lemma 2 it follows that $FN \to FD0$; thus if it can be shown that $FN(x) \to FN\big(P(x)\big)$, it will follow that

$$FN(x) \to FN\big(P(x)\big) \to FN\big(P^2(x)\big) \to FD0\big(P^2(x)\big) \equiv FD2(x).$$

Thus assume $FN(x)$, that is, $\exists n(x \simeq n)$. Define $f \in P(n)^{2^n}$ by $f(y) = \chi_y$, χ_y being the characteristic function $n \to 2$. It is easily seen that f is bijective, and so $P(x) \simeq P(n) \simeq 2^n$. Thus $FN(x) \to FN\big(P(x)\big)$, and so $FN \to FD2$.

Now assume $\sim FN(x)$, and define $f \in P^2(x)^{\omega}$ by $f(n) = \{y \in P(x) ; y \simeq n\}$. Considerations similar to those in the proof of Theorem 2 show that $\mathrm{In}(f)$. Thus $\sim FD2(x)$ follows by Lemma 2, and so $FD2 \to FN$.

5.

The remaining implication is $FS \to FD1$. As stated earlier, its proof is longer than the previous proofs, and depends upon two preliminary lemmas. Thus it might be best to commence with an informal outline.

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The proof proceeds, as usual, by contradiction, and so it is assumed that $f \in P(x)^{\omega}$ is injective; the problem is to construct $g \in P(x)^{\omega}$ such that $\mathrm{In}(g)$ and either $\uparrow(g)$ or $\downarrow(g)$ – by Lemma 1, it does not matter which. An obvious construction is applied to f in an attempt to construct $h \in P(x)^{\omega}$ such that $\mathrm{In}(h)$ & $\uparrow(h)$: if it works, the proof is complete. If it fails, however, then a proper subset x_0 of x can be defined, as can a function $f_0 \in P(x_0)^{\omega}$ with $\mathrm{In}(f_0)$. The original problem has thus been relativized to x_0 . Now the process is iterated, and so after a finite number of iterations a strictly increasing function will be produced, or else an inductive definition of a strictly decreasing function can be given.

A well-ordering on the finite subsets of ω is required. Thus:-

DEFINITION 7.

(i) Define $S_{\omega} \in P^2(\omega)$ by $S_{\omega} = \{J \in P(\omega)\ ;\ FN(J)\}$.

(ii) For $J \in S_{\omega}$, $J \simeq n$, define $J^* \in \omega n$ by

$$J^*(k) = \min\left(J - \{J^*(j)\}_{j \in k}\right) \ , \text{ for } k \in n \ .$$

DEFINITION 8. Define the binary relation

$$<_{l} \ \in P\left(\left(\sum_{i \in \omega} \omega^{i}\right) \times \left(\sum_{i \in \omega} \omega^{i}\right)\right)$$

as follows. For $(f, g) \in \omega^{m} \times \omega^{n}$, set

$$f <_{l} g \equiv \Big(m \in n \vee \Big(m = n \ \ \&\ \ \exists k \forall j \Big(k \in m \ \ \&\ \ f(k) \in g(k)$$
$$\&\ \ \big(j \in k \rightarrow f(j) = g(j)\big)\Big)\Big)\Big) \ .$$

Clearly $<_{l}$ defines a well-ordering on $\sum_{i \in \omega} \omega^{i}$; a well-ordering can now be induced on S_{ω} .

DEFINITION 9. Define the binary relation $<_{\sigma} \in P\left(S_{\omega} \times S_{\omega}\right)$ as follows. For $J, K \in S_{\omega}$, set $J <_{\sigma} K \equiv J^* <_{l} K^*$.

Some terminology is required for the preliminary lemmas.

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DEFINITION 10. Let x be a set, and take $y \in P(x)$, $f \in P(x)^{\omega}$, $J \in S_{\omega}$.

(i) Define $y/f \in P(\omega)$ by $y/f = \{n \; ; \; f(n)\cdot y \neq \emptyset\}$.

(ii) Define $y\#f \in P^2(x)$ by $y\#f = \{f(n)\cdot y \in P(x) \; ; \; n \in y/f\}$.

(iii) Define $Jf \in P(x)$ by $Jf = \sum_{j \in J} f(j)$.

The two preliminary lemmas can now be given.

LEMMA 3.

$$\forall x \exists f \forall J \left[f \in P(x)^{\omega} \;\; \& \;\; \mathrm{In}(f) \;\; \& \;\; \left(J \in S_{\omega} \to FN(Jf\#f) \right) \to \sim FS(x) \right] .$$

Proof. Given x , assume that f satisfies the hypotheses of the lemma. It will be shown that $\sim FS^*(x)$, from which $\sim FS(x)$ follows by Lemma 1. Let an attempt be made to define $g \in P(x)^{\omega}$ as follows:

$$g(0) = f(0) \; , \quad g(n+1) = g(n) + f(m_n) \; , \quad m_n = \min\{k \; ; \; f(k) \notin P(g(n))\} \; .$$

In order to show that this definition is valid, it is necessary to show that for each n , m_n exists. For this it suffices to show that $\forall n \exists J (J \in S_{\omega} \;\&\; g(n) = Jf)$, for then the existence of m_n follows from the hypotheses of the lemma.

