NORMAL FITTING CLASSES
AND HALL SUBGROUPS

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It was shown by Bryce and Cossey that each Hall \( \pi \)-subgroup of a
group in the smallest normal Fitting class \( S_\pi \) necessarily lies
in \( S_\pi \), for each set of primes \( \pi \). We prove here that for each
set of primes \( \pi \) such that \( |\pi| \geq 2 \) and \( \pi' \) is not empty,
there exists a normal Fitting class without this closure
property. A characterisation is obtained of all normal Fitting
classes which do have this property.

Let \( F \) be a normal Fitting class closed under taking Hall
\( \pi \)-subgroups, in the sense of the paragraph above, and let \( S_\pi \)
denote the Fitting class of all finite soluble \( \pi \)-groups, for
some set of primes \( \pi \). The second main theorem is a
characterisation of the groups in the smallest Fitting class
containing \( F \) and \( S_\pi \) in terms of their Hall \( \pi \)-subgroups.

1. Introduction

Let \( F \) be a normal Fitting class of finite soluble groups and \( \pi \) a
set of primes. \( F \) is said to be closed under taking Hall \( \pi \)-subgroups if
each group in \( F \) possesses a Hall \( \pi \)-subgroup which lies in \( F \). Since
every normal Fitting class contains all finite nilpotent groups [3, Theorem
5.1], we avoid triviality by assuming that \( |\pi| \geq 2 \) and that \( \pi' \) is not

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empty. Bryce and Cossey showed that the smallest normal Fitting class is closed under taking Hall $\pi$-subgroups, for each set of primes $\pi$ [6, 4.15]. This fact can be more easily deduced from a result of Hauck [8, Chapter 6]. In Section 3 of this paper, we prove the following result.

**Theorem 1.** Let $\pi$ be a set of primes such that $|\pi| \geq 2$ and $\pi'$ is not empty. Then there exists a normal Fitting class which is not closed under taking Hall $\pi$-subgroups.

The concept of the join of two Fitting classes was introduced in [7]. The join of Fitting classes $X$ and $Y$ is defined to be the smallest Fitting class containing their union. For each set of primes $\pi$, let $S_{\pi}$ denote the Fitting class of all finite soluble $\pi$-groups, and recall that a subgroup $N$ of the direct product $G \times H$ of groups $G$ and $H$ is said to be subdirect in $G \times H$ if $N(1 \times H) = G \times H = (G \times 1)N$. Our second main result is proved in Section 4.

**Theorem 2.** Let $\pi$ be a set of primes and $F$ a normal Fitting class closed under taking Hall $\pi$-subgroups. Let $H$ be a Hall $\pi$-subgroup of a group $G$. Then $G$ lies in $S_{\pi} \vee F$ if and only if $(G \times H)_F$ is subdirect in $G \times H$.

That many normal Fitting classes are closed under taking Hall $\pi$-subgroups for a given set of primes $\pi$ is ensured by the characterisation of these Fitting classes obtained in Theorem 5 of Section 3.

2. Preliminaries

All groups mentioned are finite and soluble. Basic definitions and facts concerning Fitting classes and the $^*$-operation may be found in [3] and [10]. The notation is standard and is described in [7]. We point out that as a consequence of [10, Theorem 2.2c]), the normal Fitting class $S_\pi$ is contained in every normal Fitting class. We list the following results for the reader's convenience.

I [7, Corollary 2.6]. Let $X$ and $Y$ be Fitting classes such that $X \subseteq Y^*$. Then a group $G$ lies in $X \vee Y$ if and only if there exists a group $K$ in $X$ such that $(G \times K)^Y$ is subdirect in $G \times K$.

When $X = S_{\pi}$, for a set of primes $\pi$, and $Y$ is a normal Fitting
class closed under taking Hall $\pi$-subgroups, Theorem 2 will allow us to
dispens with the arbitrary choice of the group $K$ in I. The next result
can be deduced from I and Theorem 2.9 of [7].

