8
The leptonic sector

8.1 Feynman rules

We gave in Chapter 6 the Lagrange function for the fermions and the gauge bosons. In the previous chapter we defined the physical bosons with definite masses. It is now a straightforward exercise to rewrite the Lagrange density in terms of the physical bosons and read off the Feynman rules. For the rules, it is necessary to introduce quantized fields in order to keep track of the combinatorics and other factors, especially for diagrams with closed loops. The canonical quantization method in terms of Wick's theorem does not work for non-Abelian gauge theories because there are ambiguities that arise from gauge transformations. The appropriate discussion at this point is the quantization in the path-integral formalism. This will be a long digression and will delay us from arriving at physical results. We adopt a compromise. We consider the fermionic part of the Lagrange function in terms of the physical fields and read off the relevant vertices. The interested reader can compare this method with the procedure used in textbooks of quantum electrodynamics. In this way we obtain an extensive set of Feynman rules for vertices and propagators, in terms of which we discuss many physical processes.

Later on, we repeat this procedure for other parts of the Lagrangian, which include Higgses and gauge bosons. The rules that we obtain suffice when we calculate tree diagrams to any order. Difficulties occur when loop diagrams are computed, beginning with one-loop diagrams. The difficulties are solved by introducing additional diagrams with scalar particles: the Faddeev–Popov ghosts.

We saw in the previous chapter that the neutral gauge fields mix among themselves. It is appropriate to introduce a mixing angle

$$\tan\theta_{\mathrm{W}} = \frac{g'}{g}. \tag{8.1}$$

https://doi.org/10.1017/9781009402378.009 Published online by Cambridge University Press

The physical fields defined as mass eigenstates are given by

$$W_\mu^\pm = \frac{1}{\sqrt{2}}(W_1 \pm \mathrm{i}W_2), \tag{8.2}$$

$$A_\mu = \cos(\theta_\mathrm{W})B_\mu + \sin(\theta_\mathrm{W})W_\mu^3, \tag{8.3}$$

$$Z_\mu = \sin(\theta_\mathrm{W})B_\mu - \cos(\theta_\mathrm{W})W_\mu^3. \tag{8.4}$$

As stated earlier, A_μ represents the photon, while Z_μ and $W_\mu^\pm$ represent the neutral and charged intermediate gauge bosons, whose masses satisfy the relation

$$\frac{M_\mathrm{W}^2}{M_\mathrm{Z}^2} = \cos^2\theta_\mathrm{W}. \tag{8.5}$$

In the leptonic Lagrangian (6.18) we substitute for W_μ^i and B_μ in terms of the fields $W_\mu^\pm$, Z_μ, and A_μ. Using again the identity

$$\vec{\tau}\cdot\vec{W}_\mu = \sqrt{2}(\tau^+ W_\mu^- + \tau^- W_\mu^+) + \tau_3 W_\mu^3, \tag{8.6}$$

we can read off the couplings of the $\mathrm{W}^\pm$ bosons to charged currents.

The couplings of Z_μ and A_μ to their respective currents follow after some algebra. The neutral gauge couplings are

$$\begin{aligned}\mathcal{L}_\mathrm{NC} &= -\bar{\psi}_\mathrm{R}\gamma^\mu g'\frac{Y}{2}B_\mu\psi_\mathrm{R} - \bar{\psi}_\mathrm{L}\gamma^\mu\left(g'\frac{Y}{2}B_\mu + g\frac{\tau_3}{2}W_\mu^3\right)\psi_\mathrm{L} \\ &= -gs\left[\bar{\psi}_\mathrm{R}\gamma^\mu\frac{Y}{2}\psi_\mathrm{R} + \bar{\psi}_\mathrm{L}\gamma^\mu\left(\frac{Y}{2}+\frac{\tau_3}{2}\right)\psi_\mathrm{L}\right]A_\mu \\ &\quad + \frac{g}{c}\left[-s^2\left(\bar{\psi}_\mathrm{R}\gamma^\mu\frac{Y}{2}\psi_\mathrm{R} + \bar{\psi}_\mathrm{L}\gamma^\mu\frac{Y}{2}\psi_\mathrm{L}\right) + c^2\bar{\psi}_\mathrm{L}\gamma^\mu\frac{\tau_3}{2}\psi_\mathrm{L}\right]Z_\mu,\end{aligned} \tag{8.7}$$

with $(c, s) = (\cos\theta_\mathrm{W}, \sin\theta_\mathrm{W})$. The weak hypercharge Y can be replaced according to (6.11) by

$$\bar{\psi}_\mathrm{R}\frac{Y}{2}\gamma^\mu\psi_\mathrm{R} = \bar{\psi}_\mathrm{R}Q\gamma^\mu\psi_\mathrm{R}, \tag{8.8}$$

$$\bar{\psi}_\mathrm{L}\frac{Y}{2}\gamma^\mu\psi_\mathrm{L} = \bar{\psi}_\mathrm{L}\left(Q - \frac{\tau_3}{2}\right)\gamma^\mu\psi_\mathrm{L}, \tag{8.9}$$

giving finally

$$\begin{aligned}\mathcal{L}_\mathrm{NC} &= -gs\left(\bar{\psi}_\mathrm{R}\gamma^\mu Q\psi_\mathrm{R} + \bar{\psi}_\mathrm{L}\gamma^\mu Q\psi_\mathrm{L}\right)A_\mu \\ &\quad + \frac{g}{c}\left[\bar{\psi}_\mathrm{L}\gamma^\mu\frac{\tau_3}{2}\psi_\mathrm{L} - s^2(\bar{\psi}_\mathrm{R}\gamma^\mu Q\psi_\mathrm{R} + \bar{\psi}_\mathrm{L}\gamma^\mu Q\psi_\mathrm{L})\right]Z_\mu.\end{aligned} \tag{8.10}$$

