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Short Time Behavior of Solutions to Linear and Nonlinear Schrödinger Equations

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Abstract. We examine the fine structure of the short time behavior of solutions to various linear and nonlinear Schrödinger equations $u_t = i\Delta u + q(u)$ on $I \times \mathbb{R}^n$, with initial data $u(0,x) = f(x)$. Particular attention is paid to cases where f is piecewise smooth, with jump across an $(n-1)$-dimensional surface. We give detailed analyses of Gibbs-like phenomena and also focusing effects, including analogues of the Pinsky phenomenon. We give results for general n in the linear case. We also have detailed analyses for a broad class of nonlinear equations when $n = 1$ and 2, with emphasis on the analysis of the first order correction to the solution of the corresponding linear equation. This work complements estimates on the error in this approximation.

1 Introduction

This paper continues work of [24], analyzing the behavior near $t = 0$ of the solution to an initial value problem

$$\frac{\partial u}{\partial t} = i\Delta u + q(u), \quad u(0,x) = f(x), \tag{1.1}$$

where f is a function on $\mathbb{R}^n$ and $q\colon \mathbb{R}^2 \to \mathbb{R}^2 \approx \mathbb{C}$ is a smooth map. In [24] this was investigated for f satisfying

$$\|f\|_{H^{\sigma,2}(\mathbb{R}^n)} \le A, \quad \|e^{it\Delta} f\|_{L^\infty(\mathbb{R}^n)} \le A, \tag{1.2}$$

with $n = 1$ or 2. It was assumed that $\sigma \ge 0$ for $n = 1$ and $\sigma \in (0,1)$ for $n = 2$, and that $q(0) = 0$, $Dq(0) = 0$, where $Dq(u)$ is the 2×2 matrix of partial derivatives of q, one family of examples being $q(u) = \lambda |u|^{2k} u$, for some $k \in \mathbb{N}$, $\lambda \in \mathbb{C}$. In such cases it was shown that there exists $T_0 > 0$ such that for $t \in [0, T_0]$ the solution to (1.1) exists and satisfies

$$u(t) = u_0(t) + \int_0^t e^{i(t-s)\Delta} q(u_0(s))\, ds + w(t), \quad u_0(t) = e^{it\Delta} f, \tag{1.3}$$

with the remainder estimate $\|w(t)\|_{L^\infty(\mathbb{R}^n)} \le Ct^\alpha$, where $\alpha = 3/2$ for $n = 1$, and one can take any $\alpha < 1 + \sigma$ for $n = 2$. For related work we mention [9], which treated the case $n = 1$, $q(u) = -i|u|^2 u$, and $f = \chi_I$, the characteristic function of an interval. We also mention [12], treating the short time behavior for $n \ge 3$, with $\sigma > (n-2)/2$ in (1.2).

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As shown in [24], the hypotheses in (1.2) apply when $n = 1$ or 2 and f is compactly supported and piecewise smooth with jump discontinuity. We are particularly interested here in a precise analysis of the right side of (1.3) for such f. This analysis has two parts. First one wants a precise description of $u_0(t) = e^{it\Delta} f$. Then one wants a precise description of the integrand in (1.3). To change notation slightly, we want to understand

$$v_0(s,t) = e^{it\Delta}\, q(e^{is\Delta} f), \tag{1.4}$$

uniformly for $s, t \in [0, T_0]$.

Certainly one useful tool in such an analysis is the integral formula

$$e^{is\Delta} f(x) = (4\pi i s)^{-n/2} \int\limits_{\mathbb{R}^n} e^{i|x-y|^2/4s} f(y)\, dy. \tag{1.5}$$

We will see that applying this gives rise to a number of interesting problems involving oscillatory integrals. One basic, but illuminating, case is $f = \chi_I$, the characteristic function of the interval $I = [-1, 1] \subset \mathbb{R}$. In such a case, we have

$$e^{is\Delta} \chi_I(x) = (4\pi i s)^{-1/2} \int_{x-1}^{x+1} e^{iy^2/4s}\, dy = \mathrm{Fr}\Big(\frac{x+1}{\sqrt{4s}}\Big) - \mathrm{Fr}\Big(\frac{x-1}{\sqrt{4s}}\Big),$$

where $\mathrm{Fr}(x)$ is the Fresnel integral:

$$\mathrm{Fr}(x) = (\pi i)^{-1/2} \int_0^x e^{iy^2}\, dy. \tag{1.6}$$

This is a smooth, odd function of x, tending to the limits $\pm 1/2$ as $x \to \pm\infty$. More precisely, one has

$$\mathrm{Fr}(x) = \frac{1}{2}\operatorname{sgn} x + e^{ix^2}\Phi(x), \tag{1.7}$$

where $\Phi(x)$ is smooth except at $x = 0$, with $\lim_{x\to\pm 0} \Phi(x) = \mp 1/2$ and

$$\Phi(x) \sim \sum_{\nu \geq 0} a_\nu\, x^{-1-2\nu}, \quad |x| \to \infty. \tag{1.8}$$

These results on $\mathrm{Fr}(x)$ are classical [15, pp. 16–23]. They also follow from material presented in Section 2. It will also be shown in Section 2 that if f is compactly supported and $2f(x) - \operatorname{sgn} x$ is smooth, then

$$e^{it\Delta} f(x) = v(t,x) + e^{ix^2/4t}\,\Phi\Big(\frac{x}{\sqrt{4t}}\Big), \tag{1.9}$$

where Φ is as in (1.7) and

$$v(t,x) = f(x) + g(t,x), \tag{1.10}$$

with $g(t,x)$ smooth and rapidly decreasing in x. More general point singularities will be treated in Section 2.

