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A Representation Theorem for Archimedean Quadratic Modules on $*$-Rings

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Abstract. We present a new approach to noncommutative real algebraic geometry based on the representation theory of C^*-algebras. An important result in commutative real algebraic geometry is Jacobi's representation theorem for archimedean quadratic modules on commutative rings. We show that this theorem is a consequence of the Gelfand–Naimark representation theorem for commutative C^*-algebras. A noncommutative version of Gelfand–Naimark theory was studied by I. Fujimoto. We use his results to generalize Jacobi's theorem to associative rings with involution.

1 Introduction

Jacobi's representation theorem [13, Theorem 5] is important in the study of positive polynomials on compact semialgebraic sets. Its history and applications are surveyed in [25]. We will give a functional-analytic proof of this theorem and extend it from commutative rings to noncommutative $*$-rings. Our motivation comes from noncommutative real algebraic geometry; see [20]. We hope that this paper will convince the reader that irreducible $*$-representations should be considered as points of this geometry. The problem of extending the Positivstellensatz to this context remains open.

Our work may also be of some interest to functional analysts. In Section 3 we characterize real C^*-algebras within the class

$$\mathcal{M} = \{(A, M) \colon M \text{ is an } m\text{-admissible wedge on an involutive ring } A\}$$

and extend the notion of an enveloping C^*-algebra from the subclass of Banach $*$-algebras to $\mathcal{M}$. In Section 5 we state and prove the real version of Fujimoto's CP-convexity Gelfand-Naimark theorem [12].

As a motivation for later sections we present our version of Jacobi's representation theorem for the special case of *commutative* $*$-rings. Let R be a commutative unital ring with involution $*$, write

$$\mathrm{Sym}(R) = \{a \in R \colon a = a^*\} \quad \text{and} \quad R^+ = \left\{ \textstyle\sum_i a_i a_i^* : a_i \in R \right\}.$$

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A subset M of $\mathrm{Sym}(R)$ is an *archimedean quadratic module* if $-1 \notin M$, $1 \in M$, $M + M \subseteq M$, $R^+M \in M$, and for every $a \in \mathrm{Sym}(R)$ there exists $n \in \mathbb{N}$ such that $n \pm a \in M$. Write

$$\mathrm{Arch}(M) = \{a \in \mathrm{Sym}(R) \colon \forall n \in \mathbb{N}\ \exists k \in \mathbb{N} \colon k(1 + na) \in M\}.$$

The conjugation $\phi \mapsto \overline{\phi}$, $\overline{\phi}(a) = \overline{\phi(a)}$, is an automorphism of order 2 on

$$X_M = \{\phi \colon R \to \mathbb{C} \colon \phi \text{ is a } *\text{-ring homomorphism such that } \phi(M) \geq 0\}.$$

We equip X_M with the topology of pointwise convergence. Finally, let

$$\mathcal{C}(X_M, -) = \{f \in \mathcal{C}(X_M, \mathbb{C}) \colon f(\overline{\phi}) = \overline{f(\phi)} \text{ for every } \phi \in X_M\}$$

with the natural involution $f \mapsto f^*$, $f^*(\phi) = f(\overline{\phi})$. In Jacobi's original theorem, $*$ was identity, and hence $\mathcal{C}(X_M, -) = \mathcal{C}(X_M, \mathbb{R})$, and all elements of X_M are real valued.

Theorem 1.1 *Let M be an archimedean quadratic module on a commutative unital $*$-ring R. Then the space X_M is nonempty and compact. Moreover, the mapping*

$$\Phi \colon R \to \mathcal{C}(X_M, -), \quad \Phi(a)(\phi) = \phi(a),$$

is a homomorphism of unital $*$-rings, $\mathbb{Q} \cdot \Phi(R)$ is dense in $\mathcal{C}(X_M, -)$, and

$$\Phi^{-1}(\mathcal{C}^+(X_M, -)) = \mathrm{Arch}(M).$$

Proof Let R and M be as above. For every $a \in R$ write

$$\mathrm{n}_M(a) = \inf\Big\{ \frac{r}{s} \colon r, s \in \mathbb{N},\ r^2 - s^2 aa^* \in M \Big\}.$$

We will prove in Section 3 that $\mathrm{I}(M) = \{a \in R \colon \mathrm{n}_M(a) = 0\}$ is a $*$-ideal of R and that n_M induces a norm on $R/\mathrm{I}(M)$. Moreover, the completion R_M of $R/\mathrm{I}(M)$ in this norm is an abelian real C^*-algebra. Also, the canonical mapping $j \colon R \to R_M$ is a homomorphism of $*$-rings and $j^{-1}((R_M)^+) = \mathrm{Arch}(M)$.

Let Y_M be the set of all real $*$-algebra homomorphisms $R_M \to \mathbb{C}$ with the topology of pointwise convergence. We will see in Section 4 that the mapping $Y_M \to X_M$, $\psi \mapsto \psi \circ j$ has an inverse $r \colon X_M \to Y_M$, which factors an element of X_M through $R/\mathrm{I}(M)$ and extends it by continuity to an element of Y_M. The mapping r is a homeomorphism with respect to the topologies of pointwise convergence on X_M and Y_M and it commutes with the conjugations on X_M and Y_M. It induces a mapping $\tilde{r} \colon \mathcal{C}(Y_M, -) \to \mathcal{C}(X_M, -)$, $f \mapsto f \circ r$, which is one-to-one and onto, an isometry, and satisfies $\tilde{r}^{-1}(\mathcal{C}^+(X_M, -)) = \mathcal{C}^+(Y_M, -)$.

Note that Y_M coincides with the spectral space $\Omega(R_M)$; see [17, Definition 2.7.1, Theorem 5.2.10 and Theorem 3.2.3 (7) $\Rightarrow$ (4)]. Since $\Omega(R_M)$ is nonempty by [17, Theorem 2.7.3] and compact by [17, Theorem 2.7.2 (4)], so also are X_M and Y_M. The Gelfand transform $\Gamma \colon R_M \to \mathcal{C}(Y_M, -)$, $\Gamma(a)(\psi) = \psi(a)$ is a $*$-isomorphism by

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[17, Proposition 5.1.4] and satisfies $\Gamma^{-1}(\mathcal{C}^+(Y_M, -)) = (R_M)^+$ by [17, Proposition 5.2.2 (3) and Theorem 2.7.2 (4)].