Finally, on substituting for

$$\psi_\mathrm{L} = \begin{pmatrix} \nu \\ \mathrm{e} \end{pmatrix}_\mathrm{L} \quad \text{and} \quad \psi_\mathrm{R} = \mathrm{e}_\mathrm{R},$$

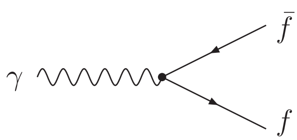
we obtain

$$\begin{aligned}
\mathcal{L}_\mathrm{F} = \mathrm{i}\bar{e}\,\partial\!\!\!/ e + \mathrm{i}\bar{\nu}\,\partial\!\!\!/ \nu - gsA_\mu \bar{e}\gamma^\mu e \\
+ \frac{g}{2\sqrt{2}}\left[\bar{\nu}\gamma^\mu(1-\gamma_5)eW_\mu^+ + \bar{e}\gamma^\mu(1-\gamma_5)\nu W^-\right] \\
+ \frac{g}{4c}\left[\bar{\nu}\gamma^\mu(1-\gamma_5)\nu + \bar{e}\gamma^\mu\gamma_5 e - (1-4s^2)\bar{e}\gamma^\mu e\right]Z_\mu.
\end{aligned} \tag{8.11}$$

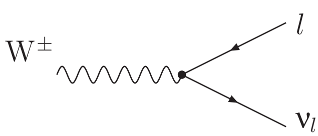
The third term above is the coupling of a massless vector particle to the electromagnetic current of electrons. Its coupling is evidently the electromagnetic charge

$$e = g\sin\theta_\mathrm{W}. \tag{8.12}$$

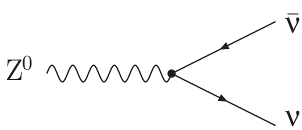
We can read off the following vertices:



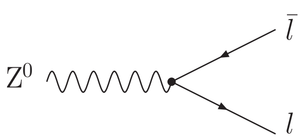
$$-\mathrm{i}eQ_f\gamma^\mu$$



$$-\frac{\mathrm{i}g}{2\sqrt{2}}\gamma^\mu(1-\gamma_5)$$



$$-\frac{\mathrm{i}g}{4\cos\theta_\mathrm{W}}\gamma^\mu(1-\gamma_5)$$



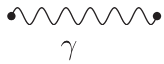
$$\frac{\mathrm{i}g}{4\cos\theta_\mathrm{W}}\gamma^\mu(1-4\sin^2\theta_\mathrm{W}-\gamma_5)$$

The propagators for particles are introduced in many textbooks. For the fermions,

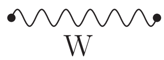
$$\frac{\mathrm{i}}{p\!\!\!/ - m + \mathrm{i}\varepsilon}$$

https://doi.org/10.1017/9781009402378.009 Published online by Cambridge University Press

For vector mesons the propagator depends on the gauge. We will frequently use the Feynman gauge, for which the three relevant propagators are



$$-\frac{\mathrm{i}g_{\mu\nu}}{p^2+\mathrm{i}\varepsilon}$$



$$-\frac{\mathrm{i}g_{\mu\nu}}{p^2-M_{\mathrm{W}}^2+\mathrm{i}\varepsilon}$$

$$-\frac{\mathrm{i}g_{\mu\nu}}{p^2-M_{\mathrm{Z}}^2+\mathrm{i}\varepsilon}$$

In an arbitrary gauge the vector boson propagator depends on the gauge parameter ξ as follows:

$$\Delta_{\mu\nu}(p) = -\mathrm{i}\frac{g_{\mu\nu}+(\xi-1)p_\mu p_\nu/(p^2-\xi M^2)}{p^2-M^2+\mathrm{i}\varepsilon}.$$

In special gauges we obtain

$$\xi = 1 \qquad \Delta_{\mu\nu} = -\mathrm{i}\frac{g_{\mu\nu}}{p^2-M^2+\mathrm{i}\varepsilon} \qquad \text{(Feynman gauge)}$$

$$\xi = 0 \qquad \Delta_{\mu\nu} = -\mathrm{i}\frac{g_{\mu\nu}-p_\mu p_\nu/p^2}{p^2-M^2+\mathrm{i}\varepsilon} \qquad \text{(Landau gauge)}$$

$$\xi = \infty \qquad \Delta_{\mu\nu} = -\mathrm{i}\frac{g_{\mu\nu}-p_\mu p_\nu/M^2}{p^2-M^2+\mathrm{i}\varepsilon} \qquad \text{(unitary gauge)}$$