The result (1.9) gives us a taste of what sorts of functions arise in (1.4). For example, if

$$q(u) = \lambda\, u^{\ell}\overline{u}^{m}, \quad \ell, m \in \mathbb{N}, \tag{1.11}$$

and if $e^{it\Delta} f$ is as in (1.9), then

$$q(e^{is\Delta} f) = \lambda(v + e^{ix^2/4s}\Phi_s)^{\ell}(\overline{v} + e^{-ix^2/4s}\overline{\Phi}_s)^{m}, \tag{1.12}$$

where $\Phi_s(x) = \Phi(x/\sqrt{4s})$. A binomial expansion yields

$$q(e^{is\Delta} f) = \lambda \sum_{j=0}^{\ell}\sum_{k=0}^{m}\binom{\ell}{j}\binom{m}{k} v^{\ell-j}\overline{v}^{m-k}\Phi_s^j\overline{\Phi}_s^k\, e^{i(j-k)x^2/4s}, \tag{1.13}$$

which is a sum of terms of the form

$$v_\nu(s,x)\Phi^{jk}\Big(\frac{x}{\sqrt{4s}}\Big)\, e^{i\nu x^2/4s}, \tag{1.14}$$

where $\Phi^{jk}(x) = \Phi(x)^j\overline{\Phi}(x)^k$ and $-m \le \nu \le \ell$. Here each $v_\nu(x)$ is smooth except at $x = 0$, where it might have a jump discontinuity, and it is rapidly decreasing as $|x| \to \infty$. The dependence of v_ν on s is innocuous. Thus the task of analyzing (1.4) comes down to analyzing

$$A^{jk}(s,t,x) = e^{it\Delta}\big(v_\nu\Phi_s^{jk}e^{i\nu x^2/4s}\big), \tag{1.15}$$

uniformly for $s, t \in [0, T_0]$, where $\Phi_s^{jk}(x) = \Phi^{jk}(x/\sqrt{4s})$ and $\nu = j - k$. For smooth q more general than (1.11), we can write

$$q(e^{is\Delta} f) = q\big(v + e^{ix^2/4s}\Phi_s\big) = \sum_{\nu=-\infty}^{\infty} w_\nu(s,x)e^{i\nu x^2/4s},$$

with

$$w_\nu(s,x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} q\big(v(s,x) + e^{i\theta}\Phi_s(x)\big)\, e^{-i\nu\theta}\, d\theta.$$

We take up the analysis of (1.15) in Section 4. We mention that there is a notable effect when $\nu = -1$, manifested near $t = s$. For example, it will be shown that

$$A^{01}(t,t,x) = e^{ix^2/4t}\hat{u}_\delta * \hat{\Psi}^{01}(t^{-1/2}x), \tag{1.16}$$

where $\delta = \sqrt{4t}$, $\hat{u}_\delta(z) = \delta^{-1}\hat{u}(\delta^{-1}z)$, $\hat{u}$ is a piecewise smooth function (perhaps with jump at $z = 0$) satisfying

$$|\hat{u}(z)| \le C(1 + |z|)^{-2}, \tag{1.17}$$

and $\hat{\Psi}^{01}$ is C^∞ on $\mathbb{R} \setminus 0$ and has the small z behavior

$$\hat{\Psi}^{01}(z) = A \log |z| + B \operatorname{sgn} z + R(z), \tag{1.18}$$

where $R(z)$ is Hölder continuous, and the large z behavior $\hat{\Psi}^{01}(z) \sim b_{01} z^{-1}$, $|z| \to \infty$. See Section 4 for more on this, and for more subtle results dealing with a conic neighborhood of the ray $s = t$. We mention that the appearance of this logarithmic blow-up is somewhat reminiscent of a log blow-up in the asymptotics for Fejér summability, derived in [6] (see also [22, Proposition 5.2]), though the analytical details are somewhat different.

