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ON THE EXISTENCE OF FINITE CRITICAL TRAJECTORIES IN A FAMILY OF QUADRATIC DIFFERENTIALS

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Abstract

We discuss the existence of finite critical trajectories connecting two zeros in certain families of quadratic differentials. In addition, we reprove some results about the support of the limiting root-counting measures of the generalised Laguerre and Jacobi polynomials with varying parameters.

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1. Introduction and main results

Quadratic differentials appear in many areas of mathematics and mathematical physics such as orthogonal polynomials, moduli spaces of algebraic curves, univalent functions and asymptotic theory of linear ordinary differential equations. One of the most common problems in the study of a given quadratic differential is the existence of its short trajectories. We address this question, under suitable assumptions. Lastly, we present new proofs of the existence of short trajectories of quadratic differentials related to generalised Laguerre and Jacobi polynomials with varying parameters.

Let Ω be a nonempty connected subset of $\mathbb{C}$ and $Q(z) = \prod_{k=1}^{3}(z - a_k)^{m_k}$ be a polynomial with simple or double zeros (that is, $m_k \in \{1, 2\}$). Consider two continuous functions $a, b : \Omega \to \mathbb{C} \setminus \{a_1, a_2, a_3\}$ such that

$$a(t) \neq b(t) \quad \text{for all } t \in \Omega. \tag{1.1}$$

Consider the families of rational and polynomial functions R_t and P_t given by

$$R_t(z) = \frac{(z - a(t))(z - b(t))}{Q(z)}, \quad P_t(z) = (z - a(t))(z - b(t))Q(z).$$

Denote by $\mathcal{J}_{a(t),b(t)}$ the set of all Jordan arcs in $\mathbb{C} \setminus \{a_1, a_2, a_3\}$ joining $a(t)$ and $b(t)$ and suppose that there exists a continuous function (in the Hausdorff metric)

$$\Phi : \Omega \to \mathcal{J}_{a(t),b(t)}, \quad t \mapsto \phi_t,$$

where $\phi_t(\tau), \tau \in [0,1]$, is a parametrised Jordan arc such that

$$\phi_t(0) = a(t), \quad \phi_t(1) = b(t). \tag{1.2}$$

We assume that for some choice of branches of the square roots $\sqrt{R_t(z)}$ and $\sqrt{P_t(z)}$, ϕ_t satisfies the conditions

$$\Re \int_{\phi_t} \sqrt{R_t(z)}\, dz = 0, \tag{1.3}$$

$$\Re \int_{\phi_t} \sqrt{P_t(z)}\, dz = 0, \tag{1.4}$$

and we consider the quadratic differentials

$$\varpi(R_t, z) = -R_t(z)\, dz^2, \quad \varpi(P_t, z) = -P_t(z)\, dz^2.$$

Then, the following results hold.

Proposition 1.1. *Under assumptions (1.1)–(1.3), either, for every t in Ω, there exists exactly one short trajectory of the quadratic differential $\varpi(R_t, z)$ connecting $a(t)$ and $b(t)$ and homotopic to ϕ_t in $\mathbb{C} \setminus \{a_1, a_2, a_3\}$ or there is no such trajectory for any $t \in \Omega$.*

Proposition 1.2. *Under assumptions (1.1), (1.2) and (1.4), the set of all $t \in \Omega$ such that $\varpi(P_t, z)$ has a short trajectory connecting $a(t)$ and $b(t)$ is a closed subset of Ω, but may be empty.*

2. Basics of quadratic differentials

We first present some basics for quadratic differentials.

Definition 2.1. A rational quadratic differential on the Riemann sphere $\overline{\mathbb{C}}$ is a form $\varpi = \varphi(z)\, dz^2$, where φ is a rational function of a local coordinate z. If $z = z(\zeta)$ is a conformal change of variables, then $\widetilde{\varphi}(\zeta)\, d\zeta^2 = \varphi(z(\zeta))(dz/d\zeta)^2\, d\zeta^2$ represents ϖ in the local parameter ζ.

The *critical points* of ϖ are its zeros and poles. A critical point is *finite* if it is a zero or a simple pole; otherwise, it is *infinite*. All other points of $\overline{\mathbb{C}}$ are called *regular* points.

