ON THE TRUNCATED LONG-RANGE PERCOLATION ON $\mathbb{Z}^2$

BERNARDO NUNES BORGES DE LIMA,* UFMG
ARTËM SAPOZHNIKOV,** CWI

Abstract

We consider an independent long-range bond percolation on $\mathbb{Z}^2$. Horizontal and vertical bonds of length $n$ are independently open with probability $p_n \in [0,1]$. Given $\sum_{n=1}^{\infty} \prod_{i=1}^{n} (1-p_i) < \infty$, we prove that there exists an infinite cluster of open bonds of length less than or equal to $N$ for some large but finite $N$. The result gives a partial answer to the truncation problem.

Keywords: Long-range percolation; truncation; renewal theory; renormalization; mixed percolation

2000 Mathematics Subject Classification: Primary 60K35; 82B44

1. Notation and results

We consider an independent bond percolation on the graph $G = (\mathbb{Z}^2,E)$, where $E = \{\langle x, x+n e_i \rangle : x \in \mathbb{Z}^2 \text{ and } i \in \{1,2\}\}$. For a given sequence $(p_n)$ such that $p_n \in [0,1]$, we declare an edge $\langle x, x+n e_i \rangle$, $x \in \mathbb{Z}^2$ and $i \in \{1,2\}$, to be open with probability $p_n$ and closed otherwise. More formally, we consider the probability space $(\Omega, F, P)$. As sample space, we take $\Omega = \{0,1\}^E$. The elements of $\Omega$ are denoted as $\omega = \{\omega(f) : f \in E\}$. The value $\omega(f) = 1$ corresponds to $f$ being open, and the value $\omega(f) = 0$ corresponds to $f$ being closed. We take $F$ to be the $\sigma$-algebra generated by finite cylinder sets in $\Omega$. We define the product measure $P$ on $(\Omega, F)$ as $\prod_{f \in E} \mu_f$, where $\mu_f$ is the Bernoulli measure on $\{0,1\}$ given by $\mu_f(\omega(f) = 1) = 1 - \mu_f(\omega(f) = 0) = p_{|f|}$, where $|f| = \max(|x_1 - y_1|, |x_2 - y_2|)$ given $f = \langle x, y \rangle$.

Definition 1.1. We say that two sites $x, y \in \mathbb{Z}^2$ are $k$-connected, $x \leftrightarrow_k y$, if there are $v_1, \ldots, v_m \in \mathbb{Z}^2$ such that $v_1 = x$, $v_m = y$, $(v_i, v_{i+1}) \in E$ is open, and $|v_i - v_{i+1}| \leq k$ for all $i$. If $k = \infty$ then we say that $x$ and $y$ are connected, $x \leftrightarrow y$. We say that two sites $x$ and $y \in \mathbb{Z}^2$ are connected in $W \subset \mathbb{Z}^2$ if $x, y \in W$ and if there is an open path between $x$ and $y$ such that all the sites of the path are in $W$.

In this note we study the well-known truncation problem: given a sequence $(p_n)$ for which $P(0 \leftrightarrow_\infty \infty > 0)$, is it true that $P(0 \leftrightarrow_N \infty > 0)$ for some large finite $N$? The answer is no for the one-dimensional independent percolation (see [4] and [10]). It is believed that the answer is yes in dimensions $d \geq 2$. However, only partial results have been obtained so far. In [8] the

Received 2 November 2007; revision received 11 January 2008.

* Postal address: UFMG, Av. Antônio Carlos 6627, C.P. 702, 30123-970 Belo Horizonte, Brazil.
Email address: bublima@mat.ufmg.br

** Postal address: CWI, Kruislaan 413, NL-1098SJ Amsterdam, The Netherlands.
Email address: artem.sapozhnikov@cwi.nl
affirmative answer is given in the case in which the sequence \((p_n)\) is exponentially decaying. The heavy-tailed case has been considered in [1], [9], and [11]. In all the papers it is assumed that the sequence \((p_n)\) is monotone decreasing with some conditions on the speed of decay. The first results for nonmonotone sequences were obtained in [2] and [3]. In [3] the positive answer to the truncation question was given for sequences \((p_n)\) such that \(\lim sup_n p_n > 0\). For nonsummable sequences \((p_n)\) (i.e. \(\sum_n p_n = \infty\)), the affirmative answer to the truncation question was given in [2] in dimensions \(d \geq 3\). It was also conjectured that the statement was true in two dimensions. In this note we answer yes to the truncation question in two dimensions for a very general class of nonsummable sequences \((p_n)\) (see, e.g. (1.2), below), which supports the conjecture in [2].

