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NILPOTENT ORBIT VARIETIES AND THE ATOMIC DECOMPOSITION OF THE Q-KOSTKA POLYNOMIALS

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ABSTRACT. We study the coordinate rings $k[\overline{C_\mu} \cap \mathfrak{t}]$ of scheme-theoretic intersections of nilpotent orbit closures with the diagonal matrices. Here μ' gives the Jordan block structure of the nilpotent matrix. de Concini and Procesi [5] proved a conjecture of Kraft [12] that these rings are isomorphic to the cohomology rings of the varieties constructed by Springer [22, 23]. The famous q-Kostka polynomial $\tilde{K}_{\lambda\mu}(q)$ is the Hilbert series for the multiplicity of the irreducible symmetric group representation indexed by λ in the ring $k[\overline{C_\mu} \cap \mathfrak{t}]$. Lascoux and Schützenberger [15, 13] gave combinatorially a decomposition of $\tilde{K}_{\lambda\mu}(q)$ as a sum of "atomic" polynomials with non-negative integer coefficients, and Lascoux proposed a corresponding decomposition in the cohomology model.

Our work provides a geometric interpretation of the atomic decomposition. The Frobenius-splitting results of Mehta and van der Kallen [19] imply a direct-sum decomposition of the ideals of nilpotent orbit closures, arising from the inclusions of the corresponding sets. We carry out the restriction to the diagonal using a recent theorem of Broer [3]. This gives a direct-sum decomposition of the ideals yielding the $k[\overline{C_\mu} \cap \mathfrak{t}]$, and a new proof of the atomic decomposition of the q-Kostka polynomials.

1. **Introduction.** The *q-Kostka polynomials* $K_{\lambda\mu}(q)$, also called Kostka-Foulkes or Foulkes-Green polynomials, have been central to numerous developments over the last two decades at the crossroads of combinatorics, algebra, and geometry. Defined [18] through the expansion

$$s_\lambda(\mathbf{x}) = \sum_\mu K_{\lambda\mu}(q) P_\mu(\mathbf{x}; q) \tag{1.1}$$

expressing the Schur function $s_\lambda(\mathbf{x})$ as a linear combination of Hall-Littlewood polynomials $P_\mu(\mathbf{x}; q)$, they are polynomials in q which specialize upon setting $q = 1$ to the ordinary Kostka numbers $K_{\lambda\mu}$. Here λ and μ are partitions of some positive integer n; we write $\lambda \vdash n$.

Lascoux and Schützenberger [20] proved a conjecture of Foulkes [6] by expressing $K_{\lambda\mu}(q)$ as a sum

$$K_{\lambda\mu}(q) = \sum_{T \in CS(\lambda,\mu)} q^{c(T)}, \tag{1.2}$$

in which T varies over the set $CS(\lambda, \mu)$ of column-strict (also called semi-standard) tableaux of shape λ containing μ_1 ones, μ_2 twos, etc., and $c(T)$, the *charge* of T,

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$\lambda \backslash \mu$	6	51	42	411	33	321
6	1	1	1	1	1	1
51		q	q	$q+q^2$	q	$q+q^2$
42			q^2	q^2	q^2	q^2+q^3
411				q^3	0	q^3
33					q^3	q^3
321						q^4

$\lambda \backslash \mu$	6	51	42	411	33	321
6	1	0	0	0	0	0
51		q	0	q^2	0	0
42			q^2	0	0	q^3
411				q^3	0	0
33					q^3	0
321						q^4

TABLE 1: $\tilde{K}_{\lambda\mu}(q)$ AND $R_{\lambda\mu}(q)$, RESPECTIVELY, FOR $\lambda, \mu \geq (321)$.

is an intricate combinatorial statistic. Our topic, the *atomic decomposition* of the q-Kostka polynomials, comes from their subsequent work [14, 15, 13], which reveals the remarkable combinatorial structures underlying the q-Kostka polynomials. To be precise, they write the variant polynomial

$$\tilde{K}_{\lambda\mu}(q) = q^{n(\mu)} K_{\lambda\mu}(1/q) = \sum_{T \in CS(\lambda,\mu)} q^{\hat{c}(T)}, \tag{1.3}$$

where $n(\mu) = \sum_i (i-1)\mu_i$ and $\hat{c}(T) = n(\mu) - c(T)$, as

$$\tilde{K}_{\lambda\mu}(q) = \sum_{\nu \geq \mu} R_{\lambda\nu}(q), \tag{1.4}$$

in which the *atom polynomials* $R_{\lambda\nu}(q)$ themselves have non-negative coefficients.

EXAMPLE 1.5. Table 1 gives $\tilde{K}_{\lambda\mu}(q)$ and $R_{\lambda\mu}(q)$ for the partitions $\lambda, \mu \geq (321)$. Note that the only incomparable pair in this set is $(411, 33)$.