The horizontal trajectories (or just trajectories) are the zero loci of the equation

$$\Im \int^z \sqrt{\varphi(t)}\, dt = \text{constant}, \tag{2.1}$$

or, equivalently,

$$\varphi(z)\, dz^2 > 0.$$

The vertical (or orthogonal) trajectories are obtained by replacing $\Im$ by $\Re$ in the equation above. The horizontal and vertical trajectories of ϖ produce two pairwise orthogonal foliations of the Riemann sphere $\overline{\mathbb{C}}$. A critical trajectory is a trajectory

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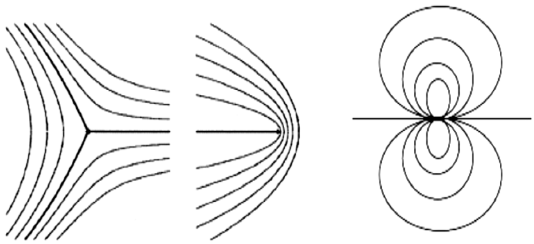

Figure 1. Trajectories near a simple zero (left), simple pole (middle) and fourth-order pole (right).

passing through a critical point. A finite critical trajectory or *short trajectory* is a critical trajectory connecting two finite critical points of ϖ. It is called *unbroken* if it does not pass through any finite critical points except its two endpoints; otherwise, it is *broken*. The set of finite and infinite critical trajectories of ϖ together with their limit points (critical points of ϖ) is called the *critical graph* of ϖ. If $z(t), t \in \mathbb{R}$, is a trajectory of (2.1), then the function

$$t \mapsto \Im \int^t \sqrt{\varphi(z(u))}\, z'(u)\, du$$

is monotone. For more details, we refer the reader to [11].

Trajectories have the following local properties.

(a) At any regular point, horizontal (respectively, vertical) trajectories behave locally like simple analytic arcs passing through this point, and through every regular point of ϖ there passes a uniquely determined horizontal (respectively, vertical) trajectory of ϖ. These horizontal and vertical trajectories are locally orthogonal at this point.

(b) From a zero of multiplicity r of ϖ, there emanate $(r+2)$ horizontal (respectively, vertical) trajectories. The angle between any two adjacent trajectories equals $\pi/(r+2)$.

(c) At a simple pole, there emanates only one trajectory (see Figure 1).

(d) At a double pole, the local behaviour of the trajectories depends on the vanishing of the real or imaginary part of the residue and the trajectories have either radial, circular or log-spiral form (see Figure 2).

(e) At a pole of order r greater than two, there are $(r-2)$ asymptotic directions (called *critical directions*), equally spaced at angle $2\pi/(r-2)$, and a neighbourhood $\mathcal{U}$, such that each trajectory entering $\mathcal{U}$ stays in $\mathcal{U}$ and tends to this pole in one of the critical directions (see Figure 1).

The main trouble in the global behaviour of trajectories comes from the so-called recurrent trajectories which are dense in some domain in $\mathbb{C}$. Jenkins' three pole theorem asserts that such a situation cannot happen for a quadratic differential that has at most three poles.

A necessary condition for the existence of a short trajectory connecting two finite critical points a and b of a quadratic differential $\varphi(z)\, dz^2$ is the existence of a Jordan

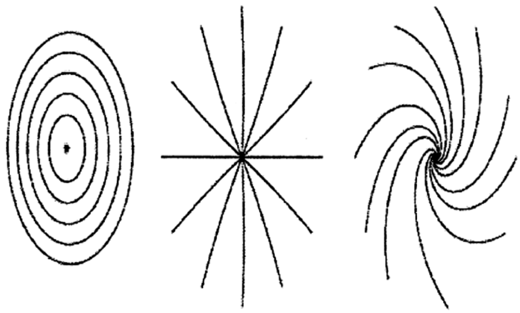

FIGURE 2. Trajectories near a double pole: circle (left), radial (middle) and log-spiral (right) forms.