In addition to electromagnetism, the theory describes weak interactions mediated by charged $\mathrm{W}^{\pm}$ bosons and the neutral boson Z. Charged-current interactions are mediated by the $\mathrm{W}^{\pm}$ bosons, whose mass satisfies the relation

$$M_{\mathrm{W}} = \frac{1}{2}gv. \tag{8.13}$$

In low-energy reactions the W masses can be factored out – or integrated out – giving an effective four-fermion interaction with the coupling

$$\frac{G_{\mathrm{F}}}{\sqrt{2}} = \frac{g^2}{8M_{\mathrm{W}}^2}. \tag{8.14}$$

The electroweak theory goes beyond the V–A theory and predicts the existence of neutral currents mediated by the Z bosons, whose mass

$$M_{\mathrm{Z}} = \frac{1}{2}gv\frac{1}{\cos\theta_{\mathrm{W}}} \tag{8.15}$$

https://doi.org/10.1017/9781009402378.009 Published online by Cambridge University Press

is related to M_{W} through (8.5). At low energies the neutral-current interaction also reduces to an effective interaction, whose overall strength is determined by

$$\left(-\frac{ig}{4\cos\theta_{\mathrm{W}}}\right)^2 \frac{1}{p^2 - M_{\mathrm{Z}}^2} = \frac{g^2}{16M_{\mathrm{W}}^2} = \frac{G_{\mathrm{F}}}{2\sqrt{2}}. \tag{8.16}$$

The Lagrangian (8.11) defines the weak and electromagnetic interactions of electrons and neutrinos. It contains four unknown quantities: g, $\sin^2\theta_{\mathrm{W}}$, and the masses M_{W} and M_{Z} which occur in the propagators of the bosons. In addition there is the mass of the electron, which is to be taken from experiment. All four quantities are not independent, since three of them are related through (8.5). We can use three experimental quantities to determine them. Electromagnetic measurements determine the fine-structure constant

$$\alpha = \frac{e^2}{4\pi} = \frac{1}{137.036\ldots}. \tag{8.17}$$

The muon decay is used to determine G_{F}. Neutral-current measurements, discussed in this and subsequent chapters, determine

$$\sin^2\theta_{\mathrm{W}} = 0.222 \pm 0.010. \tag{8.18}$$

From the three low-energy measurements we now determine the other parameters:

$$M_{\mathrm{W}}^2 = \frac{\pi\alpha}{\sqrt{2}G_{\mathrm{F}}\sin^2\theta_{\mathrm{W}}}, \quad M_{\mathrm{Z}}^2 = \frac{M_{\mathrm{W}}^2}{\cos^2\theta_{\mathrm{W}}}, \tag{8.19}$$

and

$$v = \frac{1}{\left(\sqrt{2}G_{\mathrm{F}}\right)^{1/2}} = 246\,\mathrm{GeV}. \tag{8.20}$$

8.2 Predictions in the leptonic sector

We have now a theory that enables us to compute many processes at the tree level. In this chapter we compute three leptonic processes.

Boson decays Among the many predictions of the model, the decays of the gauge bosons are simple to discuss. We begin with the decay

$$\mathrm{W}^- \to \mathrm{e}^- \bar{\nu}. \tag{8.21}$$

The diagram in Fig. 8.1 gives the amplitude

$$\mathcal{M} = \frac{ig}{2\sqrt{2}}\bar{u}(k_-)\gamma_\mu(1-\gamma_5)v(k_+)\varepsilon^\mu, \tag{8.22}$$

https://doi.org/10.1017/9781009402378.009 Published online by Cambridge University Press

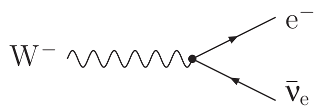


Figure 8.1. The Feynman diagram for a W^- decay.

with ε^μ being the polarization of the intermediate boson. For the decay rate we sum over the polarization states with

$$\sum_{\text{Pol}} \varepsilon_\mu^* \varepsilon_\nu = -g_{\mu\nu} + \frac{q_\mu q_\nu}{M_W^2}. \tag{8.23}$$

The contribution from $q_\mu q_\nu$ is proportional to the masses of the leptons and can be neglected. We also ignore terms proportional to lepton masses in the trace computation. The mass of the neutrino being small or zero does not cause any difficulties, because in the spinor normalization $\bar{u}u = 2m$ they simply do not occur in the formulas. Had we used another normalization, i.e. $\bar{u}u = 1$, then we could give a small mass to the neutrinos and proceed to calculate decay rates and cross sections, but we will find in the end that the neutrino masses drop out of the formulas.