II. Let $X$, $Y$ and $Z$ be Fitting classes such that $X \subseteq Y^*$. 
1. If $X \subseteq Z$, then $(X \vee Y) \cap Z = X \vee (Y \cap Z)$.
2. If $Y \subseteq Z$, then $(X \vee Y) \cap Z = (X \cap Z) \vee Y$.

We now introduce a notation of Hauck [8]. Let $F$ be a Fitting class
and $\pi$ a set of primes. Then $Y(S_{\pi}, F)$ denotes the Fitting class of
groups in which each Hall $\pi$-subgroup lies in $F$. The following theorem
is a consequence of Hilfssatz 3 of [1].

III. $Y(S_{\pi}, F)$ is a normal Fitting class, for each set of primes $\pi$
and normal Fitting class $F$.

Finally, we have a theorem collated from various sources, which will
be crucial to the proof of Theorem 1.

IV. Let $p$ and $q$ be distinct primes. There exists a group
$H(p, q)$ such that $O_p(H(p, q)) = H(p, q)^{S_{\pi}}_{S_{\pi}}$ and $|H(p, q)/H(p, q)^{S_{\pi}}_{S_{\pi}}| = q$.

If $q \mid p-1$, then the existence of $H(p, q)$ is established in [?].
The existence of $H(p, q)$ when $q \mid p-1$ is a consequence of the main
theorems of [5] and [9]. Details of the construction of a suitable group
$H(p, q)$ may be found in [4, Chapter 3.7].

3. Normal Fitting classes closed under taking Hall $\pi$-subgroups

Let $\pi$ be a non-trivial set of primes, in the sense of Theorem 1.
Choose distinct primes $p$, $q$ and $r$ such that $p$ and $q$ are in $\pi$, and
$r$ is in $\pi'$. Set $K = H(p, q)$, $L = H(r, q)$ and denote by $G$ the
normal subgroup $(K^{S_{\pi}} \times L^{S_{\pi}})\langle (k, l) \rangle$ of $K \times L$, where $k$ and $l$ are
elements of order $q$ in $K$ and $L$ respectively. Set $F = \text{Fit}(G) \vee S_{\pi}$.

Certainly $F$ is a normal Fitting class, since $S_{\pi} \subseteq F \subseteq S$ [10].

Proof of Theorem 1. The candidate is $F$. Since $G$ lies in $F$ and
each Hall $\pi$-subgroup of $G$ is isomorphic to $K$ it is sufficient to prove
that $K$ is not in $F$. We begin by examining $G$. 

If $G$ is in $S_*$, then $G \leq (K \times L)_{S_*}$ . This implies that $K \times L = (K \times 1)(K \times L)_{S_*}$ and it follows from the definition of a Fitting class that $L$ lies in $S_{\pi} \vee S_*$ . Let $Q$ be a Sylow $q$ -subgroup of $L$ . Certainly $Q$ is a Hall $\pi$ -subgroup of $L$ , and so by Theorem 2, $(L \times Q)_{S_*}$ is subdirect in $L \times Q$ . Since $Q$ is nilpotent, $Q$ lies in $S_*$ , which leads to the contradiction that $L$ is in $S_*$ . We conclude that $G$ does not lie in $S_*$ , and consequently that $G_{S_*} = (K \times L)_{S_*}$ .

Suppose now that $K$ lies in $F$ . By [7, Corollary 2.5], $G$ possesses a characteristic subgroup $N$ such that $(K \times N)_{S_*}$ is subdirect in $K \times N$ . If $N \leq G_{S_*}$ , then $K$ must lie in $S_*$ , a contradiction. We may therefore assume that $NG_{S_*} = G$ . It follows that $(K \times G)_{S_*}$ is subdirect in $K \times G$ . There exists, therefore, an element $x$ of $G$ , of order $q$ , such that $(k, x)$ is an element of $(K \times G)_{S_*}$ . Each element of order $q$ in $G$ is a conjugate of $(k, 1)^n$ , for some integer $n$ lying between 1 and $q$ . Since $G/G_{S_*}$ is abelian, we have