Having an analysis of (1.15), we can turn our attention to $v_0(s, t, x)$, given by (1.4), and to

$$v(t, x) = \frac{1}{t} \int_0^t v_0(s, t - s, x)\, ds = \frac{1}{t} \int_0^t e^{i(t-s)\Delta} q(u_0(s))\, ds.$$

Certainly one has

$$\lim_{s,t \to 0} v_0(s, t, x) = q(f(x)), \tag{1.19}$$

and

$$\lim_{t \to 0} v(t, x) = q(f(x)), \tag{1.20}$$

in some sense. In fact, given hypothesis (1.2), it is not hard to deduce convergence in L^2-norm (see Section 5 for details). On the other hand, one certainly does not expect convergence in L^∞-norm. It is of interest to know if there is a uniform L^∞-bound on $v_0(s, t, \cdot)$ or on $v_0(t, \cdot)$. In fact, it follows from (1.16)–(1.18) that, for typical initial data f on $\mathbb{R}$, piecewise smooth and compactly supported,

$$\|v_0(t, t, \cdot)\|_{L^\infty(\mathbb{R})} \geq C \log \frac{1}{t}, \quad t \ll 1,$$

if $q(u)$ is a polynomial in u and $\overline{u}$. Such behavior is concentrated quite near such a ray, and, as shown in Section 5, one has $\|v(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq C$ when f is a piecewise smooth, compactly supported function on $\mathbb{R}$. Other senses in which (1.19)–(1.20) hold are discussed below.

The structure of the rest of this paper is as follows. Sections 2–3 are devoted to analysis of detailed properties of $e^{it\Delta} f(x)$, for t in a neighborhood of 0, for some special classes of functions f. Section 2 considers functions on $\mathbb{R}^n$ singular at one point, with either an algebraic or a logarithmic singularity. Section 3 considers piecewise smooth functions on $\mathbb{R}^n$, with a jump across a smooth hypersurface Σ. These classes largely overlap for $n = 1$; the additional consideration of logarithmic singularities in Section 2 will be of technical use in Section 4. In both Sections 2 and 3 we encounter variants of the Gibbs phenomenon on a neighborhood of the singularity. To treat the Gibbs phenomenon in higher dimensions, we borrow a wave equation technique from [17]. In higher dimension, there are also focusing effects, including a variant of the Pinsky phenomenon, which was introduced in [16] (and also studied in [17]) in the context of Fourier inversion and involves a failure of pointwise convergence. This Pinsky phenomenon starts in dimension 3 for Fourier inversion, but it starts in dimension 2 for the short time behavior of $e^{it\Delta} f$ when f has a jump across a sphere in $\mathbb{R}^n$. The Pinsky phenomenon is associated with a perfect focus caustic, produced by wave fronts issuing from the hypersurface Σ. When Σ is not a sphere, other types of caustics form, such as folds, etc. We discuss the behavior of $e^{it\Delta} f$ on and near such caustics.

In Section 4 we analyze the behavior of $e^{it\Delta}$ acting on a family of functions on $\mathbb{R}$, given as a product of a singular factor and an oscillatory factor, as in the family of functions $A^{jk}(s,t,x)$ defined by (1.15). There is a great deal of structure in the behavior of $A^{jk}(s,t,x)$, as $s,t \to 0$ in various regimes. As indicated above, the behavior on the ray $s = t \geq 0$ and a small neighborhood thereof has a particularly delicate structure. Also the regions $s \ll t$ and $t \ll s$ require special attention.

In Section 5 we turn our attention to the second term on the right side of (1.3), and investigate various ways in which convergence holds in (1.19)–(1.20). In particular, we show that if f is compactly supported and piecewise smooth on $\mathbb{R}^n$, $n = 1$ or 2, then (1.19)–(1.20) hold in L^2-norm and weak* in the Besov space $B^{1/2}_{2,\infty}(\mathbb{R}^n)$. If $n = 1$ and $q(u)$ is a sum of terms of the form (1.11), we also have $v(t) \to q(f)$ boundedly, and furthermore $v(t) \to q(f)$ locally uniformly on $\mathbb{R} \setminus S$, where S is the singular set of f.

Finally, Section 6 gives a further recapitulation of how the results of Sections 1–5 bear on the analysis of the short time behavior of solutions to (1.1).

In the course of our analysis, we bring in a number of function spaces. In particular, we use "symbol spaces":

$$S^m(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n) : |\partial_x^\alpha f(x)| \leq C_\alpha (1 + |x|)^{m-|\alpha|},\ \forall \alpha\},$$

and $S^m_{\mathrm{cl}}(\mathbb{R}^n)$, consisting of $f \in S^m(\mathbb{R}^n)$ having an asymptotic expansion

$$f(x) \sim \sum_{k\geq 0} f_k(x), \quad |x| \to \infty,$$

where $f_k \in C^\infty(\mathbb{R}^n \setminus 0)$ is homogeneous of degree $m - k$ in x. We also make use of L^p-Sobolev spaces $H^{s,p}(\mathbb{R}^n)$, which can be characterized for $p \in (1,\infty)$ $s \in \mathbb{R}$ as

$$H^{s,p}(\mathbb{R}^n) = (1 - \Delta)^{-s/2} L^p(\mathbb{R}^n).$$

Furthermore, we make use of Besov spaces $B^s_{p,q}(\mathbb{R}^n)$. These spaces are closely related to $H^{s,p}(\mathbb{R}^n)$ but differ in subtle (and useful) ways. They can be defined via real interpolation of L^p-Soblev spaces, or via Littlewood–Paley decomposition. We refer to [25] for basic material.