The mapping Φ can be decomposed as $\Phi = i \circ \Gamma \circ \tilde{r}$. Since j, Γ, and $\tilde{r}$ are homomorphisms, so is Φ. Since $\mathbb{Q} \cdot j(R)$ is dense in R_M and Γ and $\tilde{r}$ are isometries, it follows that $\mathbb{Q} \cdot \Phi(R) = \tilde{r}(\Gamma(\mathbb{Q} \cdot j(R)))$ is dense in $\mathcal{C}(X_M, -)$. Since $\tilde{r}^{-1}(\mathcal{C}^+(X_M, -)) = \mathcal{C}^+(Y_M, -)$, $\Gamma^{-1}(\mathcal{C}^+(Y_M, -)) = (R_M)^+$ and $j^{-1}((R_M)^+) = \operatorname{Arch}(M)$, it follows that $\Phi^{-1}(\mathcal{C}^+(X_M, -)) = \operatorname{Arch}(M)$. ■

The main difference in the noncommutative case is that we replace homomorphisms by topologically irreducible representations on a Hilbert space of a sufficiently high dimension. A noncommutative version of Gelfand's theory is provided by Fujimoto's CP-convexity theory. In Section 6 we will compare our theory with the theory of $*$-orderings on $*$-rings. Recent generalizations of Jacobi's theorem by M. Marshall [21, Theorem 2.3] and I. Klep [16] are not considered here.

2 Quadratic Modules, Definition, and Examples

Let A be a unital ring with involution and $\operatorname{Sym}(A) = \{a \in A | a = a^*\}$. A subset $M \subset \operatorname{Sym} A$ is called a *quadratic module* if

(i) $-1 \notin M$,
(ii) $1 \in M$,
(iii) $M + M \subseteq M$,
(iv) $aMa^* \subseteq M$ for every $a \in A$.

In [27], the term *m-admissible wedge* is used. If $* =$ identity, then our definition coincides with the definition of a quadratic module in [24].

Write A^+ for the set of all finite sums $\sum_i a_i a_i^*$. This is consistent with the notation $\mathbb{Z}^+, \mathbb{Q}^+, \mathbb{R}^+, \mathbb{C}^+$. Clearly, $A^+ \subseteq M$ for every quadratic module M. Thus:

Lemma 2.1 *The following are equivalent:*

(i) $-1 \notin A^+$,
(ii) *A^+ is a quadratic module on A,*
(iii) *A has at least one quadratic module.*

A quadratic module M on A is *archimedean* if for every $a \in A$ there exist $n \in \mathbb{N}$ such that $n - aa^* \in M$.

Example 1 If $A = \mathbb{R}[X_1, \dots, X_n]$ with $* =$ identity, then $-1 \notin A^+$. The quadratic module A^+ is not archimedean. A quadratic module $M \subset A$ is archimedean if and only if there exists $m \in \mathbb{N}$ such that $m - \sum_{i=1}^n X_i^2 \in M$; see [19, Corollary 5.2.4].

Example 2 Let A be a real or complex Banach $*$-algebra. Then A^+ is an archimedean quadratic module on A.

Example 3 Let $A = k[G]$ where G is any group and k is $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$. For every element $a = \sum_i \alpha_i g_i \in A$, write

$$a^* = \sum_i \overline{\alpha}_i g_i^{-1}, \quad \|a\|_1 = \sum_i |\alpha_i|.$$

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Clearly, $a \mapsto a^*$ is an involution on A and $\|\cdot\|_1$ is a norm on the $*$-ring A. Since for every $a \in A$,

$$\|a\|_1^2 - aa^* = \sum_{i<j} |\alpha_i \alpha_j| \Big(1 - \frac{\alpha_i \overline{\alpha}_j}{|\alpha_i \alpha_j|} g_i g_j^{-1}\Big) \Big(1 - \frac{\alpha_i \overline{\alpha}_j}{|\alpha_i \alpha_j|} g_i g_j^{-1}\Big)^* \in M,$$

A^+ is an archimedean quadratic module on A.

Finally, we have several general constructions for producing new quadratic modules from old ones.

Example 4 For every quadratic module M on A and for every subset $S \subset \mathrm{Sym}(A)$ write

$$M_S := \Big\{ \sum_{i,j} a_{ij} c_i a_{ij}^* : a_{ij} \in A, c_i \in M \cup S \Big\}.$$

Note that $M(S)$ is a quadratic module if and only if $-1 \notin M_S$. In this case M_S is the smallest quadratic module which contains M and S.

Example 5 Let M be a quadratic module in A. Then

$$M^e := \{a \in A : ka \in M \text{ for some } k \in \mathbb{N}\}$$

is a quadratic module on A,

$$M \otimes \mathbb{Q}^+ := \Big\{ \sum_i m_i \otimes r_i : m_i \in M, r_i \in \mathbb{Q}^+ \Big\}$$

is a quadratic module on $A \otimes \mathbb{Q}$, and $(M \otimes \mathbb{Q}^+) \cap A = M^e$. This example shows that we may always assume without loss of generality that $\mathbb{Q} \subset A$ and $M = M^e$. (This works even if $(A, +)$ has nonzero torsion.)

Example 6 Let A be a unital $*$-ring. The complexification A° of A is the set $A \times A$ with the following operations:

(i) $(x, y) + (u, v) = (x + u, y + v)$,
(ii) $-(x, y) = (-x, -y)$,
(iii) $(x, y)(u, v) = (xu - yv, xv + yu)$,
(iv) $(x, y)^* = (x^*, -y^*)$.