For $n = 0$ this is trivial. Assume that $g(p) = J_0 f$ for some $J_0 \in S_{\omega}$. Since $\mathrm{In}(f)$ & $FN(J_0 f\#f)$, it follows that $\exists j \left[f(j) \notin P(g(p)) \right]$. Let j_0 be the least such j , and put $J_1 = J_0 + \{j_0\}$; then $g(p+1) = J_1 f$. Thus g is well-defined, and it is clear that $\mathrm{In}(g)$ & $\uparrow(g)$.

LEMMA 4. $\forall x \forall f \exists J \exists n \left[f \in P(x)^{\omega} \;\; \& \;\; J \in S_{\omega} \;\; \& \;\; Jf\#f \simeq \omega \to f(n)\#f \simeq \omega \right]$.

Proof. Given x, f , assume the hypotheses, and put $U = \{J \in S_{\omega} \; ; \; Jf\#f \simeq \omega\}$, $X = \min_{\sigma} U$. Then $|X| = 1$. For if not, put $X_0 = \{\min X\}$, $X_1 = X - X_0$. Then $|X_i| \geq 1$, and $FN(X_i f\#f)$, for $i < 2$. But $|Xf\#f| \leq |X_0 f\#f| + |X_1 f\#f| < \omega$, a contradiction. Thus $|X| = 1$.

THEOREM 4. $FS \to FD1$.

Proof. Assume $FS(x)$ & $\sim FD1(x)$, and let $f \in P(x)^{\omega}$ be injective. The construction referred to in the outline of this proof above is that described in the proof of Lemma 3; it follows from the assumption $FS(x)$ that this construction must fail. Thus, in accordance with the outline, a set $x_0 \in P(x)-\{x\}$ and an injection $f_0 \in P(x_0)^{\omega}$ must be defined. It may of course be assumed the $x \notin f''\omega$; and the analogous assumption can and will be made with respect to each pair (x_i, f_i) to be defined. From the assumption $FS(x)$ it follows, by Lemma 3, that $Jf\#f \simeq \omega$ for some $J \in S_{\omega}$; hence, by Lemma 4, $f(j)\#f \simeq \omega$ for some $j \in \omega$. Let j_0 be the least such j , and put $x_0 = f(j_0)$; clearly $x_0 \in P(x)-\{x\}$.

Now define $f_0 \in P(x_0)^{\omega}$ as follows:

$$f_0(n) = x_0 \cdot f(k_n) \ , \quad k_n = \min\left\{m \ ; \ x_0 \cdot f(m) \notin \{\phi, x_0, f_0(0), \ldots, f_0(n-1)\}\right\} .$$

For each n , such a k_n exists, since $x_0\#f \simeq \omega$; thus f_0 is well-defined. From the definition of f_0 it is clear that $\mathrm{In}(f_0)$ and that $\{x_0, \phi\} \notin P(f_0''\omega)$.

Now assume that for each $m \le q$, $q \in \omega$ fixed, x_m, f_m have been defined such that the following conditions hold:

(i) $\forall m\left(m < q \to x_{m+1} \in P(x_m)-\{x_m\}\right)$;

(ii) for each $m < q$, $f_{m+1} \in P(x_{m+1})^{\omega}$ has been defined in terms of f_m and x_m such that $\mathrm{In}(f_{m+1})$ & $\{x_{m+1}, \phi\} \notin P(f_{m+1}''\omega)$.

It follows from the assumption $FS(x)$ that $FS(x_m)$ for $m \le q$; hence an application of Lemmas 3 and 4 to the pair (x_q, f_q) guarantees the existence of $j \in \omega$ such that $f_q(j)\#f_q \simeq \omega$; let j_{q+1} be the least such j , and put $x_{q+1} = f_q(j_{q+1})$. It is easy to see that (i) carries over.

Now define $f_{q+1} \in P(x_{q+1})^{\omega}$ as follows:

$$f_{q+1}(n) = x_{q+1} \cdot f_q(k_n) ,$$

where

$$k_n = \min\left\{m \; ; \; x_{q+1} \cdot f_q(m) \notin \{\emptyset, x_{q+1}, f_{q+1}(0), \ldots, f_{q+1}(n-1)\}\right\} .$$

The verification that f_{q+1} is well-defined goes through as before, and it is routine to check that (ii) is carried over. Thus induction ensures the existence of x_n for all n .

But then the function $g \in P(x)^\omega$ given by $g(n) = x_n$ is well-defined and strictly decreasing, from which $\sim FS(x)$ follows. This contradiction proves the theorem.

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