arc γ connecting a and b in $\mathbb{C} \setminus \{\text{poles of } \varphi\}$, such that

$$\Im \int_\gamma \sqrt{\varphi(t)}\, dt = 0,$$

but this condition is not sufficient. Figure 3 illustrates the critical graph of the quadratic differential $Q(z) = -(z^4 - 1)\, dz^2$; in particular, there is no short trajectory connecting the zeros $\pm i$. Suppose γ is an oriented Jordan arc joining $\pm i$ in $\mathbb{C} \setminus [-1, 1]$ and $\sqrt{z^4 - 1}$ is chosen in $\mathbb{C} \setminus ([-1, 1] \cup \gamma)$ with the condition $\sqrt{z^4 - 1} \sim z^2, z \to \infty$, then, from the Laurent expansion $\sqrt{z^4 - 1} = z^2 + O(z^{-2})$ at ∞, the residue of $\sqrt{z^4 - 1}$ at ∞ is zero. For $t \in [-1, 1] \cup \gamma$, we denote by $(\sqrt{t^4 - 1})_+$ and $(\sqrt{t^4 - 1})_-$ the limits from the $+$ and $-$ sides, respectively. (As usual, the $+$ side of an oriented curve lies to the left and the $-$ side lies to the right, if one traverses the curve according to its orientation.) Let

$$I = \int_{-1}^{1} (\sqrt{t^4 - 1})_+\, dt + \int_\gamma (\sqrt{t^4 - 1})_+\, dt.$$

Since $(\sqrt{t^4 - 1})_+ = -(\sqrt{t^4 - 1})_-$, for $t \in [-1, 1] \cup \gamma$,

$$2I = \int_{[-1,1]\cup\gamma} [(\sqrt{t^4 - 1})_+ - (\sqrt{t^4 - 1})_-]\, dt = \oint_{\Gamma_1 \cup \Gamma_2} \sqrt{z^4 - 1}\, dz,$$

where Γ_1 and Γ_2 are closed contours, respectively encircling the curve $[-1, 1]$ and γ once in a clockwise direction. Deform the contour to pick up the residue at $z = \infty$, giving

$$I = \frac{1}{2} \oint_{\Gamma_1 \cup \Gamma_2} \sqrt{z^4 - 1}\, dz = \pm i\pi \operatorname{res}_\infty(\sqrt{z^4 - 1}) = 0.$$

On the other hand, $\Re \int_{-1}^{1} (\sqrt{t^4 - 1})_+\, dt = 0$, which implies $\Re \int_\gamma (\sqrt{t^4 - 1})_+\, dt = 0$.

The quadratic differential $\varphi(z)\, dz^2$ defines a φ-metric with the differential element $\sqrt{|\varphi(z)|}|dz|$. If γ is a rectifiable arc in $\overline{\mathbb{C}}$, then its φ-length is defined by

$$|\gamma|_\varphi = \int_\gamma \sqrt{|\varphi(z)|}|dz|.$$

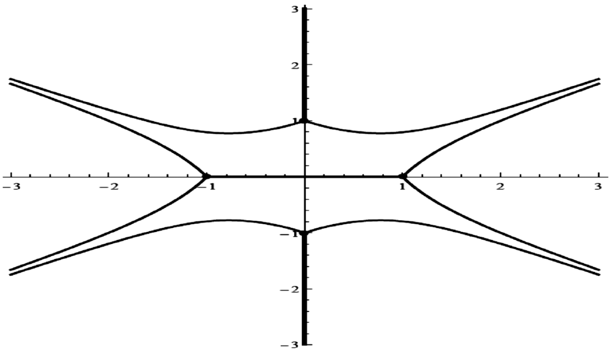

FIGURE 3. Critical graph of the quadratic differential $Q(z) = -(z^4 - 1)\, dz^2$.

A trajectory of $\varphi(z)\, dz^2$ is called finite if its φ-length is finite; otherwise it is infinite. In particular, a critical trajectory is finite if and only if both its end points are finite critical points.