2 Data Singular at a Point

Here we study $e^{it\Delta} f$ near $t = 0$ for a function $f \in C^\infty(\mathbb{R}^n \setminus 0)$, with a "conormal" singularity at $x = 0$. We make various hypotheses on the behavior of $f(x)$ for large x.

One sort is that f have compact support, but the analysis of that case naturally leads us to consider $f(x)$ behaving like a homogeneous function for large $|x|$. Our analysis has points in common with [23, §5], but here we take the analysis much further.

To start, assume $f \in C^\infty(\mathbb{R}^n \setminus 0)$ is compactly supported and that for x in a neighborhood $\mathcal{O}$ of 0, f is homogeneous:

$$f(x) = h_\alpha(x) \text{ on } \mathcal{O}, \quad h_\alpha(rx) = r^\alpha h_\alpha(x), \quad \alpha > -n. \tag{2.1}$$

Then $f \in L^1(\mathbb{R}^n)$ and h_α is a tempered distribution. Let us write

$$h_\alpha(x) = f(x) + u_b(x),$$

with $u_b \in C^\infty(\mathbb{R}^n)$, homogeneous of degree α for large $|x|$. Hence

$$e^{it\Delta} f(x) = e^{it\Delta} h_\alpha(x) - e^{it\Delta} u_b(x).$$

The behavior of the last term on the right is simple.

Lemma 2.1 *We have $e^{it\Delta} u_b(x)$ smooth jointly in t and x, and*

$$e^{it\Delta} u_b(x) \sim \sum_{k\geq 0} \frac{(it)^k}{k!} \Delta^k u_b(x), \quad |x| \to \infty, \tag{2.2}$$

locally uniformly in t.

Proof The left side of (2.2) is the inverse Fourier transform of

$$e^{-it|\xi|^2} \hat{u}_b(\xi), \tag{2.3}$$

where $\hat{u}_b(\xi)$ is smooth on $\mathbb{R}^n \setminus 0$, rapidly decreasing as $|\xi| \to \infty$, and has a conormal singularity at $\xi = 0$. Writing (2.3) as

$$\sum_{k=0}^{N} \frac{(it)^k}{k!} (-|\xi|^2)^k \hat{u}_b(\xi) + R_N^t(\xi),$$

where $R_N^t \in C^\infty(\mathbb{R}^n \setminus 0)$ is rapidly decreasing as $|\xi| \to \infty$ and fairly smooth near $\xi = 0$ for N large, and taking inverse Fourier transform, one obtains (2.2). ■

Remark. Lemma 2.1 is a special case of much stronger results [14, §3].

It is immediate from (1.5) that $e^{it\Delta} f \in C^\infty(\mathbb{R}^n)$ for each $t \neq 0$. We deduce that $e^{it\Delta} h_\alpha \in C^\infty(\mathbb{R}^n)$ for each $t \neq 0$. Considerations of homogeneity yield

$$e^{it\Delta} h_\alpha(x) = t^{\alpha/2} h^\#(t^{-1/2} x),$$

where $h^\#(x) = e^{i\Delta} h_\alpha(x) = e^{i\Delta} f(x) + e^{i\Delta} u_b(x)$. As noted above, we have $h^\# \in C^\infty(\mathbb{R}^n)$. We want to understand its behavior as $|x| \to \infty$. The behavior of

$$g(x) = e^{i\Delta} u_b(x) \sim \sum_{k\geq 0} \frac{i^k}{k!} \Delta^k u_b(x), \quad |x| \to \infty \tag{2.4}$$

is a special case of (2.2). As for $e^{i\Delta} f(x)$, write

$$(2.5)\qquad \begin{aligned} e^{i\Delta} f(x) &= (4\pi i)^{-n/2} \int_{\mathbb{R}^n} e^{i|x-y|^2/4} f(y)\,dy \\ &= Ce^{i|x|^2/4} \int_{\mathbb{R}^n} e^{-ix\cdot y/2} e^{i|y|^2/4} f(y)\,dy \\ &= e^{i|x|^2/4} \hat{f}_2\Big(\frac{x}{2}\Big), \end{aligned}$$

where

$$(2.6)\qquad f_2(y) = Ce^{i|y|^2/4} f(y).$$

We have $\hat{f}_2 \in C^\infty(\mathbb{R}^n)$, asymptotically a sum of homogeneous functions as $|x| \to \infty$, more precisely

$$(2.7)\qquad \hat{f}_2 \in S_{\text{cl}}^{-n-\alpha}(\mathbb{R}^n).$$

We summarize the results obtained so far.

Proposition 2.2 *Assume $f \in C^\infty(\mathbb{R}^n)$ is compactly supported and satisfies* (2.1). *Then*

$$(2.8)\qquad e^{it\Delta} f(x) = t^{\alpha/2} h^\#(t^{-1/2}x) - e^{it\Delta} u_b(x),$$

where $h^\#(x) = g(x) + e^{i|x|^2/4} \hat{f}_2(\frac{x}{2})$, with g given by (2.4) *and $\hat{f}_2$ by* (2.6)–(2.7), *and where the last term on the right side of* (2.8) *is described by Lemma* 2.1.