$xG_{S_*} = (k, 1)^nG_{S_*}$ , for some integer $n$ . The fact that $G$ is a normal subgroup of $K \times L$ now establishes that $(k, k^n, l^n)$ is an element of $(K \times K \times L)_{S_*}$ . By definition of the $\ast$ -operation [10],

$$(K \times K \times L)_{S_*} = (K_{S_*} \times K_{S_*} \times 1)(g^{-1}, g, 1) \mid g \in K .$$

We therefore have

$$(1, k^{n+1}, l^n) = (k^{-1}, k, 1)(k, k^n, l^n) \in (K \times K \times L)_{S_*} .$$

Certainly $(1, k^{n+1}, l^n)$ is an element of $1 \times K \times L$ , and so $(k^{n+1}, l^n) \in (K \times L)_{S_*}$ . Since $(K \times L)_{S_*} = K_{S_*} \times L_{S_*}$ , the choice of $k$ and $l$ implies that $q$ divides both $n$ and $n + 1$ . This contradiction leads us to conclude that $K$ does not lie in $F$ .

The characterisation of those normal Fitting classes closed under taking Hall $\pi$ -subgroups depends on the following two results.
**Lemma 3.** Let \( \pi \) be a set of primes and \( F \) a normal Fitting class which is closed under taking Hall \( \pi \)-subgroups. Let \( H \) be a Hall \( \pi \)-subgroup of a group \( G \) in \( FS_\pi \). Then \( G \) lies in \( Y(S_\pi, F) \) if and only if \( G = H^F_G \).

**Proof.** Suppose that \( G \) is in \( Y(S_\pi, F) \). Certainly \( G = H^F_G \), and \( H = H^F_F \). It is immediate that \( G = H^F_G \). Conversely, suppose that \( G = H^F_G \). Then \( H = H^F_F \) \( (H \cap G_F) \), and by hypothesis \( H \cap G_F \) lies in \( F \). Thus \( H = H^F_F \), ensuring that \( G \) is in \( Y(S_\pi, F) \).

**Theorem 4.** \( S_\pi \cup Y(S_\pi, S_A) = S \), for each set of primes \( \pi \).

**Proof.** Let \( H \) be a Hall \( \pi \)-subgroup and \( K \) a Hall \( \pi' \)-subgroup of a group \( G \) in \( S_A S_\pi \). Then \( G = H S_A \), and so \( H \leq G S_A \). Since \( S_A \) is closed under taking Hall \( \pi \)-subgroups, this ensures that \( H \) lies in \( S_A \). Thus \( S_A S_\pi \), is contained in \( Y(S_\pi, S_A) \).

Suppose now that \( H \) is a Hall \( \pi \)-subgroup of a group \( G \) in \( S_A S_\pi \). Then \( G \times H \) is in \( S_A S_\pi \), and it follows from Lemma 3 that \( (H \times H) S_A \) \( (G \times H) S_A \) is the \( Y(S_\pi, S_A) \)-radical of \( G \times H \). Since \( (H \times H) S_A \) \( (G \times H) S_A \) contains \( (H S_A G S_A \times H S_A \) \( (h^{-1}, h) \) | \( h \in H \) \), and \( G = H S_A \), the \( Y(S_\pi, S_A) \)-radical of \( G \times H \) is subdirect in \( G \times H \). We conclude from I that \( S_A S_\pi \) is contained in \( S_\pi \cup Y(S_\pi, S_A) \). It follows from [1, Theorem 2.1] that \( S_A S_\pi \cup S_A S_\pi = S \), and consequently \( S_\pi \cup Y(S_\pi, S_A) = S \).

**Theorem 5.** Let \( F \) be a normal Fitting class and \( \pi \) a set of primes. Then \( F \) is closed under taking Hall \( \pi \)-subgroups if and only if \( F = (S_\pi \cap F) \cup (Y(S_\pi, S_A) \cap F) \).