Several other algebraic and geometric interpretations of $\tilde{K}_{\lambda\mu}(q)$ have been given. Hotta and Springer [9], using a result of Spaltenstein [21], identified $\tilde{K}_{\lambda\mu}(q)$ as the Poincaré series for the multiplicities of the irreducible symmetric group representation V_λ in the cohomology groups of an algebraic variety X_μ defined by Springer [22]. Lusztig [16, 17] described $\tilde{K}_{\lambda\mu}(q)$ as the local intersection homology Poincaré series of a nilpotent orbit variety, as an affine Kazhdan-Lusztig polynomial, and as a q-analog of weight multiplicities. Our concern here will be with one further interpretation, introduced by Kraft and de Concini-Procesi [12, 5], and recently given a simple and purely elementary treatment by Garsia and Procesi [7], which we now pause to describe.

In brief, the Kraft-de Concini-Procesi approach forms rings giving the q-Kostka polynomials—and isomorphic to the cohomology rings of Springer's varieties—from the *scheme-theoretic intersections* of *nilpotent orbit varieties* with the diagonal matrices $\mathfrak{t}$. Here we work over the field $k = \mathbb{C}$; the nilpotent orbit variety $\overline{C_\mu}$ is the Zariski closure of C_μ, the conjugacy class of nilpotent matrices having Jordan block sizes $\mu'_1, \mu'_2, \ldots, \mu'_n$, with $\mu \vdash n$. The geometric relation

$$\overline{C_\mu} = \bigcup_{\nu \geq \mu} C_\nu \tag{1.6}$$

will be fundamental to this work. Denoting by $k[X]$ the ring of polynomial functions on an algebraic variety X, and by $\mathcal{I}(X)$ the ideal of X, the scheme-theoretic intersection $\overline{C_\mu} \cap \mathfrak{t}$ may be defined by its ring of functions

$$k[\overline{C_\mu} \cap \mathfrak{t}] = k[\mathfrak{gl}_n] / \big(\mathcal{I}(C_\mu) + \mathcal{I}(\mathfrak{t})\big). \tag{1.7}$$

(We note that [5] and [25] gave generators for $\mathcal{I}(C_\mu) + \mathcal{I}(\mathfrak{t})$, while [26] gave generators for $\mathcal{I}(C_\mu)$.) We view $k[\mathcal{N} \cap \mathfrak{t}]$ as an S_n-module by restricting to the permutation matrices the $\mathrm{GL}_n(k)$ conjugation action on $\mathfrak{gl}_n(k)$, and write

$$\tilde{K}_{\lambda\mu}(q) = \sum_d \mathrm{mult}\Big(\chi^\lambda, \big(k[\overline{C_\mu} \cap \mathfrak{t}]\big)_d\Big) q^d; \tag{1.8}$$

here $(k[\overline{C_\mu} \cap \mathfrak{t}])_d$ denotes the homogeneous component of degree d, and $\mathrm{mult}(\chi^\lambda, V)$ denotes the multiplicity of the irreducible character χ^λ in the character of an S_n module V. This identification first arose as a consequence of the work of Kraft, de Concini, Procesi, Spaltenstein, Springer, and Hotta and Springer [12, 5, 21, 22, 23, 9], while Garsia and Procesi [7] gave an elementary proof. Other important papers discussing aspects of this construction include Kostant [10], Steinberg [24], Bergeron and Garsia [2], and Carrell [4].

In this paper we interpret the atomic decomposition (1.4) in the Kraft-de Concini-Procesi setting, thus proving anew the non-negativity of the atom polynomials $R_{\lambda\nu}$. Our proof departs from the purely elementary spirit of Garsia and Procesi, relying fairly heavily upon algebraic geometry. Using a Frobenius-splitting result of Mehta and van der Kallen we deduce that there is a direct-sum decomposition of the coordinate rings of the nullcone, $\mathcal{N}$,

$$k[\mathcal{N}] = \bigoplus_{\nu \vdash n} \hat{A}_\nu \tag{1.9}$$

compatible with the ideals of the nilpotent orbit varieties:

$$\mathcal{I}(C_\mu) = \bigoplus_{\nu \not\geq \mu} \hat{A}_\nu \quad \text{for all } \mu \vdash n. \tag{1.10}$$

This decomposition holds in prime characteristics (Corollary 4.2) and hence also in characteristic zero (Theorem 5.12). Then, a recent result of Broer (intended by him for

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quite different purposes) enables us to carry the decomposition down to the diagonal matrices. More precisely, we show that there is a direct-sum decomposition

$$k[\mathcal{N} \cap \mathfrak{t}] = \bigoplus_{\nu} A_\nu \tag{1.11}$$

of graded S_n-modules, such that for all μ,

$$I(\overline{C_\mu} \cap \mathfrak{t})/I(\mathcal{N} \cap \mathfrak{t}) = \bigoplus_{\nu \not\geq \mu} A_\nu. \tag{1.12}$$

From this it is immediate that (1.4) holds with

$$R_{\lambda\nu}(q) = \sum_d \operatorname{mult}\big(\chi^\lambda, (A_\nu)_d\big) q^d. \tag{1.13}$$

Lascoux [13, Theorem 6.5] anticipated our result by stating, without proof, a similar algebraic version of the atomic decomposition, in the setting of the cohomology ring of the flag variety.