The two Jordan arcs $\alpha, \beta : [0, 1] \to \mathbb{C}$ which join a point p_1 to a point p_2 in $\mathbb{C} \setminus \{\text{poles of } \varphi\}$ are called homotopic if there exists a continuous function H mapping $[0, 1] \times [0, 1] \to \mathbb{C} \setminus \{\text{poles of } \varphi\}$ such that

$$H(t, 0) = \alpha(t), \quad H(t, 1) = \beta(t), \quad t \in [0, 1].$$

Homotopy is an equivalence relation on the set $\mathcal{J}_{p_1,p_2}$ of all Jordan arcs joining p_1 to p_2 in $\mathbb{C} \setminus \{\text{poles of } \varphi\}$. If $|\{\text{poles of } \varphi\}| = m$, it is well known that $\mathbb{C} \setminus \{\text{poles of } \varphi\}$ and the wedge of m circles have the same homotopy type. In particular, there are 2^m homotopy equivalence classes on $\mathcal{J}_{p_1,p_2}$.

DEFINITION 2.2. A locally rectifiable (in the spherical metric) curve γ_0 is called a φ-geodesic if it is locally shortest in the φ-metric. It is called a critical geodesic it passes through a critical point of the quadratic differential $\varphi(z)\, dz^2$.

PROPOSITION 2.3 [11, Theorem 16.2]. *Let γ be a φ-geodesic arc joining p_1 to p_2 in $\mathbb{C} \setminus \{\text{poles of } \varphi\}$. For every $\gamma_1 \in \mathcal{J}_{p_1,p_2}$ which is homotopic to γ on $\mathbb{C} \setminus \{\text{poles of } \varphi\}$, $|\gamma_1|_\varphi \geq |\gamma|_\varphi$, with equality if and only if $\gamma_1 = \gamma$.*

We finish this section with the so-called Teichmuller lemma.

DEFINITION 2.4. A domain in $\mathbb{C}$ bounded by segments of φ-geodesics and/or horizontal and/or vertical trajectories of the quadratic differential $\varphi(z)\, dz^2$ (and their endpoints) is called a φ-polygon.

LEMMA 2.5 (Teichmuller). *Let Ω be a φ-polygon, let z_j be the singular points of $\varphi(z)\, dz^2$ on the boundary $\partial\Omega$ of Ω, with respective multiplicities n_j, and let $\theta_j \in [0, 2\pi]$ be the interior angle at the vertex z_j. Then*

$$\sum \left(1 - \theta_j \frac{n_j + 2}{2\pi}\right) = 2 + \sum n_i, \tag{2.2}$$

where the n_i are the multiplicities of the singular points inside Ω.

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3. Proofs

Lemma 3.1. *In the notation of Proposition 1.2:*

(a) *there exists at most one unbroken short trajectory of the quadratic differential $\varpi(P_t, z)$ connecting $a(t)$ and $b(t)$; and*
(b) *if there exist two short trajectories of the quadratic differential $\varpi(R_t, z)$ connecting $a(t)$ and $b(t)$, then they are not homotopic in $\mathbb{C} \setminus \{a_1, a_2, a_3\}$.*

Proof. (a) Suppose that γ_1 and γ_2 are two unbroken short trajectories of $\varpi(P_t, z)$ connecting $a(t)$ and $b(t)$ and let Ω be the ϖ-polygon with vertices $a(t)$ and $b(t)$ and edges γ_1 and γ_2. From Lemma 2.5, the left-hand side of (2.2) is smaller than two, whereas the right-hand side is clearly at least two, which is a contradiction.

(b) Let γ_1 and γ_2 to be two short trajectories of $\varpi(R_t, z)$ connecting $a(t)$ and $b(t)$. If they are homotopic in $\mathbb{C} \setminus \{a_1, a_2, a_3\}$, then there is no pole of R_t inside Ω and, again, we reach a contradiction, by Lemma 2.5. □

Remark 3.2. The number of unbroken short geodesics of $\varpi_P(t, z)$ can be any integer between $\deg(P_t(z)) - 1$ and $\binom{\deg(P_t(z))}{2}$. We refer the reader to [9] for the proof.

Remark 3.3. Note that $\mathcal{J}_{a(t),b(t)}$ consists of Jordan curves. One can easily observe that there exist eight homotopy classes of curves in $\mathcal{J}_{a(t),b(t)}$. Using the same approach as in the proof of Lemma 3.1 together with Proposition 2.3, we see that there exist at most eight unbroken short geodesics of $\varpi(R_t, z)$ joining $a(t)$ and $b(t)$.