The square of the matrix element summed over spins is

$$\begin{aligned}\sum_{\text{Spins, Pol}} \mathcal{M}\mathcal{M}^* &= \frac{g^2}{8} \cdot 2 \cdot \mathrm{Tr}[\gamma_\mu (1-\gamma_5)(\not{k}_+ + m_e)\gamma_\nu(\not{k}_- - m_\nu)] \cdot (-g^{\mu\nu}) \\ &= 2g^2 k_+ \cdot k_-. \end{aligned} \tag{8.24}$$

The decay rate is given following standard rules:

$$\begin{aligned}\Gamma &= \frac{1}{2M_W} \int g^2 k_+ \cdot k_- (2\pi)^4 \delta^4(p - k_+ - k_-) \frac{1}{2E_e} \frac{d^3k_-}{(2\pi)^3} \frac{1}{2E_\nu} \frac{d^3k_+}{(2\pi)^3} \\ &= \frac{g^2}{16\pi} M_W = \frac{G_F M_W^3}{2\sqrt{2}\pi}. \end{aligned} \tag{8.25}$$

We must still average over the initial polarizations of the gauge bosons and obtain

$$\overline{\Gamma} = \frac{1}{(2s+1)} \Gamma = \frac{G_F M_W^3}{6\sqrt{2}\pi}, \tag{8.26}$$

which gives 211.3 MeV for $M_W = 80$ GeV (235.9 MeV for $M_W = 83$ GeV).

The total width is obtained by adding the additional decays into $\mu\bar{\nu}_\mu$, $\tau\bar{\nu}_\tau$, and quark pairs. We introduce the vertices of the gauge bosons to quarks in the next chapter, but we mention here that the decay of W^- into a quark pair of definite color is also given by (8.26). Thus, for three generations of quarks and leptons the total

https://doi.org/10.1017/9781009402378.009 Published online by Cambridge University Press

width is obtained by multiplying the width by 12: (3 lepton families) + [(3 quark families) × (3 colors)] = 12;

$$\Gamma_{\text{total}} = \frac{2G_{\text{F}}M_{\text{W}}^3}{\sqrt{2}\pi}. \tag{8.27}$$

In leptonic decays all that can be observed is the charged lepton. In hadronic decays the W bosons are inferred by reconstructing the hadronic jets, which imitate to some extent the kinematic characteristics of the original quarks.

In contrast, the leptonic decays of Z bosons into charged leptons are identified by the invariant mass of the pairs. By comparing the coupling constants we obtain the partial decay widths

$$\Gamma(\text{Z} \to \nu\bar{\nu}) = \frac{G_{\text{F}}M_{\text{Z}}^3}{12\sqrt{2}\pi} \tag{8.28}$$

and

$$\Gamma(\text{Z} \to \text{e}^+\text{e}^-) = \frac{G_{\text{F}}M_{\text{Z}}^3}{12\sqrt{2}\pi}\left(1 - 4\sin^2\theta_{\text{W}} + 8\sin^4\theta_{\text{W}}\right). \tag{8.29}$$

For the total width, we need in addition the decay width into quarks (see Problem 1). Summing again over three generations,

$$\Gamma_{\text{total}}(\text{Z}) = \frac{G_{\text{F}}M_{\text{Z}}^3}{3\sqrt{2}\pi}\left(\frac{21}{4} - 10\sin^2\theta_{\text{W}} + \frac{40}{3}\sin^4\theta_{\text{W}}\right). \tag{8.30}$$

It is worth noting that the total width is sensitive to the total number of quarks and leptons lighter than M_{Z} and precise measurements of Γ_{total} could produce exotic surprises. The width of the Z boson has been determined in the CERN experiments to be

$$\Gamma_{\text{Z}} = (2.490 \pm 0.007)\,\text{GeV}.$$

The width in turn limits the number of neutrinos to

$$N_\nu = 3.09 \pm 0.13,$$

which is very close to the number of neutrinos allowed by nucleosynthesis arguments, $N_\nu \simeq 3$–4.

8.3 Leptonic neutral currents

A striking piece of evidence for the electroweak theory was the discovery of neutral currents. The Lagrangian in (8.11) describes both charged- and neutral-current

https://doi.org/10.1017/9781009402378.009 Published online by Cambridge University Press

interactions of neutrinos and electrons. All the couplings depend on the SU(2) coupling constant g and the Weinberg angle θ_{W}. At low energies, the overall strength of the neutral-current interaction is determined by G_{F} through Eq. (8.16). Thus all neutral-current interactions must depend on a single parameter, $\sin^2\theta_{\mathrm{W}}$. There is a large number of neutral-current reactions that have been measured and the agreement after two and a half decades of research is indeed impressive. This section describes in detail leptonic neutral-current reactions. The reader will find this section very useful also for the semileptonic interactions discussed in Chapters 10–12, since many of the formulas can be taken over. We consider first neutrino–electron scattering. Six reactions of this type are shown in Fig. 8.2.