**Proof.** If. Certainly \( S_\pi \cap F \) and \( Y(S_\pi, S_A) \) are contained in \( Y(S_\pi, F) \). Thus \( F \) is contained in \( Y(S_\pi, F) \), and so is closed under taking Hall \( \pi \)-subgroups.
ONLY IF. Suppose that $F$ is closed under taking Hall $\pi$-subgroups. In other words, $F \subseteq Y(S_\pi, F)$. Since $Y(S_\pi, S_\lambda) \subseteq Y(S_\pi, F)$, it follows from II and Theorem 4 that

$$Y(S_\pi, F) = (S_\pi \cup Y(S_\pi, S_\lambda)) \cap Y(S_\pi, F) = (S_\pi \cap Y(S_\pi, F)) \cup Y(S_\pi, S_\lambda) = (S_\pi \cap F) \cup Y(S_\pi, S_\lambda).$$

A further application of II yields that

$$F = F \cap Y(S_\pi, F) = F \cap ((S_\pi \cap F) \cup Y(S_\pi, S_\lambda)) = (S_\pi \cap F) \cup (Y(S_\pi, S_\lambda) \cap F).$$

4. The proof of Theorem 2

**Lemma 6.** Let $\pi$ be a set of primes and $F$ a normal Fitting class closed under taking Hall $\pi$-subgroups. Let $H$ be a Hall $\pi$-subgroup of a group $G$ in $S_\pi \cup F$. Then $G_P$ contains $H_P$.

Proof. Certainly $S_\pi \cup F \subseteq FS_\pi$, and so Lemma 3 implies that $H_P G_P$ is the $Y(S_\pi, F)$-radical of $G$. Since $F \subseteq Y(S_\pi, F)$, we may apply II to obtain $(S_\pi \cup F) \cap Y(S_\pi, F) = (S_\pi \cap Y(S_\pi, F)) \cup F = F$. Thus $H_P G_P$ lies in $F$, establishing the result.

Proof of Theorem 2. IF. This follows immediately from I.

ONLY IF. Both $F$ and $S_\pi$ are contained in $FS_\pi$, so $S_\pi \cup F$ is contained in $FS_\pi$. Let $T$ denote the set of groups $G$ in $FS_\pi$ such that for some Hall $\pi$-subgroup $H$ of $G$, $(G \times H)_F$ is subdirect in $G \times H$. Since $F$ is closed under taking Hall $\pi$-subgroups, $F \subseteq T$, and by definition of the $\ast$-operation $S_\pi \subseteq T$. That $T \subseteq S_\pi \cup F$ is ensured by I, and it is thus sufficient to show that $T$ is a Fitting class.

Let $H$ be a Hall $\pi$-subgroup of a group $G$ in $T$. Certainly $G \times H$ is in $S_\pi \cup F$, and it follows from Lemma 6 that $(G \times H)_F$ contains $(H \times H)_F$. The definition of the $\ast$-operation, and the fact that $G = H G_P$, allow us to write $(G \times H)_F = (G_F \times H_P)(h^{-1}, h) \mid h \in H$. Suppose now that $N$ is a normal subgroup of $G$. Then $N = (N \cap H)N_P$ and

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\[(N \times (N \cap H))_F = (N \times (N \cap H)) \cap \left( (G_F \times H_F) \langle h^{-1}, h \mid h \in H \rangle \right) \]
\[= (N_F \times (N \cap H)_F) \langle h^{-1}, h \mid h \in N \cap H \rangle. \]

Thus \(N\) lies in \(T\).

If \(N\) and \(M\) are normal subgroups, and \(H\) is a Hall \(\pi\)-subgroup, of a group \(G\), such that \(N\) and \(M\) are in \(T\) and \(G = NM\), then certainly \(H = (H \cap N)(H \cap M)\). Let \(h\) be an element of \(H\). Then there exist elements \(n\) of \(N\) and \(m\) of \(M\) such that \(h = nm\). By hypothesis, \((n^{-1}, n) \in (N \times (N \cap H))_F\) and \((m^{-1}, m) \in (M \times (M \cap H))_F\). Since \(G/G_F\) is abelian, \(mn^{-1}m^{-1}n^{-1} \in G_F\). Thus
\[(h^{-1}, h) = (m^{-1}n^{-1}, nm) = (n^{-1}, n)(m^{-1}, m)(mn^{-1}m^{-1}, 1)(G \times H)_F,\]
ensuring that \(G\) lies in \(T\). This completes the proof that \(T\) is a Fitting class.

References


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