Proof of Proposition 1.1. Let Λ be the subset of Ω of all t for which there exists a short trajectory of $\varpi(R_t, z)$ homotopic to ϕ_t in $\mathbb{C} \setminus \{a_1, a_2, a_3\}$.

Assume that Λ is nonempty and choose $t_0 \in \Lambda$. By the continuity of the quadratic differential $\varpi(R_t, z)$, for every $\varepsilon > 0$ there exists $\delta > 0$ such that, for any $t \in \Omega$ with $|t - t_0| < \delta$, there exists a trajectory of $\varpi(R_t, z)$, say, γ_t, emanating from $a(t)$ and intersecting the ε-neighbourhood $\mathcal{U}_\varepsilon$ of $b(t)$. If γ_t does not pass through $b(t)$, we may assume that $\delta > 0$ is small enough so that γ_t intersects an orthogonal trajectory σ_t emanating from $b(t)$ at some point $c(t)$. Denote by φ_t the path that follows the arc of γ_t from $a(t)$ to $c(t)$ and then continues to $b(t)$ along σ_t. Clearly, the arcs ϕ_t and φ_t are homotopic in $\mathbb{C} \setminus \{a_1, a_2, a_3\}$ and, by the definition of orthogonal trajectories, the real part of the integral along φ_t of $\sqrt{R_t(z)}$ cannot vanish. This contradiction shows that there is a whole small neighbourhood of t_0 in Λ and so Λ is an open subset of Ω.

Suppose now that (t_n) is a sequence of points in Λ converging to $t \in \Omega$, so that $a(t_n)$ and $b(t_n)$ converge, respectively, to $a(t)$ and $b(t)$. For each t_n, there exists a unique short trajectory γ_n joining $a(t_n)$ and $b(t_n)$ and each γ_n is homotopic to ϕ_{t_n} in $\mathbb{C} \setminus \{a_1, a_2, a_3\}$. It is obvious that the limit set of the sequence γ_n (in the Hausdorff metric) is either another short trajectory connecting $a(t)$ and $b(t)$ or a union of two infinite critical trajectories γ_a and γ_b emanating, respectively, from $a(t)$ and $b(t)$. Each of these trajectories diverges to some pole of the quadratic differential $\varpi(R_t, z)$. If γ_a and γ_b do not diverge to the same pole, or if one of them diverges to a simple pole, then

$$\inf_{x\in\gamma_a,\, y\in\gamma_b} |x-y| = \operatorname{dist}(\gamma_a, \gamma_b) > 0,$$

which contradicts the fact that $\lim_{n\to\infty} \gamma_n = \gamma_a \cup \gamma_b$.

Now let $c \in \{a_1, a_2, a_3\} \cup \{\infty\}$ be a double pole which is the common pole of divergence of γ_a and γ_b. We assume, first, that the residue of the quadratic differential $\varpi(R_t, z)$ at the pole c is not real, so that γ_a and γ_b diverge to c as a log-spiral. Let σ be an orthogonal trajectory that diverges (also in log-spiral form) to c. Then σ intersects γ_a and γ_b an infinite number of times. Considering three consecutive points of intersection, it is obvious that we can construct two paths, γ and γ', joining $a(t)$ and $b(t)$, formed by three parts from γ_a, σ and γ_b. Clearly, γ and γ' are not homotopic in $\mathbb{C} \setminus \{c\}$ and, by continuity of the family ϕ_{t_n}, one of them must be homotopic to ϕ_{t_n} for all $n \geq n_0$ sufficiently large. But then

$$\Re \int_{\gamma} \sqrt{R_t(z)}\, dz \neq 0 \quad \text{and} \quad \Re \int_{\gamma'} \sqrt{R_t(z)}\, dz \neq 0,$$

which contradicts (1.3). Thus, the limit set of the sequence γ_n is a short trajectory joining $a(t)$ and $b(t)$ which implies that Λ is a closed subset of Ω. The cases when the residues at c are real (positive or negative) can be treated in the same way.