Our approach is different from the one in [3]. It is based on Blackwell’s renewal theorem and renormalization techniques.

**Theorem 1.1.** Given a sequence \((p_n)\) such that \(p_n \in [0, 1]\) and \(\sum_{n=1}^\infty p_n = \infty\), if
\[
\limsup_{n \to \infty} P(0 \text{ and } n \text{ are connected in } [0, n]) > 0
\]
then there exists \(N\) such that
\[
P(0 \leftrightarrow N \to \infty) > 0.
\]

**Remark 1.1.** If \(\limsup_n p_n > 0\) then (1.1) is trivially satisfied. In particular, the result from [3] follows.

In the next theorem we give a sufficient condition for (1.1).

**Theorem 1.2.** If
\[
\sum_{n=1}^\infty \prod_{i=1}^n (1 - p_i) < \infty
\]
then condition (1.1) holds.

2. Proofs

**Proof of Theorem 1.1.** For convenience, we assume that the greatest common divisor \(\gcd\{k : p_k > 0\} = 1\). The condition ensures that the infinite open cluster is unique (see [5, Theorem 12.3]). If \(\gcd\{k : p_k > 0\} = m > 1\) then we consider the bond percolation on \(m\mathbb{Z}^2\) with
\[
P(mx, m(x + ne_1)) \text{ is open} = P(mx, m(x + ne_2)) \text{ is open} = p_{mx}.
\]
For the sake of notation, we also assume that there exist \(p > 0\) and \(n_0\) such that, for all \(n \geq n_0\),
\[
P(0 \text{ and } n \text{ are connected in } [0, n]) \geq p.
\]
(2.1)

The general case when (2.1) is only satisfied for an infinite subsequence \((n_k)\) can be treated in the same way.

The proof is based on a renormalization argument. Let \(l\) and \(L\) be positive integers such that \(l \leq L\). For any \(x \in \mathbb{Z}^2\), the event \(A_x\) occurs if
- any two sites from the set \(2Lx + [-l, l] \times \{0\}\) are connected in \(2Lx + [-L, L] \times \{0\}\) (see Definition 1.1); and
- any two sites from the set \(2Lx + \{0\} \times [-l, l] \) are connected in \(2Lx + \{0\} \times [-L, L] \).

https://doi.org/10.1234/jap/1208358969 Published online by Cambridge University Press
The events $A_0$ and $A_{e_1}$ are illustrated in Figure 1. From the space homogeneity, it follows that $P(A_x) = P(A_0)$ for all $x \in \mathbb{Z}^2$. Moreover, since $P(\text{all the sites } [-l, l] \text{ are connected in } \mathbb{Z}^2) = 1$ for all $l \in \mathbb{N}$ (we use the assumption that $\gcd\{i : p_i > 0\} = 1$), then, for any $\epsilon > 0$ and for all $l \in \mathbb{N}$, there exists $L_1 = L_1(\epsilon, l)$ such that, for all $L \geq L_1$,

$$P(A_0) > 1 - \epsilon.$$ 

For $x \in \mathbb{Z}^2$ and $y = x + (1, 0)$, we say that the event $B_{x,y}$ occurs if there exists $k \in [-l, l] \setminus \{0\}$ such that the sites $2Lx + (0, k)$ and $2Ly + (0, k)$ are connected in $[2Lx + (0, k), 2Ly + (0, k)] = 2Lx + (0, k) + [0, 2L] \times [0, 2L]$. In Figure 1 the event $B_{0,e_1}$ occurs with $k = -1$. We assume that $B_{y,x} = B_{x,y}$. Similarly, for $x \in \mathbb{Z}^2$ and $y = x + (0, 1)$, the events $B_{x,y}$ and $B_{y,x}$ occur if there exists $k \in [-l, l] \setminus \{0\}$ such that the sites $2Lx + (k, 0)$ and $2Ly + (k, 0)$ are connected in $[2Lx + (k, 0), 2Ly + (k, 0)] = 2Lx + (k, 0) + [0, 2L] \times [0, 2L]$. Space homogeneity and symmetry of the model imply that, for any $x \sim y$ (i.e. $x$ and $y$ are nearest neighbors in $\mathbb{Z}^2$) and $u \sim v$, $P(B_{x,y}) = P(B_{u,v})$.

