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ON THE TRUNCATED LONG-RANGE PERCOLATION ON $\mathbb{Z}^2$

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Abstract

We consider an independent long-range bond percolation on $\mathbb{Z}^2$. Horizontal and vertical bonds of length n are independently open with probability $p_n \in [0, 1]$. Given $\sum_{n=1}^{\infty} \prod_{i=1}^{n}(1 - p_i) < \infty$, we prove that there exists an infinite cluster of open bonds of length less than or equal to N for some large but finite N. The result gives a partial answer to the truncation problem.

Keywords: Long-range percolation; truncation; renewal theory; renormalization; mixed percolation

2000 Mathematics Subject Classification: Primary 60K35; 82B44

1. Notation and results

We consider an independent bond percolation on the graph $\mathcal{G} = (\mathbb{Z}^2, \mathcal{E})$, where $\mathcal{E} = \{\langle x, y\rangle \in \mathbb{Z}^2 \times \mathbb{Z}^2 : x \neq y \text{ and } x_1 = y_1 \text{ or } x_2 = y_2\}$. For a given sequence (p_n) such that $p_n \in [0, 1]$, we declare an edge $\langle x, x + ne_i\rangle$, $x \in \mathbb{Z}^2$ and $i \in \{1, 2\}$, to be open with probability p_n and closed otherwise. More formally, we consider the probability space $(\Omega, \mathcal{F}, \mathrm{P})$. As sample space, we take $\Omega = \{0, 1\}^{\mathcal{E}}$. The elements of Ω are denoted as $\omega = \{\omega(f) : f \in \mathcal{E}\}$. The value $\omega(f) = 1$ corresponds to f being open, and the value $\omega(f) = 0$ corresponds to f being closed. We take $\mathcal{F}$ to be the σ-algebra generated by finite cylinder sets in Ω. We define the product measure P on $(\Omega, \mathcal{F})$ as $\prod_{f\in\mathcal{E}} \mu_f$, where μ_f is the Bernoulli measure on $\{0, 1\}$ given by

$$\mu_f(\omega(f) = 1) = 1 - \mu_f(\omega(f) = 0) = p_{|f|},$$

where $|f| = \max(|x_1 - y_1|, |x_2 - y_2|)$ given $f = \langle x, y\rangle$.

Definition 1.1. We say that two sites $x, y \in \mathbb{Z}^2$ are *k-connected*, $x \stackrel{k}{\longleftrightarrow} y$, if there are $v_1, \ldots, v_m \in \mathbb{Z}^2$ such that $v_1 = x$, $v_m = y$, $\langle v_i, v_{i+1}\rangle \in \mathcal{E}$ is open, and $|v_i - v_{i+1}| \leq k$ for all i. If $k = \infty$ then we say that x and y are *connected*, $x \longleftrightarrow y$. We say that two sites x and y of $\mathbb{Z}^2$ are *connected in* $W \subset \mathbb{Z}^2$ if $x, y \in W$ and if there is an open path between x and y such that all the sites of the path are in W.

In this note we study the well-known truncation problem: given a sequence (p_n) for which $\mathrm{P}(0 \longleftrightarrow \infty) > 0$, is it true that $\mathrm{P}(0 \stackrel{N}{\longleftrightarrow} \infty) > 0$ for some large finite N? The answer is no for the one-dimensional independent percolation (see [4] and [10]). It is believed that the answer is yes in dimensions $d \geq 2$. However, only partial results have been obtained so far. In [8] the

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affirmative answer is given in the case in which the sequence (p_n) is exponentially decaying. The heavy-tailed case has been considered in [1], [9], and [11]. In all the papers it is assumed that the sequence (p_n) is monotone decreasing with some conditions on the speed of decay. The first results for nonmonotone sequences were obtained in [2] and [3]. In [3] the positive answer to the truncation question was given for sequences (p_n) such that $\limsup_n p_n > 0$. For nonsummable sequences (p_n) (i.e. $\sum_n p_n = \infty$), the affirmative answer to the truncation question was given in [2] in dimensions $d \geq 3$. It was also conjectured that the statement was true in two dimensions. In this note we answer yes to the truncation question in two dimensions for a very general class of nonsummable sequences (p_n) (see, e.g. (1.2), below), which supports the conjecture in [2]. Our approach is different from the one in [3]. It is based on Blackwell's renewal theorem and renormalization techniques.

Theorem 1.1. *Given a sequence* (p_n) *such that* $p_n \in [0, 1]$ *and* $\sum_{n=1}^{\infty} p_n = \infty$, *if*

$$\limsup_{n\to\infty} \mathrm{P}(0 \textit{ and } n \textit{ are connected in } [0, n]) > 0 \tag{1.1}$$

then there exists N *such that*

$$\mathrm{P}(0 \stackrel{N}{\longleftrightarrow} \infty) > 0.$$

Remark 1.1. If $\limsup_n p_n > 0$ then (1.1) is trivially satisfied. In particular, the result from [3] follows.

In the next theorem we give a sufficient condition for (1.1).