Finally, since Ω is a connected subset of $\mathbb{C}$ which is both open and closed in $\mathbb{C}$, either $\Lambda = \Omega$ or $\Lambda = \emptyset$.

Proof of Proposition 1.2. Denote by $\Gamma_{a(t)}$ and $\Gamma_{b(t)}$ the union of the three critical trajectories that emanate, respectively, from $a(t)$ and $b(t)$ and consider the Euclidean distance

$$\operatorname{dist}(\Gamma_{a(t)}, \Gamma_{b(t)}) = \inf_{x\in\Gamma_{a(t)},\, y\in\Gamma_{b(t)}} |x-y|.$$

We claim that the quadratic differential $\varpi(P_t, z)$ has a short trajectory connecting $a(t)$ and $b(t)$ if and only if $\operatorname{dist}(\Gamma_{a(t)}, \Gamma_{b(t)}) = 0$. Indeed, there are $n+2$ asymptotic directions, equally spaced at angle $2\pi/(n+2)$, along which almost any horizontal (respectively, vertical) trajectory of the quadratic differential $\varpi(P_t, z)$ diverges to infinity. The asymptotic directions of the vertical trajectories are obtained from those of the horizontal trajectories by a rotation through an angle $\pi/2$. Obviously, if $\operatorname{dist}(\Gamma_{a(t)}, \Gamma_{b(t)}) > 0$, then there is no short trajectory connecting $a(t)$ and $b(t)$. Assume that $\operatorname{dist}(\Gamma_{a(t)}, \Gamma_{b(t)}) = 0$ and there is no short trajectory connecting $a(t)$ and $b(t)$. Since $\Gamma_{a(t)} \cap \Gamma_{b(t)} = \emptyset$, there exist two horizontal trajectories $\gamma_{a(t)}$ and $\gamma_{b(t)}$ that emanate from $a(t)$ and $b(t)$ and diverge to infinity in the same direction D; let σ be a vertical trajectory (not critical) diverging to infinity in the two directions adjacent to D. Obviously, σ intersects $\gamma_{a(t)}$ and $\gamma_{b(t)}$ at exactly two points, $P_{a(t)}$ and $P_{b(t)}$. Let $\gamma \in \mathcal{J}_{a(t),b(t)}$ be the union of the part of $\gamma_{a(t)}$ from $a(t)$ to $P_{a(t)}$, the part of σ from $P_{a(t)}$ to $P_{b(t)}$ and the part of $\gamma_{b(t)}$ from $P_{b(t)}$ to $b(t)$. Integrating along γ and using

$$\Re \int_{a(t)}^{P_{a(t)}} \sqrt{P_t(z)}\, dz = \Re \int_{P_{b(t)}}^{b(t)} \sqrt{P_t(z)}\, dz = 0,$$

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we find that

$$\Re \int_\gamma \sqrt{P_t(z)}\, dz = \Re \int_{P_{a(t)}}^{P_{b(t)}} \sqrt{P_t(z)}\, dz \neq 0,$$

which violates (1.4). By continuity of the function $t \mapsto \mathrm{dist}(\Gamma_{a(t)}, \Gamma_{b(t)})$, it follows that the set of all $t \in \Omega$ such that the quadratic differential $\varpi(P_t, z)$ has no short trajectory connecting $a(t)$ and $b(t)$ is an open subset of Ω.

Notice that Proposition 1.2 remains valid for polynomials Q of higher degree or with larger multiplicities of the zeros.

4. Connection with Laguerre and Jacobi polynomials

The rescaled generalised Laguerre polynomials $L_n^{nC}(nz)$ with varying parameter nC and the Jacobi polynomials $P_n^{(nA,nB)}(z)$ with varying parameters nA and nB are given explicitly by (see [12])

$$L_n^{nC}(nz) = \sum_{k=0}^{n} \binom{n+nC}{n-k} \frac{(-z)^k}{k!},$$

$$P_n^{(nA,nB)}(z) = 2^{-n} \sum_{k=0}^{n} \binom{n+nA}{n-k}\binom{n+nB}{k} (z-1)^k (z+1)^{n-k}.$$

Jacobi or Laguerre polynomials with (real) parameters, depending on the degree n appear naturally as polynomial solutions of hypergeometric differential equations or in the expressions of the wave functions of many classical systems in quantum mechanics (see [2]).