Reactions (1) and (2) can proceed only through neutral currents. Other reactions like

$$\nu_{\mathrm{e}}(k) + \mathrm{e}^-(p) \to \nu_{\mathrm{e}}(k') + \mathrm{e}^-(p') \tag{8.31}$$

involve both charged- and neutral-current diagrams. For low-energy reactions it is convenient to write the Feynman amplitude in the form

$$\mathcal{M} = -\mathrm{i}\frac{G_{\mathrm{F}}}{\sqrt{2}}\left[\bar{\nu}\gamma_\mu(1-\varepsilon\gamma_5)\nu\bar{e}\gamma^\mu(g_{\mathrm{V}} - g_{\mathrm{A}}\gamma_5)e\right]. \tag{8.32}$$

In this form both vertices retain the charge of the lepton. Evidently, not all reactions are of this form, because several of them include also charged-current reactions. Charged- and neutral-current reactions have different propagators and in addition the order of the spinors is different. Charged-current reactions can be transformed into the charge-retaining form by Fierz's reordering theorem. A special form of the theorem states that, when at least one of the couplings is $(1 \pm \gamma_5)$, then we can interchange the first and the third (or second and fourth) spinors. All examples in Fig. 8.2 involve a neutrino or antineutrino whose vertex is $\gamma_\mu(1-\varepsilon\gamma_5)$ with $\varepsilon = 1$ for neutrinos and $\varepsilon = -1$ for antineutrinos. As an illustration, consider the reaction (8.31) to which the two diagrams in Fig. 8.3 contribute. The amplitude is

$$\begin{aligned}
\mathcal{M} &= \left(\frac{\mathrm{i}g}{4c}\right)^2 \frac{-\mathrm{i}}{q^2 - M_{\mathrm{Z}}^2 + \mathrm{i}\varepsilon}\bar{u}(k')\gamma_\mu(1-\gamma_5)u(k)\cdot\bar{u}(p')\left[\gamma^\mu\gamma_5 - (1-4s^2)\gamma^\mu\right]u(p) \\
&\quad + \left(\frac{\mathrm{i}g}{2\sqrt{2}}\right)^2 \frac{-\mathrm{i}}{q^2 - M_{\mathrm{W}}^2 + \mathrm{i}\varepsilon}\bar{u}(p')\gamma_\mu(1-\gamma_5)u(k)\cdot\bar{u}(k')\gamma^\mu(1-\gamma_5)u(p) \\
&\xrightarrow{q^2 \ll M_{\mathrm{W}}^2} -\mathrm{i}\,\frac{G_{\mathrm{F}}}{\sqrt{2}}\bar{u}(k')\gamma_\mu(1-\gamma_5)u(k)\cdot\bar{u}(p')\left[\left(\frac{1}{2}+2s^2\right)\gamma^\mu - \frac{1}{2}\gamma^\mu\gamma_5\right]u(p) \\
&= -\mathrm{i}\frac{G_{\mathrm{F}}}{\sqrt{2}}\bar{u}(k')\gamma_\mu(1-\gamma_5)u(k)\cdot\bar{u}(p')\,(g_{\mathrm{V}}\gamma^\mu - g_{\mathrm{A}}\gamma^\mu\gamma_5)\,u(p).
\end{aligned} \tag{8.33}$$

https://doi.org/10.1017/9781009402378.009 Published online by Cambridge University Press