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2. **Main theorems.** We will state our results using the terminology of lattice theory. For k a field of characteristic 0, the homogeneous GL_n-submodules of the ring $k[\mathcal{N}]$, because each graded component is finite-dimensional, by complete reducibility form a complemented modular lattice under the operations of $\cap$ and $+$. We denote $\mathcal{L}(\{I_1, \ldots, I_r\})$ the sublattice generated by the ideals $\{I_1, \ldots, I_r\}$. A lattice $\mathcal{L}$ is *distributive*—a stronger property than modular—if

$$I \cap (J + K) = (I \cap J) + (I \cap K) \quad \text{for all } I, J, K \in \mathcal{L}. \tag{2.1}$$

For example, a set of subvarieties $\{X_1, \ldots, X_r\}$ of $\mathcal{N}$ generates a distributive lattice, denoted $\mathcal{L}(\{X_1, \ldots, X_r\})$, under $\cap$ and $\cup$. The map I sends such a lattice to a collection of radical ideals, which do not necessarily themselves form a lattice.

REMARK 2.2. In particular, for any n, the underlying set of $\mathcal{L}(\{\overline{C_\mu} : \mu \vdash n\})$ is the set of all unions of the nilpotent orbit varieties $\overline{C_\mu}$. This follows from (1.6), which yields $\overline{C_\mu} \cap \overline{C_\nu} = \overline{C_{\mu\vee\nu}}$, where $\vee$ represents the least upper bound operation in the lattice P_n of partitions of n under the dominance order (see [18, p. 11]).

Finally, a bijection $\varphi : A \longrightarrow B$ between the underlying sets of two lattices A, B is a *(lattice) isomorphism* if for all $x, y \in A$ we have $x \leq y$ iff $\varphi(x) \leq \varphi(y)$, and a *(lattice) anti-isomorphism* if $x \leq y$ iff $\varphi(x) \geq \varphi(y)$. We are now in a position to state our main results.

THEOREM 2.3. *Let k be an algebraically closed field, and fix $n > 0$. Then*

1. the map I induces a lattice anti-isomorphism

$$L\left(\{\overline{C_\mu} : \mu \vdash n\}\right) \to L\left(\{I(C_\mu) : \mu \vdash n\}\right). \tag{2.4}$$

If char $k = 0$, *then*

2. $L\left(\{I(C_\mu) : \mu \vdash n\} \cup \{I(\mathfrak{t})\}\right)$ is distributive, and

3. the map $\overline{C_\mu} \mapsto I(\overline{C_\mu} \cap \mathfrak{t})/I(N \cap \mathfrak{t})$ induces a lattice anti-isomorphism

$$L\left(\{\overline{C_\mu} : \mu \vdash n\}\right) \to L\left(\{I(\overline{C_\mu} \cap \mathfrak{t})/I(N \cap \mathfrak{t}) : \mu \vdash n\}\right). \tag{2.5}$$

Geometrically, the first property means that any intersection of nilpotent orbit varieties is *scheme-theoretically reduced*: that

$$I(C_\mu) + I(C_\nu) = I(C_{\mu\vee\nu}). \tag{2.6}$$

The second property implies that the projection $k[N] \longrightarrow k[N \cap \mathfrak{t}]$ induces a lattice isomorphism, and thus, together with the first, implies the third property. In Section 3 we prove that the desired results (1.8)–(1.9) and (1.10)–(1.11) follow from Theorem 2.3. We should point out that the direct-sum decompositions are not canonical; see the general construction in the proof of Proposition 3.1 for the choices involved.

Before proceeding to the full details, we sketch the proofs here. We identify the lattice generated by $\{I(C_\mu) : \mu \vdash n\}$ by showing, in section 4, that every element of the lattice is radical. This fact is a consequence of a recent geometric result:

THEOREM 2.7 (MEHTA–VAN DER KALLEN [19]). *Let k be an algebraically closed field of positive characteristic. There is a Frobenius splitting ϕ of $\mathfrak{gl}_n$ such that every nilpotent orbit variety $\overline{C_\mu}$ is compatibly Frobenius split.*

Since the distributivity is proved in positive characteristic, we must make a standard technical argument to extend the result to characteristic 0 (section 5). This proves Theorem 2.3 (1). Finally, in section 6, we address the intersection with the diagonal, showing that the lattice generated by the $I(C_\mu)$ together with $I(\mathfrak{t})$ is still distributive, even though its elements are no longer radical.

This argument relies on a result of Broer, who in [3] extends Chevalley's restriction theorem to modules of covariants. His theorem holds in great generality, but we will need it only for the Lie group GL_n and Lie algebra $\mathfrak{gl}_n$. Let $T \subset \mathrm{GL}_n$ be the subgroup of invertible diagonal matrices, and $\mathfrak{t} \subset \mathfrak{gl}_n$ its Lie algebra. Any GL_n-module M has a representation of the Weyl group S_n on the fixed-point set M^T of the T-action. Broer's theorem applies to *small* GL_n-modules: those which do not have the T-weight 2ϕ, where ϕ is the highest root of GL_n.