With each polynomial p_n, we associate its normalised zero-counting measure μ_n,

$$\mu_n = \mu(p_n) = \frac{\sum_{p_n(z)=0} \delta_z}{n}.$$

For a compact subset K in $\mathbb{C}$,

$$\int_K d\mu_n = \frac{\text{number of zeros of } p_n \text{ in } K}{n},$$

where the zeros are counted with their multiplicities.

Following Gonchar–Rakhmanov [3] and Stahl [10], it was shown that the sequence μ_n converges (as $n \to \infty$) in the weak-* topology to a measure, supported on short trajectories of related quadratic differentials (for the case of Laguerre polynomials, see [1, 6, 7], and for the case of Jacobi polynomials, see [4, 5, 8]).

The related quadratic differential for Laguerre polynomials is

$$\varpi_C = -\frac{D_C(z)}{z^2}\, dz^2, \tag{4.1}$$

where

$$D_C(z) = z^2 - 2(C+2)z + C^2.$$

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The zeros of $D_C(z)$ are

$$a(C) = C + 2 + 2\sqrt{C+1}, \quad b(C) = C + 2 - 2\sqrt{C+1}.$$

The related quadratic differential for Jacobi polynomials is

$$\varpi_{A,B} = -\frac{D_{A,B}(z)}{(z^2-1)^2}\, dz^2, \tag{4.2}$$

where

$$D_{A,B}(z) = (A+B+2)^2 z^2 + 2(A^2 - B^2)z + (A-B)^2 - 4(A+B+1).$$

The zeros of $D_{A,B}(z)$ are

$$a(A,B) = \frac{-A^2 + B^2 + 4\sqrt{(A+1)(B+1)(A+B+1)}}{(A+B+2)^2},$$

$$b(A,B) = \frac{-A^2 + B^2 - 4\sqrt{(A+1)(B+1)(A+B+1)}}{(A+B+2)^2}.$$

Proposition 4.1 [1]. *Assume that $C \in \mathbb{C}_+$, and that γ is a Jordan arc connecting the zeros of $D_C(z)$ in the punctured plane $\mathbb{C} \setminus \{0\}$. Denote by $\sqrt{D_C(z)}$ the single-valued branch of this function in $\mathbb{C} \setminus \gamma$ determined by the condition*

$$\sqrt{D_C(z)} \sim z, \quad z \to \infty,$$

and let $(\sqrt{D_C(z)})_+$ stand for its boundary values on the $+$-side of γ. Then

$$\int_\gamma \frac{(\sqrt{D_C(t)})_+}{t}\, dt \in \pm 2\pi i\{1, C+1\}. \tag{4.3}$$

Moreover, the integral on the left-hand side of (4.3) takes the value $\pm 2\pi i$ if and only if γ is such that it can be continuously deformed in $\mathbb{C} \setminus \{0\}$ to an arc not intersecting the positive real axis.

Write $\Omega = \{C \in \mathbb{C} : \Im C \geq 0\}$ and $R_C(z) = -D_C(z)/z^2$. Then conditions (1.1), (1.2) and (1.3) are fulfilled. For $C \in (-1, +\infty)$, the zeros $a(C)$ and $b(C)$ satisfy

$$0 < b(C) < a(C)$$

and the segment $[b(C), a(C)]$ is a short trajectory of the quadratic differential (4.1) (see Figure 4). The short trajectory exists for any $C \in \Omega$.