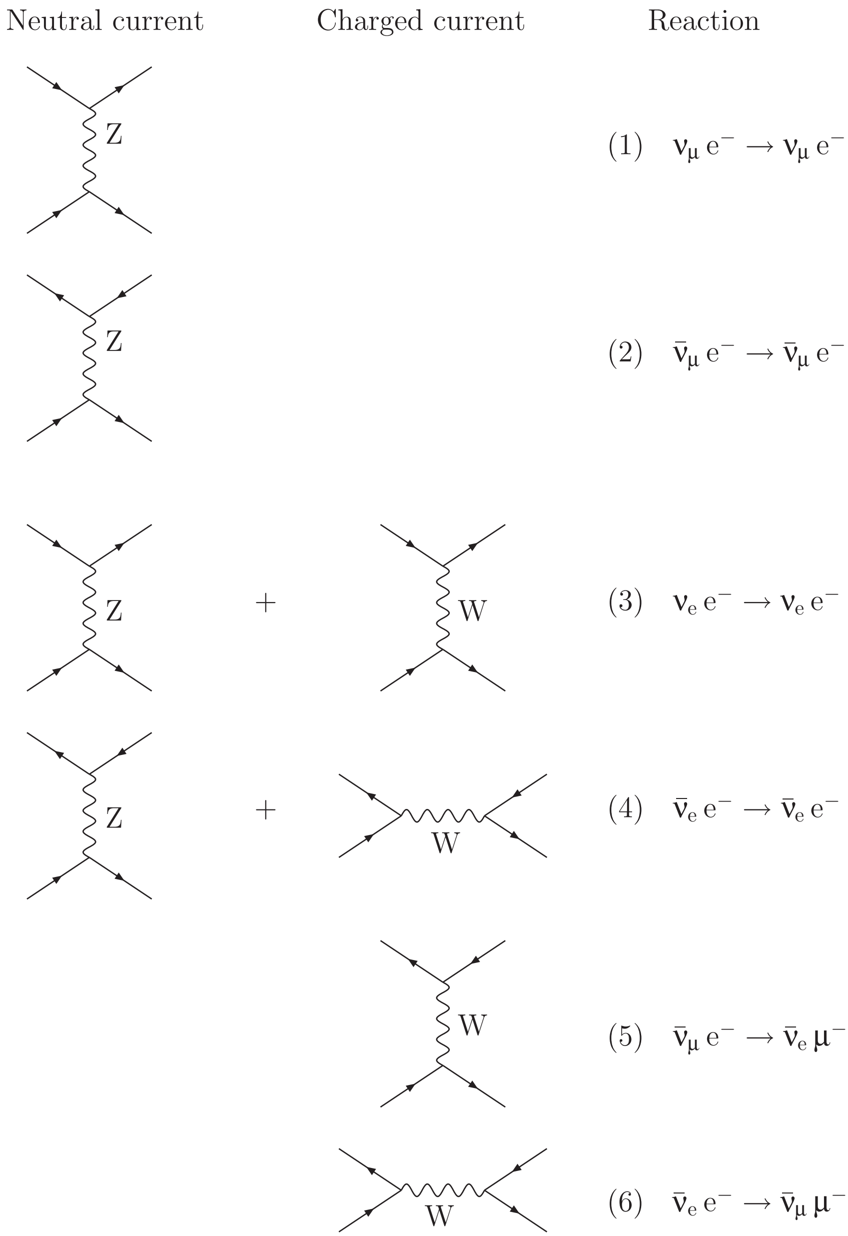


Figure 8.2. Diagrams for neutrino–electron scattering.

In this way we can write all reactions in the form (8.32). The explicit values for ε, g_V, and g_A for five reactions are given in Table 8.1. We can now calculate the cross section for the amplitude (8.32) and obtain the specific reaction by substituting the values from Table 8.1.

Table 8.1. *Effective couplings for several reactions*

		Electroweak theory		V–A theory	
Reaction	ε	g_V	g_A	g_V	g_A
$\nu_\mu + e^- \to \nu_\mu + e^-$	$+1$	$-\frac{1}{2} + 2s^2$	$-\frac{1}{2}$	0	0
$\bar{\nu}_\mu + e^- \to \bar{\nu}_\mu + e^-$	-1	$-\frac{1}{2} + 2s^2$	$-\frac{1}{2}$	0	0
$\nu_e + e^- \to \nu_e + e^-$	$+1$	$+\frac{1}{2} + 2s^2$	$+\frac{1}{2}$	1	1
$\bar{\nu}_e + e^- \to \bar{\nu}_e + e^-$	-1	$+\frac{1}{2} + 2s^2$	$+\frac{1}{2}$	1	1
$\nu_\mu + e^- \to \mu^- + \nu_e$	$+1$	1	1	1	1

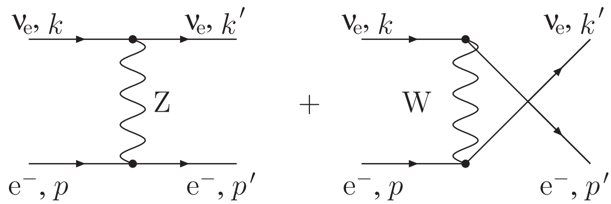


Figure 8.3. Z and W exchange in neutrino–electron scattering.

We begin with the amplitude in (8.32) and compute the differential cross section. For simplicity, for the moment we set $\varepsilon = 1$ and the mass of the electron to zero (whenever allowed). At the end we give the complete formula with ε and a small term proportional to the electron mass. The square of the amplitude summed over final spins and averaged over initial spins is

$$
\begin{aligned}
|\overline{\mathcal{M}}^2| &= \frac{1}{2}\sum_{\text{Spins}} \mathcal{M}\mathcal{M}^* \\
&= \frac{G_F^2}{4}\mathrm{Tr}\big[\gamma_\mu(1-\gamma_5)\not{k}\gamma_\nu(1-\gamma_5)\not{k}'\big] \\
&\quad\times \mathrm{Tr}\big[\gamma^\mu(g_V - g_A\gamma_5)(\not{p}+m)\gamma^\nu(g_V - g_A\gamma_5)(\not{p}'+m)\big]. \qquad (8.34)
\end{aligned}
$$

Averaging over initial spins brings in a factor of 1/2 because the neutrinos are always left-handed. Evidently the expression factorizes into two tensors,

$$|\overline{\mathcal{M}}^2| = \frac{G_F^2}{2}\mathcal{L}_{\mu\nu}\ell^{\mu\nu}, \qquad (8.35)$$

with

$$
\begin{aligned}
\mathcal{L}_{\mu\nu} &= \mathrm{Tr}\big[\gamma_\mu(1-\gamma_5)\not{k}\gamma_\nu\not{k}'\big] \\
&= 4(k_\mu k'_\nu + k_\nu k'_\mu - k\cdot k' g_{\mu\nu} + \mathrm{i}\varepsilon_{\mu\nu\alpha\beta}k^\alpha k'^\beta) \qquad (8.36)
\end{aligned}
$$

https://doi.org/10.1017/9781009402378.009 Published online by Cambridge University Press

and

$$\ell_{\mu\nu} = \mathrm{Tr}\big[(g_V^2 + g_A^2)\gamma^\mu \not{p} \gamma^\nu \not{p}' + 2 g_V g_A \gamma_5 \gamma^\mu \not{p} \gamma^\nu \not{p}'\big], \tag{8.37}$$

where we neglected terms proportional to m_e^2. The computation of the $\ell_{\mu\nu}$ tensor is similar to that of $\mathcal{L}_{\mu\nu}$. In contracting the two tensors, we observe that products with different symmetries in μ and ν vanish:

$$|\overline{\mathcal{M}}|^2 = 16 G_F^2 \big[(g_V + g_A)^2 (k \cdot p)(k' \cdot p') + (g_V - g_A)^2 (k \cdot p')(k' \cdot p)\big]. \tag{8.38}$$