THEOREM 2.8 (BROER [3]). *Let k be an algebraically closed field of characteristic zero, and let M be a small GL_n-module. For any nilpotent orbit variety $\overline{C_\mu}$, the map*

$$\mathrm{Hom}_{\mathrm{GL}_n}\left(M, k[\overline{C_\mu}]\right) \to \mathrm{Hom}_{S_n}\left(M^T, k[\overline{C_\mu} \cap \mathfrak{t}]\right) \tag{2.9}$$

induced from the restriction $k[\overline{C_\mu}] \to k[\overline{C_\mu} \cap \mathfrak{t}]$ is an isomorphism of graded vector spaces.

		$A_\nu(q)$					
		6	51	42	411	33	321
λ	6	1					
	51		x_6		x_6^2		
	42			x_5x_6			$e_3(x_3, x_4, x_5, x_6)$
	411				$x_5x_6^2$		
	33					$x_4x_5x_6$	
	321						$x_4x_5x_6^2$

TABLE 2: GENERATORS OF IRREDUCIBLES OF TYPE λ IN $A_\nu(q)$, FOR $\nu \geq (321)$.

We note as well that the identification (1.8) combined with Broer's theorem gives

$$\sum_{d\geq 0} \dim \operatorname{Hom}\Big(M_\lambda, \big(k[\overline{C_\mu}]\big)_d\Big) = \tilde{K}_{\lambda\mu}(q), \tag{2.10}$$

where the representations M_λ are defined in Section 6; and hence the atomic decomposition (1.4) of the q-Kostka polynomials follows directly from the direct-sum decomposition (1.9)–(1.10) of $k[\mathcal{N}]$. We have chosen the slightly less direct route of obtaining the decomposition inside the intersection with the diagonal, (1.11)–(1.12), because of the important role of the latter in the previous work of Springer [22], Kraft [12], de Concini-Procesi [5], Garsia-Procesi [7], and others.

EXAMPLE 2.11. Using Macaulay 2 [1], we found bases of the spaces $A_\nu \subset k[\mathcal{N} \cap \mathfrak{t}]$ whose characters are described by the atomic polynomials $R_{\lambda\nu}$ of example 1.5. In Table 2 we give one vector in each irreducible S_n-representation. This example is somewhat trivial, since there are only two incomparable partitions, and thus the corresponding lattice is automatically distributive; in addition, the graded components of the atoms are multiplicity-free. The example would have to be significantly larger ($n = 10$) to exhibit greater complexity.

3. Distributive lattices.

We begin our proof with a general statement on the connection between distributive lattices and direct sums. The cases which occur may be treated together as follows. Let G be a reductive group, and M a graded G-module. (We assume each component of a graded module to be finite-dimensional.) Then the graded submodules of M form a modular lattice under $\cap$ and $+$, with least element $\{0\}$ and greatest element M, which by complete reducibility is a complemented modular lattice.

In the following proposition, $\hat{0}$ and $\hat{1}$ are the generic symbols for the least and greatest elements of a lattice when they exist. We use $\cap$ and $+$ for the meet and join operations in the lattice. We also use $\bigoplus_{i\in A} I_i$, analogous to a direct sum of modules, to mean $\sum_{i\in A} I_i$ with the condition that $I_j \cap \sum_{i\in(A-\{j\})} I_i = \hat{0}$ for every $j \in \mathcal{A}$.

PROPOSITION 3.1. *Let $\hat{\mathcal{L}}$ be a complemented modular lattice with $\hat{0}$ and $\hat{1}$. Let $\mathcal{L}$ be the sub-lattice of $\hat{\mathcal{L}}$ generated by $I_1, I_2, \ldots, I_r$. Then the following statements are equivalent:*

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(1) There exists a family $\{V_i\}_{i\in A}$ *of elements of* $\hat{\mathcal{L}}$, *and subsets* $A_L \subset A$ *for each* $L \in \mathcal{L}$, *such that* $I_1 + \cdots + I_r = \bigoplus_{i\in A} V_i$ *and*

$$L = \bigoplus_{i\in A_L} V_i \quad \text{for all } L \in \mathcal{L}. \tag{3.2}$$