A useful alternative formula is

$$(2.9)\qquad e^{it\Delta} f(x) = t^{\alpha/2} e^{i|x|^2/4t} \hat{f}_2\Big(\frac{x}{\sqrt{4t}}\Big) + t^{\alpha/2} e^{i\Delta} u_b(t^{-1/2}x) - e^{it\Delta} u_b(x).$$

Another useful alternative formula is obtained by adding to (2.8)

$$0 = -t^{\alpha/2} h_\alpha(t^{-1/2}x) + h_\alpha(x) = -t^{\alpha/2} h_\alpha(t^{-1/2}x) + f(x) + u_b(x).$$

We get

$$(2.10)\qquad e^{it\Delta} f(x) = f(x) + (I - e^{it\Delta}) u_b(x) + t^{\alpha/2} \psi_\alpha(t^{-1/2}x),$$

with

$$(2.11)\qquad \psi_\alpha(x) = h^\#(x) - h_\alpha(x) = g(x) - h_\alpha(x) + e^{i|x|^2/4} \hat{f}_2\Big(\frac{x}{2}\Big).$$

Note that if $n = 1$ and $\alpha = 0$, then $g(x) - h_\alpha(x)$ is rapidly decreasing as $|x| \to \infty$. Also

$$n = 1,\ \alpha = 0 \quad\Longrightarrow\quad (I - e^{it\Delta}) u_b = it \frac{I - e^{it\Delta}}{it\Delta} (\Delta u_b) \in \mathcal{S}(\mathbb{R}^n),\ \forall t.$$

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Thus we recover the result (1.9)–(1.10).

We next examine $e^{it\Delta}L_0(x)$ and $e^{it\Delta}L(x)$, where

$$L(x) = \log|x| = L_0(x) + L_b(x), \tag{2.12}$$

with L_0 compactly supported and $L_b \in C^\infty(\mathbb{R}^n)$. Just as in Lemma 2.1 we have $e^{it\Delta}L_b(x)$ smooth jointly in t and x, and

$$e^{it\Delta}L_b(x) \sim \sum_{k\geq 0} \frac{(it)^k}{k!}\,\Delta^k \log|x|, \quad |x| \to \infty, \tag{2.13}$$

locally uniformly in t. It follows that $e^{it\Delta}L(x)$ is smooth for $t \neq 0$, and this time homogeneity considerations give $e^{it\Delta}L(x) = H(t^{-1/2}x) - \frac{1}{2}\log\frac{1}{t}$, where

$$H(x) = e^{i\Delta}L(x) = e^{i\Delta}L_0(x) + e^{i\Delta}L_b(x).$$

We have $H \in C^\infty(\mathbb{R}^n)$, and we can analyze its behavior as $|x| \to \infty$ as in (2.4)–(2.5). First, as a special case of (2.13),

$$G(x) = e^{i\Delta}L_b(x) \sim \sum_{k\geq 0} \frac{i^k}{k!}\,\Delta^k \log|x|, \quad |x| \to \infty. \tag{2.14}$$

Next, $e^{i\Delta}L_0(x) = e^{i|x|^2/4}\hat{L}_2(\frac{x}{2})$, where

$$L_2(y) = Ce^{i|y|^2/4}L_0(y), \quad \text{so } \hat{L}_2 \in S^{-n}_{\mathrm{cl}}(\mathbb{R}^n). \tag{2.15}$$

Consequently, parallel to (2.8), we have $e^{it\Delta}L_0(x) = H(t^{-1/2}x) - \frac{1}{2}\log\frac{1}{t} - e^{it\Delta}L_b(x)$, where $H(x) = G(x) + e^{i|x|^2/4}\hat{L}_2(\frac{x}{2})$, with G given by (2.14) and $\hat{L}_2$ by (2.15). Alternatively,

$$e^{it\Delta}L_0(x) = e^{i|x|^2/4t}\hat{L}_2\Big(\frac{x}{\sqrt{4t}}\Big) - \frac{1}{2}\log\frac{1}{t} + e^{i\Delta}L_b(t^{-1/2}x) - e^{it\Delta}L_b(x).$$

Note the logarithmic blowup at $x = 0$: $e^{it\Delta}L_0(0) = -\frac{1}{2}\log\frac{1}{t} + H(0) - e^{it\Delta}L_b(0)$. This blowup is localized, however. If, say, $\operatorname{supp} L_0 \subset \{x : |x| \leq 1\}$, then

$$\begin{aligned} &|x| \geq 1,\ t \in (0,1] \quad \Longrightarrow \\ &\qquad -\frac{1}{2}\log\frac{1}{t} + e^{i\Delta}L_b(t^{-1/2}x) - e^{it\Delta}L_b(x) = r_1(t^{-1/2}x) - t r_t(x), \end{aligned} \tag{2.16}$$

where

$$r_t = \frac{1}{t}\big(e^{it\Delta}L_b - L_b\big) \tag{2.17}$$

is a smooth function of x with values in $S^{-1}(\mathbb{R}^n)$.