Condition (2.1) implies that, for all $L \geq n_0$,

$$P(0 \text{ and } 2L \text{ are connected in } [0, 2L]) \geq p > 0.$$ 

Therefore, for any $\epsilon > 0$, there exist $l_0 = l_0(\epsilon) \in \mathbb{N}$ and $L > \max(l_0, n_0)$ such that

$$P(B_{0,e_1}) > 1 - \epsilon.$$ 

For $\epsilon > 0$, we take $L \geq \max(L_1(\epsilon, l_0(\epsilon)), n_0)$. It follows that

$$P(A_0) > 1 - \epsilon \quad \text{and} \quad P(B_{0,e_1}) > 1 - \epsilon.$$ 

Moreover, the events $\{A_x : x \in \mathbb{Z}^2\} \cup \{B_{y,z} : y, z \in \mathbb{Z}^2, y \sim z\}$ are defined in terms of the states of edges in disjoint subsets of $\epsilon$ and, therefore, are independent.
Now it is easy to complete the proof. If $A_x$ occurs, we say that the site $x$ is open. If $B_{y,z}$ occurs, we say that the bond $(y, z)$ is open. The constructed model is an independent nearest-neighbor site-bond percolation.

We can choose small enough $\varepsilon > 0$ such that there exists an infinite open cluster in the renormalized site-bond percolation model (see, e.g. [6]). The existence of an infinite open cluster in the renormalized model implies the existence of an infinite open cluster of $2L$-connected sites in the original model (see Figure 2 for an example of a renormalized cluster). Therefore, we can take $N = 2L$.

An important result in renewal theory, which we need for the next proof, is Blackwell’s theorem (see e.g. [7, p. 73]).

**Theorem 2.1.** (Blackwell’s theorem.) Let $\{X_i\}$ be a sequence of independent and identically distributed (i.i.d.) random variables taking values in $\mathbb{Z}_+$, and let $S_k = \sum_{i=1}^k X_i$. If

$$\gcd\{k : P(X_1 = k) > 0\} = 1$$

then

$$P(\text{there exists } k \text{ such that } S_k = n) \to \frac{1}{E X_1} \quad \text{as } n \to \infty.$$  

Here, if $E X_1 = \infty$ then the limit is 0.

**Proof of Theorem 1.2.** As in the proof of Theorem 1.1, we can assume, without loss of generality, that $\gcd\{k : p_k > 0\} = 1$. For any $x \in \mathbb{Z}^2$, we define

$$\xi_x = \min(n : \langle x, x + ne_1 \rangle \text{ is open}).$$

Note that $\xi_x$ are i.i.d. random variables with distribution

$$P(\xi_0 > n) = \prod_{i=1}^n (1 - p_i).$$
Since $\sum_n p_n = \infty$, the random variables are finite almost surely. Moreover,

$$E \xi_x = \sum_{n=0}^{\infty} \prod_{i=1}^{n} (1 - p_i) < \infty.$$  

From Theorem 2.1 we conclude that there exists $n_0$ such that, for all $n \geq n_0$,

$$P(0 \text{ and } n \text{ are connected in } [0, n]) \geq \frac{1}{2 E \xi_0} = \left(2 \sum_{n=0}^{\infty} \prod_{i=1}^{n} (1 - p_i)\right)^{-1} > 0.$$ 

Acknowledgements

The work started when B. N. B. L. was visiting A. S. at the Boole Centre for Research in Informatics, University College Cork. He would like to thank BCRI for hospitality. The research of B. N. B. L. was supported by CAPES (Brazilian Ministry of Education). The research of A. S. was supported by Science Foundation Ireland, grant number SFI-04-RP1-1512.

References