(2) The lattice $\mathcal{L}$ *is distributive.*

PROOF. Because the $\{V_i\}_{i\in A}$ generate a Boolean algebra containing $\mathcal{L}$, (1) implies (2). Now assume (2). We may write any element of $\mathcal{L}$ as a sum of intersections by distributing intersections over sums. Let $\mathcal{L}_0$ be the subposet of $\mathcal{L}$ consisting of all intersections of the I_j's. For (1) it suffices to find a direct-sum decomposition of $I_1 + \cdots + I_r$ such that every element of $\mathcal{L}_0$ is a direct sum of some of the summands. Index the intersections by subsets of $\{1, 2, \ldots, r\}$, so that $I_{\{i_1,i_2,\ldots,i_s\}} = I_{i_1} \cap I_{i_2} \cap \cdots \cap I_{i_s}$. Let $I_\emptyset = I_1 + \cdots + I_r$. Then associate to each I_S a complement K_S of $\sum_{T \supsetneq S} I_T$ in I_S; in particular, $K_\emptyset = \hat{0}$. Distributivity gives, for any $S \subset \{1, \ldots, n\}$,

$$I_S \cap \sum_{T\neq S} K_T = \sum_{T\neq S} (I_S \cap K_T) \subseteq \sum_{T\supsetneq S} I_T, \tag{3.3}$$

since if $T \subsetneq S$ then $I_S \cap K_T = \hat{0}$ by definition, and otherwise $I_S \cap K_T \subseteq I_{S\cup T}$. Because $K_S \cap \sum_{T\supsetneq S} I_T = \hat{0}$, we have shown $\bigoplus_S K_S$ is a direct sum; and clearly this equals $I_\emptyset = I_1 + \cdots + I_r$. Induction then gives $I_S = \bigoplus_{T\supseteq S} K_T$ for all S. ∎

Applying the proposition to the lattice $\mathcal{L}$ of graded GL_n-submodules of $k[\mathfrak{gl}_n]$, we have:

COROLLARY 3.4. *Theorem 2.3(1) is equivalent to (1.9)–(1.10).*

PROOF. Assuming first that Theorem 2.3(1) holds, we apply Proposition 3.1 to the lattice of graded submodules of $k[\mathcal{N}]$ and the sublattice $\mathcal{L}\{\mathcal{I}(C_\mu) : \mu \in P_n\}$. From the proof of the proposition, the decomposition may be written

$$k[\mathcal{N}] = \bigoplus_{S\subseteq P_n} A_S, \tag{3.5}$$

with

$$\mathcal{I}(C_\mu) = \bigoplus_{S\ni\mu} A_S. \tag{3.6}$$

Now the equation (2.6) implies

$$\bigoplus_{S\ni\mu \text{ or } S\ni\nu} A_S = \bigoplus_{S\ni\mu\vee\nu} A_S; \tag{3.7}$$

hence if $\mu \vee \nu \in S$ but $\mu \notin S$ and $\nu \notin S$ then $A_S = (0)$. Equivalently, if we write $\overline{S} = P_n - S$, then we have $A_S = (0)$ whenever $\mu \in \overline{S}$, and $\nu \in \overline{S}$, but $\mu \vee \nu \notin \overline{S}$; that is, we may take $\overline{S}$ to be closed under $\vee$. Furthermore, since $\nu \geq \mu$ implies $\mathcal{I}(C_\nu) \supseteq \mathcal{I}(C_\mu)$, we have $A_S = (0)$ unless S is an upper order ideal. Combining the two conditions, $\overline{S}$ may be taken to be a principal lower order ideal. So we re-index, writing $\hat{A}_\nu$ for $A_{\overline{\{\mu:\mu\leq\nu\}}}$; now (1.9)–(1.10) follows, since we have

$$I(C_\mu) = \bigoplus_{S\ni\mu} A_S = \bigoplus_{\nu\not\geq\mu} \hat{A}_\nu. \tag{3.8}$$

The converse is immediate. ∎

Similarly, (1.11)–(1.12) is equivalent to Theorem 2.3(3).

4. **Characteristic** p**.** In this section, let k be an algebraically closed field of characteristic $p > 0$, and use the same notation for nilpotent orbit varieties as in the Introduction. We use the method of Frobenius splitting to prove Theorem 2.3(1) over k. Let A be a commutative k-algebra. The *Frobenius map* $F: A \longrightarrow A$ is $F(a) = a^p$. Let A' be A considered as an A-module under F.

DEFINITION 4.1. A is *Frobenius split* if there exists an A-module homomorphism $\phi : A' \longrightarrow A$ such that $\phi \circ F = \mathrm{id}_A$. (Equivalently, such that $\phi(1) = 1$.)

If A is Frobenius split, A is necessarily reduced: this will be our application of the Frobenius splitting. For $I \subset A$ an ideal, $F(I) \subseteq I$, hence $I \subseteq \phi(I)$. If $\phi(I) = I$, then A/I is Frobenius-split by the map induced from ϕ, and indeed all of the objects I, A/I, and $\operatorname{Spec} A/I$ are said to be *compatibly Frobenius split* by ϕ.