Note that A° is also a unital $*$-ring with unit $(1, 0)$. The element $i = (0, 1)$ behaves as imaginary unit.

Let M be a quadratic module on A. Define

$$M^\circ := \Big\{ \sum_i (a_i, b_i)(m_i, 0)(a_i, b_i)^* : a_i, b_i \in A, m_i \in M \Big\}.$$

Note that M° is a quadratic module on A°.

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Example 7 Let A be a unital $*$-ring and $n \in \mathbb{N}$. The set $\mathrm{Mat}_n(A)$ of all $n \times n$ matrices with entries in A is a unital $*$-ring with involution $[a_{ij}]^* = [a_{ji}^*]$.

Let M be a quadratic module on A. We define

$$M_n := \Big\{ \sum_j \begin{bmatrix} a_{1j} \\ \vdots \\ a_{nj} \end{bmatrix} m_j \begin{bmatrix} a_{1j}^* & \cdots & a_{nj}^* \end{bmatrix} : m \in M, a_{ij} \in A \Big\}.$$

Clearly, M_n is a quadratic module on $\mathrm{Mat}_n(A)$.

3 The C^*-Algebra of an Archimedean Quadratic Module

From now on we assume that every $*$-ring is unital and contains $\mathbb{Q}$.

Lemma 3.1 *Let M be a quadratic module on a $*$-ring A. For every $c \in \mathrm{Sym}(A)$ and every $r \in \mathbb{Q}^+$ we have $r^2 - c^2 \in M$ if and only if $r \pm c \in M$.*

Proof If $r^2 - c^2 \in M$, then

$$r \pm c = \tfrac{1}{2r}\big((r \pm c)^2 + (r^2 - c^2)\big) \in M.$$

If $r \pm c \in M$, then

$$r^2 - c^2 = \tfrac{1}{2r}\big((r-c)(r+c)(r-c) + (r+c)(r-c)(r+c)\big) \in M.$$

■

For every element $a \in A$ write

$$\mathrm{n}_M(a) = \inf\{r \in \mathbb{Q}^+ : r^2 - aa^* \in M\}.$$

We use the convention $\inf \varnothing = \infty$.

Theorem 3.2 *Let M be a quadratic module on a $*$-ring A and $\mathrm{n} = \mathrm{n}_M$. For every $a, b \in A$ and every $t \in \mathbb{Q}$ we have*

(i) $\mathrm{n}(ta) = |t|\,\mathrm{n}(a)$,
(ii) $\mathrm{n}(a) = \mathrm{n}(a^*)$,
(iii) $\mathrm{n}(ab) \le \mathrm{n}(a)\,\mathrm{n}(b)$,
(iv) $\mathrm{n}(a+b) \le \mathrm{n}(a) + \mathrm{n}(b)$,
(v) $\mathrm{n}(aa^*) = \mathrm{n}(a)^2$,
(vi) $\mathrm{n}(a)^2 \le \mathrm{n}(aa^* + bb^*)$.

If there exists an element i in the center of A such that $i^* = -i$ and $i^2 = -1$, then assertion (i) *holds for every $t \in \mathbb{Q}(i)$.*

Proof Assertion (i) is trivial, and assertion (v) is a consequence of Lemma 3.1. To prove assertion (ii), it suffices to show that $\mathrm{n}(a^*) \le \mathrm{n}(a)$ for every $a \in A$. This is clear if $\mathrm{n}(a) = \infty$. Otherwise pick any $r \in \mathbb{Q}^+$ such that $\mathrm{n}(a) < r$. Since

$$\Big(\frac{r^2}{2}\Big)^2 - \Big(\frac{r^2}{2} - a^*a\Big)^2 = a^*(r^2 - aa^*)a \in M,$$

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it follows that $\frac{r^2}{2} \pm \left(\frac{r^2}{2} - a^*a\right) \in M$ by Lemma 3.1. Hence $\mathrm{n}(a^*) \leq r$.

Assertions (iii) and (iv) are true if either $\mathrm{n}(a) = \infty$ or $\mathrm{n}(b) = \infty$. Otherwise, pick any $r, s \in \mathbb{Q}^+$ such that $\mathrm{n}(a) < r$ and $\mathrm{n}(b) < s$. Since $r^2 - aa^* \in M$ and $s^2 - bb^* \in M$, it follows that

$$r^2s^2 - (ab)(ab)^* = s(r^2 - aa^*)s + a(s^2 - bb^*)a^* \in M,$$

so that $\mathrm{n}(ab) \leq rs$, proving (iii). Since $\mathrm{n}(ab^*) < rs$ and $\mathrm{n}(ba^*) < rs$ by assertions (ii) and (iii), we have that

$$\begin{aligned} 4r^2s^2 - (ab^* + ba^*)^2 = 2(r^2s^2 - ab^*ba^*) + 2(r^2s^2 - ba^*ab^*) \\ + (ab^* - ba^*)(ab^* - ba^*)^* \in M. \end{aligned}$$

As $2rs \pm (ab^* + ba^*) \in M$ by Lemma 3.1, we get

$$(r+s)^2 - (a \pm b)(a \pm b)^* = r^2 - aa^* + s^2 - bb^* + 2rs \pm (ab^* + ba^*) \in M.$$

So, $\mathrm{n}(a \pm b) \leq r + s$, proving (iv).

If $\mathrm{n}(aa^* + bb^*) < r$ for some r, then $r - aa^* - bb^* \in M$ by Lemma 3.1. Since $bb^* \in M$, it follows that $r - aa^* \in M$. Therefore $\mathrm{n}(a) \leq \sqrt{r}$, proving (vi). ∎

Let us say that an element $a \in A$ is *bounded* with respect to M if $\mathrm{n}_M(a) < \infty$, and *infinitesimal* with respect to M if $\mathrm{n}_M(a) = 0$. Write $\mathrm{B}(M)$ for the set of all bounded elements and $\mathrm{I}(M)$ for the set of all infinitesimal elements (of A with respect to M). Theorem 3.2 implies the following result.