Proposition 4.2 [4, 8]. *Assume that A, B satisfy*

$$A + 1 \neq 0, \quad B + 1 \neq 0, \quad A + B + 1 \neq 0, \quad A + B + 2 \neq 0, \tag{4.4}$$

that γ is a Jordan arc in $\mathbb{C} \setminus \{-1, 1\}$ joining the zeros of $D_{A,B}$ and that $\sqrt{D_{A,B}}$ is the single-valued branch in $\mathbb{C} \setminus \gamma$ fixed by the condition

$$\sqrt{D_{A,B}}(z) \sim (A+B+2)z, \quad z \to \infty.$$

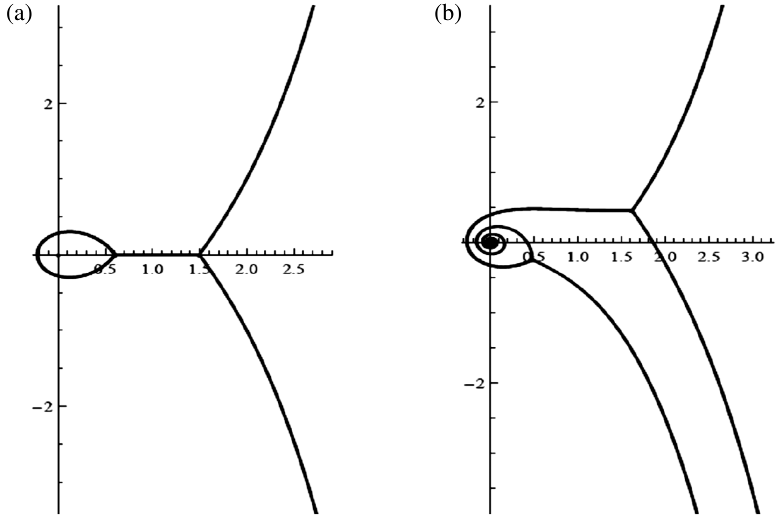

FIGURE 4. Critical graphs of $\varpi_{-0.95}$ (a) and $\varpi_{-0.95+0.1i}$ (b).

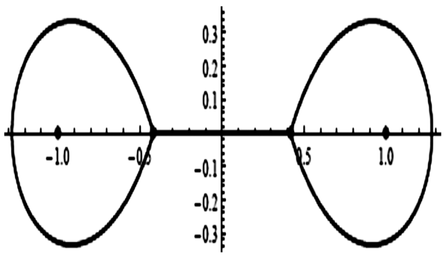

FIGURE 5. Critical graph of $\varpi_{10,10}$.

Then

$$\int_\gamma \frac{(\sqrt{D_{A,B}(t)})_+}{t^2-1}\,dt \in \pm 2\pi i\{1, (A+1), (B+1), (A+B+1)\}, \tag{4.5}$$

where $(\sqrt{D_{A,B}(t)})_+$ *is the boundary value on one of the sides of* γ.

Moreover, if, in addition to (4.4), $B > 0$*, then the integral on the left-hand side of (4.5) takes the value* $\pm 2\pi i$ *if and only if* γ *is such that*

$$\sqrt{D_{A,B}}(1) = 2A \quad \text{and} \quad \sqrt{D_{A,B}}(-1) = -2B.$$

For $B > -1$, write $\Omega = \{A \in \mathbb{C} : A+1 \neq 0, A+B+1 \neq 0, A+B+2 \neq 0\}$ and $R_A(z) = -D_{A,B}(z)/(z^2-1)^2$. Then conditions (1.1)–(1.3) are satisfied. Since, for $A \in \mathbb{R} \cap \Omega$, there exists a short trajectory of the quadratic differential (4.2), the short trajectory exists for any $A \in \Omega$. By repeating the reasoning, we reach the conclusion for any A and B satisfying (4.4) (see Figures 5 and 6).

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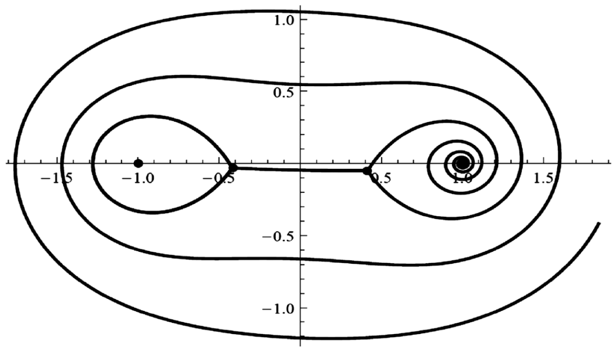

FIGURE 6. Critical graph of $\varpi_{10+i,10}$.

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