We mention that the methods developed above extend in a straightforward fashion to treat the action of $e^{it\Delta}$ on the various terms in

$$h_\alpha(x)\,\log|x| = L_{\alpha 0}(x) + L_{\alpha b}(x), \tag{2.18}$$

with h_α as in (2.1), $\alpha > -n$, $L_{\alpha 0}$ compactly supported, and $L_{\alpha b} \in C^\infty(\mathbb{R}^n)$.

We now use the asymptotic analysis developed above to analyze pointwise convergence.

Proposition 2.3 *Assume $f \in C^\infty(\mathbb{R}^n \setminus 0)$ is rapidly decreasing at infinity and equal to a homogeneous function of degree $\alpha > -n$ near 0. Then, as $t \to 0$,*

$$e^{it\Delta} f(x) \longrightarrow f(x) \ \textit{ locally uniformly on } \ \mathbb{R}^n \setminus 0, \tag{2.19}$$

provided $\alpha > -n/2$.

Proof We use (2.10) to analyze $e^{it\Delta} f(x) - f(x)$. As mentioned in Lemma 2.1, $(I - e^{it\Delta})u_b(x) \to 0$ locally uniformly on $\mathbb{R}^n$. In fact, convergence occurs in the topology of $S^{\alpha+\eta}(\mathbb{R}^n)$ for each $\eta > 0$.

It remains to analyze $t^{\alpha/2}\psi_\alpha(t^{-1/2}x)$, where $\psi_\alpha(x)$ is given by (2.11). Note that by (2.4), $|x| \ge 1 \Rightarrow |g(x) - h_\alpha(x)| \le C|x|^{\alpha-2}$, so

$$|x| \ge t^{1/2} \quad \Longrightarrow \quad t^{\alpha/2}|g(t^{-1/2}x) - h_\alpha(t^{-1/2}x)| \le Ct^{\alpha/2}|t^{-1/2}x|^{\alpha-2} = Ct|x|^{\alpha-2},$$

which tends to 0 as $t \to 0$ locally uniformly on $\mathbb{R}^n \setminus 0$. Meanwhile, by (2.7),

$$|\hat{f}_2(t^{-1/2}x/2)| \le C(1 + t^{-1/2}|x|)^{-n-\alpha} = Ct^{(n+\alpha)/2}(t^{1/2} + |x|)^{-n-\alpha},$$

so $t^{\alpha/2}|\hat{f}_2(t^{-1/2}x/2)| \le Ct^{\alpha+n/2}(t^{1/2} + |x|)^{-n-\alpha}$, which tends to 0 locally uniformly on $\mathbb{R}^n \setminus 0$ provided $\alpha > -n/2$. ■

Remark. One also has (2.19) for $f(x) = L_{\alpha 0}(x)$, given in (2.18), provided $\alpha > -n/2$.

3 Data Singular on a Hypersurface

Let $\overline{\Omega} \subset \mathbb{R}^n$ be a compact, smoothly bounded domain. We study $e^{it\Delta} f$ when $f \in C^\infty(\mathbb{R}^n \setminus \partial\Omega)$ is piecewise smooth, with a jump on $\partial\Omega$, and is rapidly decreasing at infinity. Actually, without loss of generality we can restrict attention to f of the form $F\chi_\Omega$, with $F \in C^\infty(\mathbb{R}^n)$; such a function differs from such more general f by an element of $\mathcal{S}(\mathbb{R}^n)$, and $e^{it\Delta}$ acts smoothly on $\mathcal{S}(\mathbb{R}^n)$. From here on we take $f = F\chi_\Omega$.

We start with results that apply basic integration by parts methods and stationary phase techniques, to the integral formula

$$e^{it\Delta} f(x) = (4\pi it)^{-n/2} \int_\Omega e^{i|x-y|^2/4t} f(y)\,dy = C\lambda^{n/2} \int_\Omega e^{i\lambda\psi(x,y)} f(y)\,dy,$$

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where we set

$$\psi(x,y) = |x-y|^2, \quad \lambda = \frac{1}{4t}. \tag{3.1}$$

For a while we will drop the explicit x-dependence and discuss sone basic results about $I(f,\lambda) = \int_\Omega e^{i\lambda\psi(y)} f(y)\,dy$.

Using a partition of unity we can write f as a sum of pieces supported on sets where ψ has various special properties. Note that $\nabla\psi(y) \neq 0$ except at $y = x$. If x is bounded away from $\partial\Omega$, we can isolate a piece $I(f_1,\lambda)$ where $f_1 \in C_0^\infty(\mathcal{O}_x)$ is supported on a small neighborhood $\mathcal{O}_x$ of x, disjoint from $\partial\Omega$. The behavior of $I(f_1,\lambda)$ is given by the standard stationary phase method. Equivalently, $e^{it\Delta} f_1$ converges smoothly to f_1 as $t \to 0$.