Corollary 3.3 *Take A and M as above.*

$\mathrm{B}(M)$ *is a* $*$*-subring of A and* $\mathrm{I}(M)$ *is a two-sided* $*$*-ideal in* $\mathrm{B}(M)$.

The mapping n_M *induces a norm* $\|\cdot\|$ *on* $\mathrm{B}(M)/\,\mathrm{I}(M)$. *Denote by* A_M *the completion of* $\mathrm{B}(M)/\,\mathrm{I}(M)$ *with respect to this norm. Then* A_M *is a real* C^**-algebra.*

If there exists an element i in the center of A such that $i^* = -i$ *and* $i^2 = -1$, *then* A_M *is a complex* C^**-algebra.*

Property (vi) from Theorem 3.2 is very important in the theory of real C^*-algebras because a C^*-norm with this property extends to a C^*-norm on the complexification of the algebra; see [22]. The spectral and representation theories of such real C^*-algebras work as in the complex case; we refer to [4, 5] or [17].

Example 8 Let A be either a real C^*-algebra with the property $\|a\|^2 \leq \|aa^* + bb^*\|$ for all $a, b \in A$ or a complex C^*-algebra. If $M = A^+$ then

$$\|a\| = \mathrm{n}_M(a) \text{ for every } a \in A,$$

so that $A = A_M$. Namely, [5, Corollary 4.2.1.16] says that for every $x \in \mathrm{Sym}(A)$, $\|x\| \leq r$ if and only if $r1 \pm x \in A^+$ if and only if $\sigma(x) \in [-r, r]$.

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Example 9 If M° is as in Example 6, then $(A^\circ)_{M^\circ} \cong (A_M)^\circ$. If M_n is as in Example 7, then $\mathrm{Mat}_n(A)_{M_n} \cong \mathrm{Mat}_n(A_M)$. We omit the proofs because they are straightforward and because we will not use these results in the sequel.

For every archimedean quadratic module M on a $*$-ring A, the seminorm n_M defines a topology on A with basis $B(a,\epsilon) = \{b \in A : \mathrm{n}_M(b-a) < \epsilon\}$. Write $\mathrm{Arch}(M)$ for the n_M-closure of M and $\mathrm{Int}(M)$ for the n_M-interior of M in A.

Lemma 3.4 *Let A, M, n_M be as above and $x \in \mathrm{Sym}(A)$. The following properties of x are equivalent:*

(i) $x \in \mathrm{Arch}(M)$,
(ii) $\mathrm{n}_M(r-x) \le r$ *for some* $r \in \mathbb{Q}^{>0}$ *such that* $r \ge \mathrm{n}_M(x)$,
(iii) $r + x \in M$ *for every* $r \in \mathbb{Q}^{>0}$.

Similarly, the following properties of x are also equivalent:

(i) $x \in \mathrm{Int}(M)$,
(ii) $\mathrm{n}_M(r-x) < r$ *for some* $r \in \mathbb{Q}^{>0}$ *such that* $r \ge \mathrm{n}_M(x)$,
(iii) $x \in r + M$ *for some* $r \in \mathbb{Q}^{>0}$.

The following result is useful.

Theorem 3.5 *Let A, M be as above and denote by $j\colon A \to A_M$ the canonical mapping. For every $x \in \mathrm{Sym}(A)$,*

(i) $x \in \mathrm{Arch}(M)$ *if and only if* $j(x) \in (A_M)^+$,
(ii) $x \in \mathrm{Int}(M)$ *if and only if* $j(x) \in (A_M)^+ \cap \mathrm{inv}(A_M)$.

Proof To prove assertion (i), pick any $x \in \mathrm{Sym}(A)$. By Lemma 3.4, $x \in \mathrm{Arch}(M)$ if and only if $\mathrm{n}_M(r-x) \le r$ for some rational $r \ge \mathrm{n}_M(x)$. Since $\mathrm{n}_M(a) = \|j(a)\|$ for every $a \in A$, $\mathrm{n}_M(r-x) \le r$ is equivalent to $\|r - j(x)\| \le r$. By Example 8 and Lemma 3.4, this is equivalent to $j(x) \in (A_M)^+$. Assertion (ii) is similar. ∎

Theorem 3.5 implies the following generalization to C^*-algebras of Stone's famous characterization of rings of continuous functions [28].

Corollary 3.6 *Let M be a quadratic module on a $*$-ring A. Then A is a C^*-algebra with positive cone M if and only if*

(i) $M \cap -M = \{0\}$,
(ii) *M is archimedean,* i.e., $B(M) = A$,
(iii) $M = \mathrm{Arch}(M)$,
(iv) *A is complete in the norm* n_M.

The next two examples will follow from Theorem 4.4 in Section 4.

Example 10 If A is a real or complex unital Banach $*$-algebra and $M = A^+$ then A_M is exactly the C^*-enveloping of A; see [6, 2.7.2]. Namely, by assertion (i) of Theorem 4.4, n_M coincides with the C^*-seminorm $\|\cdot\|'$ in the sense of [6, Proposition 2.7.1].

Example 11 Let G be any group, and denote by $\mathbb{C}[G]$ its group ring and by $L^1(G)$ the completion of $\mathbb{C}[G]$ in the norm $\|\cdot\|_1$ of Example 3. Note that $L^1(G)$ is an

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involutive complex Banach algebra. Its enveloping C^*-algebra is denoted by $C^*(G)$ and called *the C^*-algebra of G*; see [6, Section 13.9]. If $A = \mathbb{C}[G]$ and $M = A^+$, then $A_M = C^*(G)$.

4 M-Positive Mappings

A *positive form* on a $*$-ring A is a mapping $f\colon A \to \mathbb{C}$ such that $f(a+b) = f(a)+f(b)$, $f(a^*) = \overline{f(a)}$ and $f(aa^*) \geq 0$ for every $a, b \in A$.