Mehta and van der Kallen proved (Theorem 2.7) that there is a splitting of $\mathfrak{gl}_n(k)$ such that the nilpotent orbit varieties are compatibly Frobenius split. Since this property is preserved by sums and intersections, as a corollary we have that every element of the lattice generated by the $I(C_\mu)$ is compatibly Frobenius split, and therefore radical. Using the order-reversing map I of Hilbert's Nullstellensatz, we write more precisely:

COROLLARY 4.2 (TO THEOREM 2.7). *Theorem 2.3(1) holds for k algebraically closed of positive characteristic.*

5. **Characteristic 0.** We must now prove that the lattice $L\big(\{I(C_\mu)\}\big)$ is distributive when k is an algebraically closed field of characteristic zero. We begin by defining an ideal $I(C_\mu)$ in any commutative ring R; the definition will coincide with the original for R an algebraically closed field. (Note that we do not concern ourselves with the set C_μ over R, though it may be defined as before.) Let $\mathbf{X}$ be the n by n matrix of indeterminates $\{x_{11}, x_{12}, \ldots, x_{nn}\}$, so that $R[\mathbf{X}]$ is the coordinate ring of $\mathfrak{gl}_n(R)$, and $R[\mathrm{GL}_n] = R[\mathbf{X}][\frac{1}{\det \mathbf{X}}]$ is the coordinate ring of $\mathrm{GL}_n(R)$.

DEFINITION 5.1. For R a commutative ring, let the ideal $I\big(C_\mu(R)\big) \subset R[\mathbf{X}]$ be the kernel of the map $R[\mathbf{X}] \longrightarrow R[\mathrm{GL}_n]$ that is derived from the map $\mathrm{GL}_n \longrightarrow \mathfrak{gl}_n$ taking $g \longmapsto gN_\mu g^{-1}$, where N_μ is the standard nilpotent matrix of Jordan block structure μ'.

We must compare these ideals with the result of a simpler "extension of scalars" from $\mathbb{Z}$ to R:

DEFINITION 5.2. Consider an ideal I in a commutative ring A. For any commutative ring R, denote by $R \cdot I$ the image of the map $I \otimes_{\mathbb{Z}} R \longrightarrow A \otimes_{\mathbb{Z}} R$.

LEMMA 5.3. *Let R be a commutative ring which is torsion-free as a $\mathbb{Z}$-module. Then* $I\big(C_\mu(R)\big) = R \cdot I\big(C_\mu(\mathbb{Z})\big)$.

PROOF. From Definition 5.1 we have the exact sequence

$$0 \longrightarrow I\big(C_\mu(\mathbb{Z})\big) \longrightarrow \mathbb{Z}[\mathbf{X}] \longrightarrow \mathbb{Z}[\mathrm{GL}_n]. \tag{5.4}$$

Since R is torsion-free, it is a flat $\mathbb{Z}$-module, and the functor $\otimes_{\mathbb{Z}} R$ is exact. Further, $\mathbb{Z}[\mathbf{X}] \otimes_{\mathbb{Z}} R = R[\mathbf{X}]$ and $\mathbb{Z}[\mathrm{GL}_n] \otimes_{\mathbb{Z}} R = R[\mathrm{GL}_n]$, so tensoring the sequence (5.4) over $\mathbb{Z}$ with R we obtain

$$0 \longrightarrow I\big(C_\mu(\mathbb{Z})\big) \otimes_{\mathbb{Z}} R \longrightarrow R[\mathbf{X}] \longrightarrow R[\mathrm{GL}_n]. \tag{5.5}$$

By definition, $R \cdot I\big(C_\mu(\mathbb{Z})\big)$ is the image of $I\big(C_\mu(\mathbb{Z})\big) \otimes_{\mathbb{Z}} R$ in $R[\mathbf{X}]$, so comparing (5.5) with Definition 5.1 yields the result. ∎

LEMMA 5.6. *Fix a partition μ, and let k be a field. The truth of the statement*

$$I\big(C_\mu(k)\big) = k \cdot I\big(C_\mu(\mathbb{Z})\big) \tag{5.7}$$

depends only on $\operatorname{char} k$*; furthermore, (5.7) holds for all but finitely many primes* $\operatorname{char} k$.

PROOF. We take a computational point of view for simplicity and to indicate the extreme generality of the arguments used for such results. Reduced Gröbner bases of $I\big(C_\mu(k)\big)$ and $k \cdot I\big(C_\mu(\mathbb{Z})\big)$ may be computed from Definition 5.1, and these bases suffice to test (5.7). The only operations from k involved in a Gröbner-basis computation are arithmetic ones: adding, subtracting, multiplying, dividing, and comparing with 0. Since the map $k[\mathbf{X}] \longrightarrow k[\mathrm{GL}_n]$ is defined over the prime field k_0, the computation will involve only arithmetic in k_0, and thus depends only on the characteristic of k.

Lemma 5.3 implies that (5.7) holds for $\operatorname{char} k = 0$. Only comparing with 0, of the arithmetic operations, depends on the characteristic. Furthermore, the computation for $k = \mathbb{Q}$, being finite, involves only a finite number of comparisons with 0. These comparisons are the only points at which the computations in various characteristics might differ. Thus there are only a finite number of characteristics, those dividing the coefficients that are compared with 0, in which (5.7) might be false. ∎

We will use the abbreviation "almost all positive characteristics" for "all but a finite number of positive characteristics." The argument of Lemma 5.6 easily adapts to prove:

LEMMA 5.8. *Let I, J, and K be ideals in $\mathbb{Z}[\mathbf{X}]$, and let k be a field. The truth of the statements*

$$k \cdot I + k \cdot J = k \cdot K \tag{5.9}$$

and

$$k \cdot I \cap k \cdot J = k \cdot K \tag{5.10}$$

depends only on the characteristic of k. Furthermore, the truth of (5.9) or (5.10) is the same in characteristic zero as in almost all positive characteristics.