The next step is to investigate $I(f,\lambda)$ where $\nabla\psi \neq 0$ on supp f. (If x is on or near $\partial\Omega$, further techniques will be required, which we will get to later in this section.) If $f \in C_0^\infty(\Omega)$ and $\nabla\psi \neq 0$ on supp f, then $I(f,\lambda)$ is rapidly decreasing as $|\lambda| \to \infty$. Thus we may assume f is supported on a small collar neighborhood U of $\partial\Omega$ in $\overline{\Omega}$. Say U is diffeomorphic to $[0,1] \times \partial\Omega$, where $\{1\} \times \partial\Omega$ is identified with $\partial\Omega \subset \overline{\Omega}$. In such a case (with slight abuse of notation),

$$I(f,\lambda) = \int_{\partial\Omega} \int_0^1 e^{i\lambda\psi(s,z)} f(s,z) J(s,z)\, ds dS(z), \tag{3.2}$$

where $J(s,z)$ is an appropriate Jacobian, and $f(s,z) = 0$ for s close to 0. Further localization is provided by the following elementary result.

Lemma 3.1 *Suppose f is supported on a set $\mathcal{O} \cap \overline{\Omega}$, with $\mathcal{O} \subset \mathbb{R}^n$ open, and suppose there exists a smooth vector field X, tangent to $\partial\Omega$, such that $X\psi \neq 0$ on $\mathcal{O}$. Then $I(f,\lambda)$ is rapidly decreasing as $|\lambda| \to \infty$.*

Proof Write $e^{i\lambda\psi} = (i\lambda X\psi)^{-1} X e^{i\lambda\psi} = L(\lambda) e^{i\lambda\psi}$, and iterate, obtaining $e^{i\lambda\psi} = L(\lambda)^k e^{i\lambda\psi}$. Then, for f supported on $\mathcal{O}$, since X is tangent to $\partial\Omega$, we have

$$I(f,\lambda) = \int_\Omega e^{i\lambda\psi(y)} (L(\lambda)^t)^k f(y)\, dy = O(\lambda^{-k}), \quad |\lambda| \to \infty,$$

as asserted. ∎

Our next step is to consider (3.2) in the case where $\partial_s\psi(s,z) \neq 0$ on supp f. In such a case, elementary Fourier analysis gives

$$\int_0^1 e^{i\lambda\psi(s,z)} f(s,z) J(s,z)\, ds \sim e^{i\lambda\psi(1,z)} \sum_{k\geq 0} a_k(z) \lambda^{-1-k}.$$

Thus (identifying $(1,z)$ with z) we have

$$I(f,\lambda) \sim \sum_{k\geq 0} \lambda^{-1-k} \int_{\partial\Omega} e^{i\lambda\psi(z)} a_k(z)\, dS(z).$$

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Restoring the x-dependence, and recalling that $\psi(y) = \psi(x, y)$ is given by (3.1), we have

$$
\begin{aligned}
e^{it\Delta} f(x) &\sim \sum_{k\geq 0} \lambda^{n/2-1-k} \int_{\partial\Omega} e^{i\lambda\psi(x,z)} a_k(x,z)\, dS(z) \\
&\sim \sum_{k\geq 0} \lambda^{n/2-1-k} e^{i\lambda|x|^2} \int_{\partial\Omega} e^{i\lambda(|z|^2-2x\cdot z)} a_k(x,z)\, dS(z),
\end{aligned}
\tag{3.3}
$$

as $t = \frac{1}{4\lambda} \to 0$. In the current setting, we are assuming $x \in K$, a compact set disjoint from supp f, which in turn is contained in a small neighborhood of a point $p \in \partial\Omega$.

The analysis of (3.3) splits into several cases. There is the "non-caustic" region $\mathcal{C}_0$, consisting of x such that $\psi_x(z) = \psi(x, z)$, as a function of $z \in \partial\Omega$, has only non-degenerate critical points (necessarily a finite number), at least in supp f, say at $p_\ell(x) \in \partial\Omega$. For $x \in \mathcal{C}_0 \cap K$ the stationary phase method gives

$$
\int_{\partial\Omega} e^{i\lambda\psi(x,z)} a_k(x,z)\, dS(z) \sim \lambda^{-(n-1)/2} \sum_{\ell} e^{i\lambda\psi(x,p_\ell(x))} \sum_{m\geq 0} a_{k\ell m}(x) \lambda^{-m}.
$$

Plugging this into (3.3) and rearranging, we have

$$
e^{it\Delta} f(x) \sim \sum_{\ell} e^{i\psi(x,p_\ell(x))/4t} \sum_{k\geq 0} b_{k\ell}(x) t^{k+1/2},
\tag{3.4}
$$

for $x \in \mathcal{C}_0 \cap K$. It is clear that $e^{it\Delta} f \to 0$ on any open set disjoint from supp f, at least in a weak sense. The expansion (3.12) shows the rate at which this happens, locally uniformly on $\mathcal{C}_0 \cap K$.