Proposition 4.1 *Let M be an archimedean quadratic module on a $*$-ring A. For every positive form f on A, the following properties are equivalent:*

(i) $f(M) \geq 0$.
(ii) $|f(s)| \leq \mathrm{n}_M(s) f(1)$ *for every* $s \in \mathrm{Sym}(A)$*,*
(iii) $|f(a)| \leq \mathrm{n}_M(a) f(1)$ *for every* $a \in A$*.*

Proof Assume that (i) is true and pick $s \in \mathrm{Sym}(A)$. For every $r \in \mathbb{Q}^+$ such that $\mathrm{n}_M(s) < r$, we have that $r^2 - s^2 \in M$, hence $r \pm s \in M$ by Lemma 3.1. Since $f(M) \geq 0$, it follows that $rf(1) \pm f(s) = f(r \pm s) \geq 0$, hence $|f(s)| \leq rf(1)$. Therefore (ii) is true. Conversely, if (ii) is true, pick $m \in M$ and $r \in \mathbb{Q}^+$ such that $\mathrm{n}_M(m) < r$. By Lemma 3.4, $\mathrm{n}_M(r-m) \leq r$. By (ii), $|rf(1) - f(m)| \leq \mathrm{n}_M(r-m) f(1)$. It follows that $f(m) \geq 0$. Hence (i) is true.

Assume now that (ii) is true and pick $a \in A$. By the Cauchy–Schwartz inequality, we have $|f(a)|^2 \leq f(aa^*) f(1)$. Applying (ii) with $s = aa^*$, we get $f(aa^*) \leq \mathrm{n}_M(aa^*) f(1)$. Finally $\mathrm{n}_M(aa^*) = \mathrm{n}_M(a)^2$ by Theorem 3.2. It follows that (ii) is true. Clearly, (iii) implies (ii). ∎

Let A be a $*$-ring and H a complex Hilbert space. A representation of A on H is a (non-unital) $*$-ring homomorphism from A to $L(H)$, where $L(H)$ is the $*$-ring of all bounded operators on H. Let us say that a representation ψ of A on H is *M-positive* if $\psi(m)$ is positive semidefinite for every $m \in M$.

Proposition 4.2 *Let M be an archimedean quadratic module on $*$-ring A and H a complex Hilbert space. Then every M-positive representation ψ of A on H satisfies $\|\psi(a)\| \leq \mathrm{n}_M(a) \|\psi(1)\|$.*

Proof Pick $\psi \in \mathrm{Rep}_{\mathbb{Z}}^M(A, H)$. For every $\xi \in H$ and $a \in A$, write $f_\xi(a) = \langle \psi(a)\xi, \xi \rangle$. Clearly, each f_ξ is a positive form and $f_\xi(M) \geq 0$. By Proposition 4.1, $|f_\xi(s)| \leq n_M(s) f_\xi(1)$ for every $s \in A$. It follows that for every $a \in A$,

$$\begin{aligned}\|\psi(a)\|^2 &= \sup_\xi \frac{\langle \psi(a)\xi, \psi(a)\xi \rangle}{\langle \xi, \xi \rangle} = \sup_\xi \frac{\langle \psi(a^*a)\xi, \xi \rangle}{\langle \xi, \xi \rangle} \\ &\leq \mathrm{n}_M(a^*a) \sup_\xi \frac{\langle \psi(1)\xi, \xi \rangle}{\langle \xi, \xi \rangle} = \mathrm{n}_M(a)^2 \|\psi(1)\|. \qquad \blacksquare\end{aligned}$$

A representation ψ of a $*$-ring A on a complex Hilbert space H is *irreducible* (resp. cyclic) if $\psi(A)\xi$ is dense in $H_\psi := \overline{\psi(A)H}$ for every (resp. for some) $\xi \in H$.

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Lemma 4.3 *Let M be a quadratic module on a $*$-ring A. For a complex Hilbert space H there are natural one-to-one correspondences between*

(i) *the set $\operatorname{Rep}_{\mathbb{Z}}^{M}(A, H)$ of all M-positive representations of A on H,*
(ii) *the set $\operatorname{Rep}_{\mathbb{R}}(A_M, H)$ of all $\mathbb{R}$-linear representations of A_M on H,*
(iii) *the set $\operatorname{Rep}((A_M)^\circ, H)$ of all $\mathbb{C}$-linear representations of $(A_M)^\circ$ on H.*

The correspondences preserve the property of being irreducible or cyclic.

Proof Every M-positive representation of A on H is continuous with respect to n_M by Proposition 4.2. Hence, it can be factored through $A/\operatorname{I}(M)$ and then extended by continuity to A_M. The continuity implies that the extension to A_M is $\mathbb{R}$-linear. The converse mapping is given by $\psi \mapsto \psi \circ j$; see Theorem 3.5.

Every $\mathbb{R}$-linear representation ψ of $B = A_M$ on H extends to a $\mathbb{C}$-linear representation ψ° of B° on H by $\psi^\circ(b', b'') = \psi(b') + i\psi(b'')$ for every $b', b'' \in B$. The converse mapping is the restriction mapping $\pi \mapsto \pi|_B$. ■

Write $\operatorname{Irr}_{\mathbb{Z}}^{M}(A, H)$, $\operatorname{Irr}_{\mathbb{R}}(A_M, H)$ and $\operatorname{Irr}((A_M)^\circ, H)$ for the corresponding sets of irreducible representations. Write $\operatorname{Irr}_{\mathbb{Z}}^{M}(A) = \bigcup_H \operatorname{Irr}_{\mathbb{Z}}^{M}(A, H)$ where H runs through all complex Hilbert spaces.