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We thus wish to characterize the condition of Theorem 2.3(1) in terms of equalities among finitely many intersections and sums of ideals.

LEMMA 5.11. *Let k be an algebraically closed field. Then the following are equivalent:*

1. *the map $\mathcal{I}$ induces a lattice isomorphism $L\left(\{C_\mu(k)\}\right) \to L\left(\left\{\mathcal{I}\left(C_\mu(k)\right)\right\}\right)$.*
2. *$L\left(\left\{\mathcal{I}\left(C_\mu(k)\right)\right\}\right)$ is distributive, and $\mathcal{I}\left(C_\mu(k)\right) + \mathcal{I}\left(C_\nu(k)\right) = \mathcal{I}\left(C_{\mu\vee\nu}(k)\right)$ for every $\mu, \nu \in P_n$.*

PROOF. That (1) implies (2) is clear from Remark 2.2 and the fact that $L\left(\{C_\mu(k)\}\right)$ is distributive. Assuming (2), the distributivity implies that every element of $L\left(\left\{\mathcal{I}\left(C_\mu(k)\right)\right\}\right)$ may be written as an intersection of sums of $\mathcal{I}(C_\mu)$'s. The sum condition implies that these may be simplified to intersections of $\mathcal{I}(C_\mu)$'s; and any such intersection $\mathcal{I}(C_{\mu^{(1)}}) \cap \cdots \cap \mathcal{I}(C_{\mu^{(l)}})$ is equal to $\mathcal{I}(C_{\mu^{(1)}} \cup \cdots \cup C_{\mu^{(l)}})$. Hence the map $\mathcal{I}$ induces a bijection, which is readily seen to be a lattice isomorphism, so (1) follows. ∎

Thus we obtain Theorem 2.3(1), in fact for "almost all" fields:

THEOREM 5.12. *Fix $n > 0$. The conditions of Lemma 5.11(2) hold for any field k (except perhaps for k not algebraically closed and of a finite number of positive characteristics).*

PROOF. The distributivity and sum conditions on $L\left(\left\{\mathcal{I}\left(C_\mu(k)\right)\right\}\right)$ constitute a finite set of equations in the lattice. We know from Corollary 4.2 that these equations hold in $L\left(\left\{\mathcal{I}\left(C_\mu(k)\right)\right\}\right)$ for k algebraically closed of characteristic $p > 0$. Lemma 5.6 implies that the equations therefore hold in $L\left(\left\{k\cdot\mathcal{I}\left(C_\mu(\mathbb{Z})\right)\right\}\right)$ for almost all char $k > 0$. Thus by Lemma 5.8, the equations must hold in $L\left(\left\{k\cdot\mathcal{I}\left(C_\mu(\mathbb{Z})\right)\right\}\right)$ for char $k = 0$ as well. Finally, this and Lemma 5.3 imply that the equations hold in $L\left(\left\{\mathcal{I}\left(C_\mu(k)\right)\right\}\right)$ for char $k = 0$. ∎

6**. Intersecting with the diagonal.** In this section we prove Theorem 2.3, parts (2) and (3). We assume throughout that k is an algebraically closed field of characteristic zero. The ring $k[\mathcal{N} \cap \mathfrak{t}] = k[\mathbf{X}]/\left(\mathcal{I}(\mathcal{N}) + \mathcal{I}(\mathfrak{t})\right)$ is naturally isomorphic to the ring

$$R = k[x_1, \ldots, x_n]/\left(e_1(\mathbf{x}), \ldots, e_n(\mathbf{x})\right), \tag{6.1}$$

where $e_i(\mathbf{x})$ is the i-th elementary symmetric function of $\mathbf{x} = \{x_1, \ldots, x_n\}$. We let the ideal I_μ be the image of $\mathcal{I}(C_\mu) + \mathcal{I}(\mathfrak{t})$ in R, and define $R_\mu = R/I_\mu$; we will tend to consider the isomorphism $R_\mu \simeq k[\overline{C_\mu} \cap \mathfrak{t}]$ an equality. Note also that $I_{(1^n)} = (0)$ and $R_{(1^n)} = R$.