We choose to compute the cross section in the laboratory frame, where the initial electron is at rest:

$$\mathrm{d}\sigma = \frac{1}{2m_e} \frac{1}{2E_\nu} |\overline{\mathcal{M}}|^2 (2\pi)^4 \delta^4(k + p - k' - p') \frac{1}{2E'_\nu} \frac{\mathrm{d}^3 k'}{(2\pi)^3} \frac{1}{2E'_e} \frac{\mathrm{d}^3 p'}{(2\pi)^3}. \tag{8.39}$$

We perform the $\mathrm{d}^3 p'$ phase-space integration with the help of the δ^4-function. The last integration over the scattering angle involves the integral

$$\int \mathrm{d}^3 k' \, \delta\big[m_e^2 - (k - k' + p)^2\big] = \pi \left(\frac{E'_\nu}{E_\nu}\right) \mathrm{d}E'_\nu. \tag{8.40}$$

The δ-function gives the relation between the scattering angle and the energy transfer:

$$1 - \cos\theta = \frac{(E - E')m_e}{EE'}. \tag{8.41}$$

Since the average value for $\langle E' \rangle \approx E_\nu/2$, we can estimate the average scattering angle $\theta \approx 2^\circ/\sqrt{E}$ with E measured in GeV. The scattered electron comes out at very small forward angles and provides a unique signature for the experiments.

From (8.38) and (8.40) we obtain the final result

$$\frac{\mathrm{d}\sigma}{\mathrm{d}E'} = \frac{G_F^2 m_e}{2\pi} \left[(g_V + g_A)^2 + (g_V - g_A)^2 \left(\frac{E'_\nu}{E_\nu}\right)^2 \right]. \tag{8.42}$$

Had we used the amplitude (8.32) and retained the mass of the electron, the final result would have been

$$\frac{\mathrm{d}\sigma}{\mathrm{d}E'} = \frac{G_F^2 m_e}{2\pi} \left[(g_V + \varepsilon g_A)^2 + (g_V - \varepsilon g_A)^2 \left(\frac{E'_\nu}{E_\nu}\right)^2 + \frac{m_e \nu}{E_\nu^2}\big(g_A^2 - g_V^2\big) \right]. \tag{8.43}$$

The last term, which is proportional to the electron mass, is small, so it can be neglected at accelerator energies. The variable $\nu = E_\nu - E'_\nu$ denotes the energy transfer.

https://doi.org/10.1017/9781009402378.009 Published online by Cambridge University Press

Formula (8.43) is useful in describing all reactions shown in Fig. 8.2 that include charged-current reactions. For instance the reaction

$$\nu_\mu + e^- \to \mu^- + \nu_e \tag{8.44}$$

reduces after substitution to

$$\frac{d\sigma}{dE'} = \frac{2G_F^2 m_e}{\pi}. \tag{8.45}$$

The same result holds for the semileptonic reaction

$$\nu_\mu + d \to \mu^- + u, \tag{8.46}$$

which we rewrite in terms of the inelasticity $y = (E - E')/E$ and the square of the center-of-mass energy, $s = 2ME_\nu$, as

$$\frac{d\sigma^\nu}{dy} = \frac{G_F^2 s}{\pi}. \tag{8.47}$$

For both of the reactions

$$\bar{\nu}_\mu + u \to \mu^+ + d, \tag{8.48}$$

$$\nu_\mu + \bar{u} \to \mu^- + \bar{d} \tag{8.49}$$

the differential cross section is

$$\frac{d\sigma^{\bar{\nu}}}{dy} = \frac{G_F^2 s}{\pi}(1-y)^2. \tag{8.50}$$

We see that, when both vertices are left-handed or both are right-handed, then $d\sigma/dy$ is independent of y as in (8.47). On the other hand, when one vertex is left-handed and the other right-handed, then $d\sigma/dy$ is proportional to $(1-y)^2$ as in (8.50).

Neutral-current reactions have a mixed y dependence. As an illustrative example, consider the neutral-current reaction

$$\nu_\mu + e^- \to \nu_\mu + e^-, \tag{8.51}$$

for which the cross section is

$$\frac{d\sigma}{dy} = \frac{G_F^2 m_e E_\nu}{2\pi}\left[(1-2\sin^2\theta_W)^2 + 4\sin\theta_W(1-y)^2\right]. \tag{8.52}$$

This reaction has been studied in several experiments. When the existence of neutral currents was still in doubt, a few events of this type were observed in the Gargamelle experiment. In spite of many attempts, one could not attribute them to any other origin. This evidence was gradually reinforced by information from semileptonic neutral-current reactions until their existence was accepted. Today there are experimental results from a few hundred events for reaction (8.51). The average slope

https://doi.org/10.1017/9781009402378.009 Published online by Cambridge University Press

from all the experiments is

$$\frac{\sigma}{E_\nu} = (1.6 \pm 0.4) \times 10^{-42}\,\mathrm{cm}^2\,\mathrm{GeV}^{-1},$$

yielding a Weinberg angle given by

$$\sin^2\theta_\mathrm{W} = 0.222 \pm 0.010.$$

Data for the other neutrino reactions are also available. In the g_V versus g_A plane each of the above total cross sections limits the physical region to an elliptical band.