Theorem 4.4 *Let M be an archimedean quadratic module on A and $a \in A$.*

(i) $\mathrm{n}_M(a) = \sup_{\psi \in \operatorname{Irr}_{\mathbb{Z}}^{M}(A)} \|\psi(a)\|$.
(ii) *$a \in \operatorname{Arch}(M)$ if and only if $\psi(a)$ is positive semidefinite for every $\psi \in \operatorname{Irr}_{\mathbb{Z}}^{M}(A)$.*
(iii) *$a \in \operatorname{Int}(M)$ if and only if $\psi(a)$ is positive definite for every $\psi \in \operatorname{Irr}_{\mathbb{Z}}^{M}(A)$.*

Proof By Lemma 4.3 and Theorem 3.5 we may assume that A is a complex C^*-algebra and $M = A^+$. In this case, the results are known from [6, Sections 2.6 and 2.7]. Namely, assertion (i) follows from [6, 2.7.1 and 2.7.3]; assertion (ii) is a variant of [6, 2.6.2] which follows from [6, 2.5.4], and Krein–Milman Theorem and assertion (iii) is another variant of [6, 2.6.2] which follows from [18, remarks after Definition 2.14.6 and Proposition 2.3.13]. ■

Let M be an archimedean quadratic module on a $*$-ring A. Write $\alpha_i(A, M) = \sup_{\pi \in \operatorname{Irr}_{\mathbb{Z}}^{M}(A)} \dim H_\pi$. Define $\alpha_c(A, M)$ similarly, just replacing irreducible by cyclic representations. If H is a complex Hilbert space with $\dim H \geq \alpha_i(A, M)$, then every ireducible M-positive representation of A can be realized on H. For such H $\operatorname{Irr}_{\mathbb{Z}}^{M}(A)$ in Theorem 4.4 can be replaced by $\operatorname{Irr}_{\mathbb{Z}}^{M}(A, H)$. Let $\mathrm{A}_u^E(\operatorname{Irr}_{\mathbb{Z}}^{M}(A, H), L(H))$ denote the set of all mappings $\gamma\colon \operatorname{Irr}_{\mathbb{Z}}^{M}(A, H) \to L(H)$ such that

(i) γ is bounded (*i.e.*, $\|\gamma\| := \sup_{\pi \in \operatorname{Irr}_{\mathbb{Z}}^{M}(A,H)} \|\gamma(\pi)\| < \infty$),
(ii) γ is equivariant (*i.e.*, $\gamma(u^*\pi u) = u^*\gamma(\pi)u$ for every $\pi \in \operatorname{Irr}_{\mathbb{Z}}^{M}(A, H)$ and every partial isometry $u \in L(H)$ such that $uu^* \geq$ the projection on H_π),
(iii) γ is uniformly continuous (with respect to the weak operator topology on $L(H)$ and the topology of pointwise convergence on $\operatorname{Irr}_{\mathbb{Z}}^{M}(A, H)$).

The set $\mathrm{A}_u^E(\operatorname{Irr}_{\mathbb{Z}}^{M}(A, H), L(H))$ is a complex C^*-algebra for pointwise algebraic operations and the norm $\gamma \mapsto \|\gamma\|$.

For every $\pi \in \operatorname{Irr}_{\mathbb{Z}}^{M}(A, H)$ define $\bar{\pi} \in \operatorname{Irr}_{\mathbb{Z}}^{M}(A, H)$ by $\bar{\pi}(a) = \pi(a)^*$ for every $a \in A$. Write $\mathrm{A}_u^E(\operatorname{Irr}_{\mathbb{Z}}^{M}(A, H), -)$ for the set of all $\gamma \in \mathrm{A}_u^E(\operatorname{Irr}_{\mathbb{Z}}^{M}(A, H), L(H))$ such that

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$\gamma(\bar{\pi}) = \gamma(\pi)^*$ for every $\pi \in \mathrm{Irr}_{\mathbb{Z}}^M(A,H)$. Note that $\mathrm{A}_u^E(\mathrm{Irr}_{\mathbb{Z}}^M(A,H),-)$ is a real C^*-subalgebra of $\mathrm{A}_u^E(\mathrm{Irr}_{\mathbb{Z}}^M(A,H),L(H))$. Its positive cone $\mathrm{A}_u^E(\mathrm{Irr}_{\mathbb{Z}}^M(A,H),-)^+$ is equal to the set of all $\gamma \in A_u^E(\mathrm{Irr}_{\mathbb{Z}}^M(A,H),-)$ such that $\gamma(\pi)$ is positive semidefinite for every $\pi \in \mathrm{Irr}_{\mathbb{Z}}^M(A,H)$.

We can rephrase Theorem 4.4 as a generalization of Jacobi's theorem.

Theorem 4.5 *Let M be an archimedean quadratic module on a $*$-ring A and H a complex Hilbert space such that* $\dim H \geq \alpha_i(A,M)$. *The evaluation mapping*

$$\Phi\colon A \to \mathrm{A}_u^E(\mathrm{Irr}_{\mathbb{Z}}^M(A,H),L(H)), \quad \Phi(a)(\pi) = \pi(a),$$

is a $*$-homomorphism and an isometry. Moreover,

$$\mathrm{Arch}(M) = \Phi^{-1}(\mathrm{A}_u^E(\mathrm{Irr}_{\mathbb{Z}}^M(A,H),-)^+).$$

When we compare Theorem 4.5 with Theorem 1.1, the following questions arise.

(i) Is $\mathbb{Q}\cdot\Phi(A)$ dense in $\mathrm{A}_u^E(\mathrm{Irr}_{\mathbb{Z}}^M(A,H),-)$?
(ii) Is $\mathrm{Irr}_{\mathbb{Z}}^M(A,H)$ compact in the topology of pointwise convergence?

The answer to question (i) is yes if $\dim H \geq \alpha_c(A,M)$; see Theorem 5.3. Note that properties (ii) and (iii) from the definition of $\mathrm{A}_u^E(\mathrm{Irr}_{\mathbb{Z}}^M(A,H),L(H))$ are not required in the proof of Theorem 4.5. However, they will be required in the proof of Theorem 5.3. We do not know the answer to question (i) if $\alpha_i(A,M) \leq \dim H < \alpha_c(A,M)$.

We believe that the answer to question (ii) is no (cf. [3]), but we do not have an explicit counterexample. There exists a natural compactification of $\mathrm{Irr}_{\mathbb{Z}}^M(A,H)$, namely its closure in the set of all additive mappings $\psi\colon A \to L(H)$ of norm ≤ 1. This follows from the fact that the unit ball of $L(H)$ is compact in the weak operator topology.