Theorem 1.2. *If*

$$\sum_{n=1}^{\infty} \prod_{i=1}^{n} (1 - p_i) < \infty \tag{1.2}$$

then condition (1.1) holds.

2. Proofs

Proof of Theorem 1.1. For convenience, we assume that the greatest common divisor $\gcd\{k\colon p_k > 0\} = 1$. The condition ensures that the infinite open cluster is unique (see [5, Theorem 12.3]). If $\gcd\{k\colon p_k > 0\} = m > 1$ then we consider the bond percolation on $m\mathbb{Z}^2$ with

$$\mathrm{P}(\langle mx, m(x + ne_1)\rangle \text{ is open}) = \mathrm{P}(\langle mx, m(x + ne_2)\rangle \text{ is open}) = p_{mn}.$$

For the sake of notation, we also assume that there exist $p > 0$ and n_0 such that, for all $n \geq n_0$,

$$\mathrm{P}(0 \text{ and } n \text{ are connected in } [0, n]) \geq p. \tag{2.1}$$

The general case when (2.1) is only satisfied for an infinite subsequence (n_k) can be treated in the same way.

The proof is based on a renormalization argument. Let l and L be positive integers such that $l < L$. For any $x \in \mathbb{Z}^2$, the event A_x occurs if

- any two sites from the set $2Lx + [-l, l] \times \{0\}$ are connected in $2Lx + [-L, L] \times \{0\}$ (see Definition 1.1); and
- any two sites from the set $2Lx + \{0\} \times [-l, l]$ are connected in $2Lx + \{0\} \times [-L, L]$.

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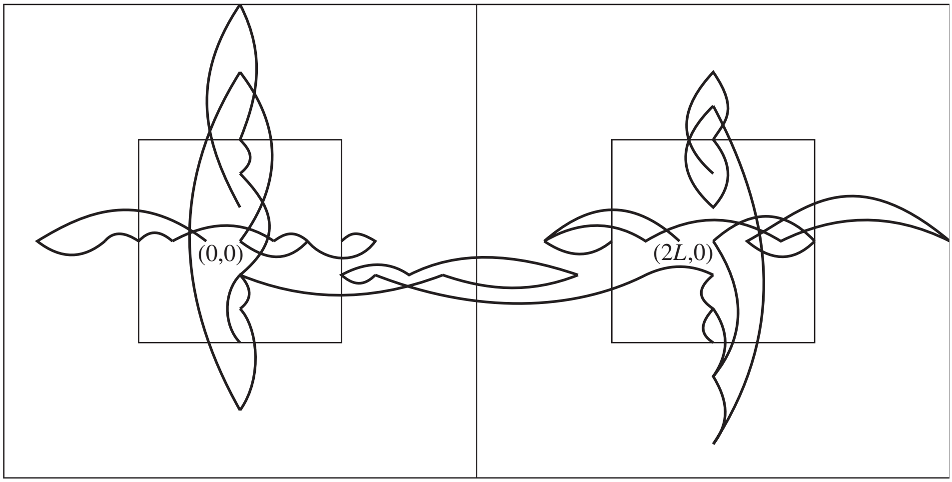

FIGURE 1: Events A_0, A_{e_1}, and B_{0,e_1}. The inner boxes are $B(0, l)$ and $B(2Le_1, l)$. The outer boxes are $B(0, L)$ and $B(2Le_1, L)$.

The events A_0 and A_{e_1} are illustrated in Figure 1. From the space homogeneity, it follows that $\mathrm{P}(A_x) = \mathrm{P}(A_0)$ for all $x \in \mathbb{Z}^2$. Moreover, since

$$\mathrm{P}(\text{all the sites } [-l, l] \text{ are connected in } \mathbb{Z}) = 1 \quad \text{for all } l \in \mathbb{N}$$

(we use the assumption that $\gcd\{i : p_i > 0\} = 1$), then, for any $\varepsilon > 0$ and for all $l \in \mathbb{N}$, there exists $L_1 = L_1(\varepsilon, l)$ such that, for all $L \geq L_1$,

$$\mathrm{P}(A_0) > 1 - \varepsilon.$$

For $x \in \mathbb{Z}^2$ and $y = x + (1, 0)$, we say that the event $B_{x,y}$ occurs if there exists $k \in [-l, l] \setminus \{0\}$ such that the sites $2Lx + (0, k)$ and $2Ly + (0, k)$ are connected in $[2Lx + (0, k), 2Ly + (0, k)] = 2Lx + (0, k) + [0, 2L] \times \{0\}$. In Figure 1 the event B_{0,e_1} occurs with $k = -1$. We assume that $B_{y,x} = B_{x,y}$. Similarly, for $x \in \mathbb{Z}^2$ and $y = x + (0, 1)$, the events $B_{x,y}$ and $B_{y,x}$ occur if there exists $k \in [-l, l] \setminus \{0\}$ such that the sites $2Lx + (k, 0)$ and $2Ly + (k, 0)$ are connected in $[2Lx + (k, 0), 2Ly + (k, 0)] = 2Lx + (k, 0) + \{0\} \times [0, 2L]$. Space homogeneity and symmetry of the model imply that, for any $x \sim y$ (i.e. x and y are nearest neighbors in $\mathbb{Z}^2$) and $u \sim v$, $\mathrm{P}(B_{x,y}) = \mathrm{P}(B_{u,v})$.