We first apply Broer's result (Theorem 2.8) to $\mathcal{N}$. Consider a family of irreducible (simple) GL_n-modules $\{M_\lambda : \lambda \vdash n\}$ such that (a) each M_λ is small; (b) the zero-weight space M_λ^T is the irreducible S_n-module with character χ^λ. (Gutkin [8] and Kostant [11], and perhaps others, observed that the usual indexing of the GL_n-irreducibles by partitions leads to the required family.) Broer's theorem implies that we can construct an S_n-module

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isomorphic to R inside $k[\mathcal{N}]$ as follows. Let P_λ be the operator of projection onto the GL_n-isotypic component of type M_λ. Let

$$\check{R} = \bigoplus_{\lambda\vdash n}\big(P_\lambda k[\mathcal{N}]\big)^T. \tag{6.2}$$

Consider $\mathrm{mult}\big(\chi^\lambda, (R)_d\big)$, the multiplicity in $(R)_d$ of an irreducible S_n-character χ^λ. This multiplicity, by Schur's Lemma, is $\dim \mathrm{Hom}_{S_n}\big(M_\lambda^T, (R)_d\big)$, which by Theorem 2.8, in the case $\mu = (1^n)$, is equal to $\dim \mathrm{Hom}_{\mathrm{GL}_n}\big(M_\lambda, (k[\mathcal{N}])_d\big)$. Thus $P_\lambda\big((k[\mathcal{N}])_d\big) \simeq M_\lambda^{\mathrm{mult}(\chi^\lambda,(R)_d)}$ as GL_n-modules, and

$$\Big(P_\lambda\Big(\big(k[\mathcal{N}]\big)_d\Big)\Big)^T \simeq (M_\lambda^T)^{\mathrm{mult}(\chi^\lambda,(R)_d)} \simeq (R)_d \tag{6.3}$$

as S_n-modules. Furthermore, the isomorphism in Theorem 2.8 is induced by the map $k[\mathcal{N}] \longrightarrow k[\mathcal{N} \cap \mathfrak{t}]$; thus we have that the restriction of this map to $\check{R}$ induces an isomorphism of graded S_n-modules

$$\check{R} \xrightarrow{\simeq} R. \tag{6.4}$$

Denote the image of $\mathcal{I}(\mathfrak{t}) + \mathcal{I}(\mathcal{N})$ in $k[\mathcal{N}]$ by L; *i.e.*, L is the ideal in $k[\mathcal{N}]$ generated by the off-diagonal matrix entries. Similarly, denote the image of $\mathcal{I}(C_\mu)$ in $k[\mathcal{N}]$ by J_μ. Then the isomorphism $R \xrightarrow{\simeq} \check{R}$ splits the short exact sequence

$$0 \longrightarrow L \longrightarrow k[\mathcal{N}] \longrightarrow R \longrightarrow 0, \tag{6.5}$$

giving

$$k[\mathcal{N}] = L \oplus \check{R}. \tag{6.6}$$

The functors $\mathrm{Hom}_{\mathrm{GL}_n}(M, \_\_)$ and $\mathrm{Hom}_{S_n}(M^T, \_\_)$ are exact on the categories of graded GL_n and S_n-modules, respectively. Thus the commutative diagram with exact rows, and vertical arrows given by reduction modulo L,

$$\begin{array}{ccccccccc} 0 \longrightarrow & J_\mu & \longrightarrow & k[\mathcal{N}] & \longrightarrow & k[\overline{C_\mu}] & \longrightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \longrightarrow & I_\mu & \longrightarrow & R & \longrightarrow & R_\mu & \longrightarrow 0 \end{array} \tag{6.7}$$

yields a commutative diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathrm{Hom}_{\mathrm{GL}_n}(M, J_\mu) & \longrightarrow & \mathrm{Hom}_{\mathrm{GL}_n}(M, k[\mathcal{N}]) & \longrightarrow & \mathrm{Hom}_{\mathrm{GL}_n}(M, k[\overline{C_\mu}]) & \longrightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \longrightarrow & \mathrm{Hom}_{S_n}(M^T, I_\mu) & \longrightarrow & \mathrm{Hom}_{S_n}(M^T, R) & \longrightarrow & \mathrm{Hom}_{S_n}(M^T, R_\mu) & \longrightarrow 0 \end{array} \tag{6.8}$$

whose rows are again exact. Theorem 2.8 says that the second and third vertical arrows are isomorphisms; thus by the Five Lemma the first is as well. Note that, for each

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$\mu \vdash n$, the intersection $J_\mu \cap \check{R}$ is $\oplus_{\lambda \vdash n}(P_\lambda J_\mu)^T$. Since the first vertical arrow in (6.8) is an isomorphism, it follows by the reasoning used for (6.4) that $J_\mu \cap \check{R} \xrightarrow{\simeq} I_\mu$. So

$$J_\mu = (J_\mu \cap L) \oplus (J_\mu \cap \check{R}). \tag{6.9}$$

Every element I of $\mathcal{L}(\{J_\mu\})$ will therefore have the property that $I = (I \cap L) \oplus (I \cap \check{R})$. Since $\mathcal{L}(\{J_\mu\})$ is distributive by Theorem 5.12, it follows that $\mathcal{L}(\{J_\mu\} \cup \{L\})$ is as well. The sublattice of elements containing L therefore projects isomorphically modulo L onto $\mathcal{L}(\{I_\mu\})$, which completes the proof of Theorem 2.3.

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