5 The Real CP-Convexity Gelfand–Naimark Theorem

The aim of this section is to prove a real version of the CP-convexity Gelfand–Naimark theorem from [12] similar to the real Gelfand–Naimark theorem from [17].

Let A be a complex C^*-algebra and H a complex Hilbert space. Let us denote by $\mathrm{A}_u^E(\mathrm{Irr}(A,H),L(H))$ the set of all mappings $\kappa\colon \mathrm{Irr}(A,H) \to L(H)$ which are equivariant, bounded, and uniformly continuous as above. Let $\alpha_c(A)$ denote the supremum of $\dim H_\pi$ where π runs through all cyclic representations of A on all complex Hilbert spaces. The CP-convexity Gelfand–Naimark theorem from [12] says the following.

Theorem 5.1 *Let A be a complex C^*-algebra and H a complex Hilbert space such that* $\dim H \geq \alpha_c(A)$. *The Gelfand transform*

$$g\colon A \to \mathrm{A}_u^E(\mathrm{Irr}(A,H),L(H)), \quad g(c)(\psi) = \psi(c),$$

is a $*$-isomorphism and an isometry.

Now let us turn our attention to the real case. Let B be a real $*$-algebra with complexification B° and H a complex Hilbert space. Write $A_u^E(\mathrm{Irr}_{\mathbb{R}}(B,H),L(H))$ for the set of all mappings $\eta\colon \mathrm{Irr}_{\mathbb{R}}(B,H)\to L(H)$ which are equivariant, bounded, and uniformly continuous. Let $s\colon \mathrm{Irr}_{\mathbb{R}}(B,H)\to \mathrm{Irr}(B^\circ,H)$ denote the natural correspondence of Lemma 4.3. The correspondence s is a homeomorphism with respect to the topologies of pointwise convergence. The mapping

$$A_u^E(\mathrm{Irr}_{\mathbb{R}}(B,H),L(H))\to A_u^E(\mathrm{Irr}(B^\circ,H),L(H)),\quad \eta\mapsto s\eta s^{-1},$$

is a $*$-isomorphism and an isometry.

For every $\rho\in\mathrm{Irr}_{\mathbb{R}}(B,H)$ we define $\bar\rho\in\mathrm{Irr}_{\mathbb{R}}(B,H)$ by $\bar\rho(b)=\rho(b)^*$ for every $b\in B$. Write $\mathrm{A}_u^E(\mathrm{Irr}_{\mathbb{R}}(B,H),-)$ for the set of all $\eta\in\mathrm{A}_u^E(\mathrm{Irr}_{\mathbb{R}}(B,H),L(H))$ such that $\eta(\bar\rho)=\eta(\rho)^*$ for every $\rho\in\mathrm{Irr}_{\mathbb{R}}(B,H)$. This is a real C^*-subalgebra of $\mathrm{A}_u^E(\mathrm{Irr}_{\mathbb{R}}(B,H),L(H))$. The mapping

$$g_{\mathbb{R}}\colon B\to\mathrm{A}_u^E(\mathrm{Irr}_{\mathbb{R}}(B,H),-),\quad g_{\mathbb{R}}(b)(\eta)=\eta(b),$$

will be called *the real Gelfand transform*. Since $g_{\mathbb{R}}(b)(\bar\rho)=\bar\rho(b)=\rho(b)^*=g_{\mathbb{R}}(b)(\rho)^*$ for every $b\in B$, it follows that $g_{\mathbb{R}}(b)\in\mathrm{A}_u^E(\mathrm{Irr}_{\mathbb{R}}(B,H),-)$ for every $b\in B$. Hence $g_{\mathbb{R}}$ is well defined. Clearly, $g_{\mathbb{R}}$ is a homomorphism of real $*$-algebras.

Theorem 5.2 is a real version of Theorem 5.1.

Theorem 5.2 *Let B be a real $*$-algebra and H a complex Hilbert space such that $\dim H\geq\alpha_c(B^\circ)$. The real Gefand transform $g_{\mathbb{R}}\colon B\to\mathrm{A}_u^E(\mathrm{Irr}_{\mathbb{R}}(B,H),-)$ is a $*$-isomorphism and an isometry.*

Proof We have a commutative diagram

$$\begin{tikzcd}
B^\circ \arrow[r, "g"] & \mathrm{A}_u^E(\mathrm{Irr}(B^\circ,H),L(H)) \\
B \arrow[u] \arrow[r, "g_{\mathbb{R}}"] & \mathrm{A}_u^E(\mathrm{Irr}_{\mathbb{R}}(B,H),-) \arrow[u, "\eta\mapsto s\eta s^{-1}"']
\end{tikzcd}$$

where the vertical arrows are one-to-one. By Theorem 5.1, g is one-to-one and onto. It follows that $g_{\mathbb{R}}$ is one-to-one. It remains to show that $g_{\mathbb{R}}$ is onto.

For every $\pi\in\mathrm{Irr}(B^\circ,H)$ write $\bar\pi$ for the mapping defined by $\bar\pi(c)=\pi(\bar c)^*$ for $c\in B^\circ$. Note that $\overline{\rho^\circ}=(\bar\rho)^\circ$ for every $\rho\in\mathrm{Irr}_{\mathbb{R}}(B,H)$. It follows that

$$\begin{aligned}
\mathrm{A}_u^E(\mathrm{Irr}(B^\circ,H),-) &:= \{s\eta s^{-1}:\eta\in\mathrm{A}_u^E(\mathrm{Irr}_{\mathbb{R}}(B,H),-)\}\\
&= \{\gamma\in\mathrm{A}_u^E(\mathrm{Irr}(B^\circ,H),L(H)):\gamma(\bar\pi)=\gamma(\pi)^*\text{ for all }\pi\in\mathrm{Irr}(B^\circ,H)\}
\end{aligned}$$

Pick any $c\in B^\circ$. Note that $g(\bar c)(\pi)^*=g(c)(\bar\pi)$ for every $\pi\in\mathrm{Irr}(A^\circ,H)$. It follows that $g(c)\in\mathrm{A}_u^E(\mathrm{Irr}(B^\circ,H),-)$ if and only if $g(c)(\pi)^*=g(\bar c)(\pi)^*$ for every $\pi\in\mathrm{Irr}(A^\circ,H)$. Since g is one-to-one, this is equivalent to $c=\bar c$. Since g is onto, it follows $g(B)=\mathrm{A}_u^E(\mathrm{Irr}(B^\circ,H),-)$. Hence, $g_{\mathbb{R}}$ is onto. ∎

The following corollary of Theorem 5.2 complements Theorem 4.5.