Condition (2.1) implies that, for all $L \geq n_0$,

$$\mathrm{P}(0 \text{ and } 2L \text{ are connected in } [0, 2L]) \geq p > 0.$$

Therefore, for any $\varepsilon > 0$, there exist $l_0 = l_0(\varepsilon) \in \mathbb{N}$ and $L > \max(l_0, n_0)$ such that

$$\mathrm{P}(B_{0,e_1}) > 1 - \varepsilon.$$

For $\varepsilon > 0$, we take $L \geq \max(L_1(\varepsilon, l_0(\varepsilon)), n_0)$. It follows that

$$\mathrm{P}(A_0) > 1 - \varepsilon \quad \text{and} \quad \mathrm{P}(B_{0,e_1}) > 1 - \varepsilon.$$

Moreover, the events $\{A_x : x \in \mathbb{Z}^2\} \cup \{B_{y,z} : y, z \in \mathbb{Z}^2,\ y \sim z\}$ are defined in terms of the states of edges in disjoint subsets of $\mathcal{E}$ and, therefore, are independent.

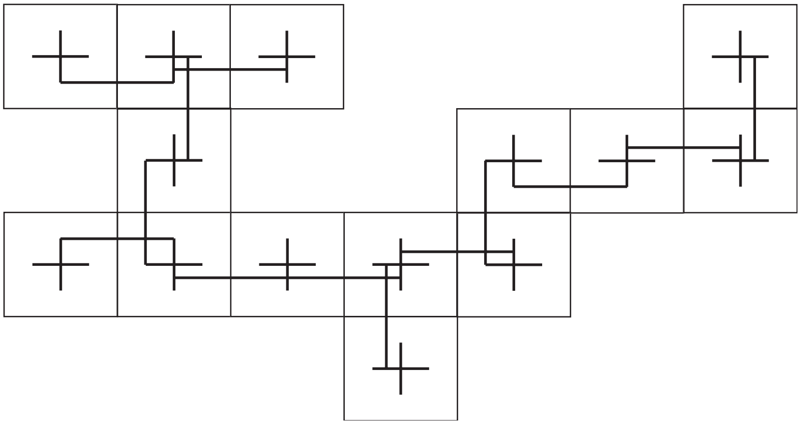

FIGURE 2: An example of a renormalized cluster. Crosses correspond to the occurence of events A_x and connections between crosses correspond to the occurence of events $B_{x,y}$.

Now it is easy to complete the proof. If A_x occurs, we say that the site x is *open*. If $B_{y,z}$ occurs, we say that the bond $\langle y, z\rangle$ is *open*. The constructed model is an independent nearest-neighbor site-bond percolation.

We can choose small enough $\varepsilon > 0$ such that there exists an infinite open cluster in the renormalized site-bond percolation model (see, e.g. [6]). The existence of an infinite open cluster in the renormalized model implies the existence of an infinite open cluster of $2L$-connected sites in the original model (see Figure 2 for an example of a renormalized cluster). Therefore, we can take $N = 2L$.

An important result in renewal theory, which we need for the next proof, is Blackwell's theorem (see e.g. [7, p. 73]).

Theorem 2.1. (Blackwell's theorem.) *Let* $\{X_i\}$ *be a sequence of independent and identically distributed (i.i.d.) random variables taking values in* $\mathbb{Z}_+$, *and let* $S_k = \sum_{i=1}^{k} X_i$. *If*

$$\gcd\{k\colon\ \mathrm{P}(X_1 = k) > 0\} = 1$$

then

$$\mathrm{P}(\textit{there exists } k \textit{ such that } S_k = n) \to \frac{1}{\mathrm{E}\, X_1} \quad \textit{as } n \to \infty.$$

Here, if $\mathrm{E}\, X_1 = \infty$ *then the limit is* 0.

Proof of Theorem 1.2. As in the proof of Theorem 1.1, we can assume, without loss of generality, that $\gcd\{k\colon\ p_k > 0\} = 1$. For any $x \in \mathbb{Z}^2$, we define

$$\xi_x = \min(n\colon\ \langle x, x + ne_1\rangle \text{ is open}).$$

Note that ξ_x are i.i.d. random variables with distribution

$$\mathrm{P}(\xi_0 > n) = \prod_{i=1}^{n}(1 - p_i).$$

Since $\sum_n p_n = \infty$, the random variables are finite almost surely. Moreover,

$$\mathrm{E}\,\xi_x = \sum_{n=0}^{\infty} \prod_{i=1}^{n} (1 - p_i) < \infty.$$

From Theorem 2.1 we conclude that there exists n_0 such that, for all $n \geq n_0$,

$$\mathrm{P}(0 \text{ and } n \text{ are connected in } [0, n]) \geq \frac{1}{2\,\mathrm{E}\,\xi_0} = \left(2 \sum_{n=0}^{\infty} \prod_{i=1}^{n} (1 - p_i)\right)^{-1} > 0.$$

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