8.4 Weak effects in electron–positron annihilation

An interesting and very important reaction occurs in electron–positron collisions,

$$\mathrm{e}^+\mathrm{e}^- \to \mu^+\mu^-,$$

which can be mediated by the exchange of a photon, as well as by the heavy boson Z^0. At low energies relative to the mass of the Z boson the photon diagram dominates. On the other hand, at a center-of-mass energy close to the Z boson's mass, the gauge boson dominates and exhibits a resonance behavior. At intermediate energies there is an interference term between electromagnetic and weak interactions, which modifies the angular distribution.

The amplitude for the process has two diagrams producing two amplitudes:

$$\begin{aligned}
\mathcal{M} &= \mathcal{M}_\gamma + \mathcal{M}_\mathrm{Z};\\
\mathcal{M}_\gamma &= -\frac{e^2}{s}\,\bar{v}(k_+)\gamma_\mu u(k_-)\,\bar{v}(p_+)\gamma^\mu u(p_-),\\
\mathcal{M}_\mathrm{Z} &= -\frac{g^2}{4\cos^2\theta_\mathrm{W}}\frac{1}{q^2 - M_\mathrm{Z}^2 + \mathrm{i}M_\mathrm{Z}\Gamma}\\
&\quad \times \bar{v}(k_+)\gamma_\mu(g_\mathrm{V} + g_\mathrm{A}\gamma_5)u(k_-)\bar{v}(p_+)\gamma^\mu(g'_\mathrm{V} + g'_\mathrm{A}\gamma_5)u(p_-).
\end{aligned} \tag{8.53}$$

We consider energies high enough that one can ignore the masses of the electron and muon. We also included the width of the Z particle in the propagator, which will lead to a cross section with a Breit–Wigner formula. In field theory the width is generated by summing the decays to all possible final states. For the couplings of the Z boson to electrons we introduced general coupling constants g_V and g_A; their dependence on the Weinberg angle follows from the Feynman rules. Similarly, g'_V and g'_A are couplings to muons, which are equal to the couplings of electrons.

https://doi.org/10.1017/9781009402378.009 Published online by Cambridge University Press

At intermediate energies, such as $\sqrt{s} = 30$–$50\,\mathrm{GeV}$, the well-known electromagnetic formula is modified by the presence of the interference term

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2}{4s}\left[(1+\cos^2\theta)\left(1+\varepsilon(s)g_{\mathrm{V}}^2\right) + 2\,\varepsilon(s)g_{\mathrm{A}}^2\cos\theta\right], \tag{8.54}$$

with s the square of the center-of-mass energy and

$$\varepsilon(s) = \frac{\sqrt{2}Gs}{4\pi\alpha}. \tag{8.55}$$

The new feature is the $\cos\theta$ term, with the consequence that the differential cross section is not symmetric in the forward–backward direction. The new term arises from the neutral current but is not parity–violating. It has a new angular dependence typical of the γ_5 coupling. We define a forward–backward asymmetry

$$A(\theta) = \frac{\mathrm{d}\sigma(\theta) - \mathrm{d}\sigma(\pi-\theta)}{\mathrm{d}\sigma(\theta) + \mathrm{d}\sigma(\pi-\theta)} = \varepsilon(s)\frac{2\cos\theta}{1+\cos^2\theta}g_{\mathrm{A}}^2\,. \tag{8.56}$$

The asymmetry was measured in many experiments and gave values for $g_{\mathrm{A}} = -\frac{1}{2}$ consistent with the standard model.

At higher energies the interference term becomes larger. As the center-of-mass energy approaches the mass

$$M_{\mathrm{Z}} = 91.188 \pm 0.002\,\mathrm{GeV}/c^2,$$

the weak term dominates and produces the cross section

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left(\frac{g^2}{8c^2}\right)^2 \frac{s}{\left(s-M_{\mathrm{Z}}^2\right)^2 + M_{\mathrm{Z}}^2\Gamma^2}\left[\left(g_{\mathrm{V}}^2+g_{\mathrm{A}}^2\right)^2(1+\cos^2\theta) + 8g_{\mathrm{V}}^2g_{\mathrm{A}}^2\cos\theta\right], \tag{8.57}$$

where $c = \cos\theta_{\mathrm{W}}$. The resonance was observed at CERN and was studied carefully to give precise values for the mass and the width of the gauge boson quoted in this chapter.

In addition to the muons, electron–positron collisions also produce $\mathrm{q}\bar{\mathrm{q}}$ pairs, which are analyzed with the same formulas. The values for g_{V}' and g_{A}' are now replaced by couplings appropriate for quarks.

Problem for Chapter 8

1. Compute for each generation the decay width of the Z boson for decays to neutrinos, charged leptons, and quark pairs separately. Then estimate the total decay width.

https://doi.org/10.1017/9781009402378.009 Published online by Cambridge University Press