Theorem 5.3 *Let A, M, H, and Φ be as in Theorem* 4.5. *If* $\dim H \geq \alpha_c(A, M)$ *then* $\mathbb{Q} \cdot \Phi(A)$ *is dense in* $A_u^E(\mathrm{Irr}_{\mathbb{Z}}^M(A, H), -)$.

Proof Let $r\colon \mathrm{Irr}_{\mathbb{Z}}^M(A, H) \to \mathrm{Irr}_{\mathbb{R}}(A_M, H)$ be the natural correspondence from Lemma 4.3. Clearly, r is a homeomorphism with respect to the topologies of pointwise convergence and it induces a mapping

$$\tilde{r}\colon A_u^E(\mathrm{Irr}_{\mathbb{Z}}^M(A, H), -) \to A_u^E(\mathrm{Irr}_{\mathbb{R}}(A_M, H), -), \quad \kappa \mapsto r\kappa r^{-1},$$

which is a $*$-isomorphism and an isometry. The diagram

$$\begin{tikzcd} A_M \arrow[r, "g_{\mathbb{R}}"] & A_u^E(\mathrm{Irr}_{\mathbb{R}}(A_M, H), -) \\ A \arrow[u, "j"] \arrow[r, "\Phi"] & A_u^E(\mathrm{Irr}_{\mathbb{Z}}^M(A, H), -) \arrow[u, "\tilde{r}"'] \end{tikzcd}$$

is commutative. Since $\dim H \geq \alpha_c(A, M) = \alpha((A_M)^\circ)$, Theorem 5.2 implies that $g_{\mathbb{R}}$ is onto. We know from Theorem 4.5 that $g_{\mathbb{R}}$ is an isometry. It is clear from the construction of A_M in Section 3 that $\mathbb{Q} \cdot j(A)$ is dense in A_M. Since $g_{\mathbb{R}}$ and $\tilde{r}$ are isometries and onto, it follows that $\mathbb{Q} \cdot \Phi(A) = \tilde{r}^{-1}(g_{\mathbb{R}}(\mathbb{Q} \cdot j(A)))$ is dense in $A_u^E(\mathrm{Irr}_{\mathbb{Z}}^M(A, H), L(H)) = \tilde{r}^{-1}(g_{\mathbb{R}}(A_M))$. ∎

6 Comments on $*$-Orderings

When functional analysts and real algebraic geometers talk about ordered complex $*$-algebras, they don't mean the same thing. For a functional analyst, an ordering on A is a cone on A, *i.e.,* a subset $C \subset \mathrm{Sym}(A)$ such that $C + C \subseteq C$ and $\mathbb{R}^+C \subseteq C$. For a real algebraic geometer, an ordering on A is usually a $*$-ordering, *i.e.,* a subset $P \subseteq \mathrm{Sym}(A)$ such that $P + P \subset P$, $aPa^* \subseteq P$ for every $a \in A$, $st + ts \in P$ for every $s, t \in P$, $P \cap -P$ is a Jordan prime ideal and $P \cup -P = \mathrm{Sym}(A)$; see [20]. Note that every $*$-ordering is a cone. The full matrix ring $\mathrm{Mat}_n(\mathbb{C})$ $(n \geq 2)$ is a typical example of a complex $*$-algebra that is ordered for a functional analyst and not orderable for a real algebraic geometer. Another example is group rings $\mathbb{C}[G]$, which are always orderable for a functional analyst, but only in special cases (for certain orderable groups) for a real algebraic geometer.

Let us recall the motivation for the definition of a $*$-ordering. The most trivial example is $(\mathbb{C}, \mathbb{R}^+)$. If A is a commutative complex $*$-algebra and $\phi\colon A \to \mathbb{C}$ is a hermitian homomorphism, then $P := \phi^{-1}(\mathbb{R}^+) \cap \mathrm{Sym}(A)$ is a natural candidate for a $*$-ordering. We list its algebraic properties ($P + P \subseteq P$, $PP \subseteq P$, $aa^* \in P$ for every $a \in P$, $P \cap -P$ is a prime ideal and $P \cup -P = \mathrm{Sym}(A)$) and take them as axioms of a $*$-ordering. The noncommutative definition is a modification that makes most of the commutative theory work.

A definition of an ordering that is not too restrictive for functional analysts and not too general for real algebraic geometers should follow the same steps as in the commutative case. Let us consider the set Π_n of all positive semidefinite hermitian matrices on $\mathrm{Mat}_n(\mathbb{C})$ as the simplest ordering. Let A be a complex $*$-algebra, $\pi\colon A \to \mathrm{Mat}_n(\mathbb{C})$ an irreducible $*$-representation, and set $P = \pi^{-1}(\Pi_n) \cap \mathrm{Sym}(A)$. The algebraic properties of P include the following:

(i) $P + P \subseteq P$,
(ii) if $a, b \in P$ commute, then $ab \in P$,
(iii) $aPa^* \subseteq P$ for every $a \in A$,
(iv) $P \cap -P$ is the symmetric part of a prime ideal,
(v) for every primitive hermitian idempotent $e \in A$, eAe is linearly ordered by $P \cap eAe$.

Similar orderings have been considered in [1]. It would be interesting to know whether an Artin–Schreier theory of such orderings can be developed.

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