The caustic set for such an oscillatory integral as (3.3) consists of points x for which $\psi_x|_{\partial\Omega}$ has degenerate critical points. The nature of such caustic sets and the behavior of such integrals on and near them is described in a number of places. Notable sources are [1, 2, 10, 11, 13]. The simplest part of the caustic set is the "fold set" $\mathcal{C}_1$. Given $q \in K \cap \mathcal{C}_1$, there is a neighborhood $\mathcal{O}_q$ of q, a smooth, real-valued function ρ, vanishing simply on $\mathcal{O}_q \cap \mathcal{C}_1$, with $\nabla\rho \neq 0$, and a smooth, real-valued θ, such that for $x \in \mathcal{O}_q$,

$$
\begin{aligned}
\int_{\partial\Omega} e^{i\lambda\psi(x,z)} a_k(x,z)\, dS(z) \sim \lambda^{1/6-(n-1)/2} \Big[& b_0(x,\lambda) Ai\big(\rho(x)\lambda^{2/3}\big) \\
& + \lambda^{-1/3} b_1(x,\lambda) Ai'\big(\rho(x)\lambda^{2/3}\big) \Big] e^{i\lambda\theta(x)},
\end{aligned}
$$

where $b_j(x,\lambda) \sim \sum_{k\geq 0} b_{jk}(x)\lambda^{-k}$. It follows that, for $x \in \mathcal{O}_q$,

$$
\begin{aligned}
e^{it\Delta} f(x) \sim e^{i\theta(x)/4t} \Big[& Ai\big((4t)^{-2/3}\rho(x)\big) \sum_{k\geq 0} b_{0k}(x) t^{k+1/3} \\
& + Ai'\big((4t)^{-2/3}\rho(x)\big) \sum_{k\geq 0} b_{1k}(x) t^{k+2/3} \Big].
\end{aligned}
\tag{3.5}
$$

Here $Ai(s)$ is the Airy function, given by $Ai(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(t^3/3+st)}\,dt$. This is a smooth function of s, with asymptotic behavior

$$Ai(s) \sim \frac{1}{2\sqrt{\pi}} s^{-1/4} e^{-(2/3)s^{3/2}}, \quad Ai(-s) \sim \frac{1}{\sqrt{\pi}} s^{-1/4} \cos\Big(\frac{2}{3}s^{3/2} - \frac{\pi}{4}\Big),$$

as $s \to +\infty$. (See also [20, Ch. 6, §4], for a discussion of the fold case.) Note in particular that for $x \in \mathcal{C}_1 \cap \mathcal{O}_q$ one has the estimate $|e^{it\Delta} f(x)| \le ct^{1/3}$, as opposed to the estimate $|e^{it\Delta} f(x)| \le Ct^{1/2}$ for $x \in K \cap \mathcal{C}_0$, which follows from (3.4).

There is a further hierarchy $\mathcal{C}_k$ of "simple caustics" of order $k \ge 2$, including cusps, swallowtails, etc. Generally, if $x \in \mathcal{C}_k \cap K$, one has $|e^{it\Delta} f(x)| \le C|t|^{1/(k+2)}$ and a corresponding asymptotic expansion for fixed $x = q$. There are uniform asymptotic expansions in a neighborhood of such $q \in \mathcal{C}_k \cap K$, of a more complicated nature than (3.5). We refer to the sources cited above for more on such simple caustics.

In addition, particularly when Ω has a continuous symmetry group, there might be caustics not of simple type. The chief paradigm is the "perfect focus" caustic, which arises when Ω is a ball; the perfect focus occurs at the center of the ball. It is worth noting that the special nature of $\psi(x, y)$ in (3.1) allows for a precise treatment of the behavior of $e^{it\Delta} f$ when $\overline{\Omega}$ is a ball, say $\overline{\Omega} = B_a = \{x \in \mathbb{R}^n : |x| \le a\}$. In such a case we have (3.3) with $|z|^2 = a^2$ on $\partial\Omega = \partial B_a$, yielding the following.

Proposition 3.2 *Assume $f = F\chi_{B_a}$ with $F \in C^\infty(\mathbb{R}^n)$ and $F \equiv 0$ on B_b (with $b \in (0, a)$). Then, locally uniformly on B_b, and also on $\mathbb{R}^n \setminus B_a$, we have*

$$e^{it\Delta} f(x) \sim e^{i(|x|^2+a^2)/4t} \sum_{k\ge 0} \hat{\alpha}_k\Big(x, \frac{x}{2t}\Big)\, t^{-n/2+1+k},$$

where $\hat{\alpha}_k(x, \xi) = \int_{\partial B_a} e^{-i\xi\cdot z} a_k(x, z)\, dS(z)$.

Note that $\hat{\alpha}_k \in C^\infty(\mathbb{R}^n)$, and the stationary phase method gives

$$\hat{\alpha}_k(x, \xi) \sim \sum_{\sigma\in\{-1,1\}} e^{-i\sigma a|\xi|} \sum_{\ell\ge 0} \alpha_{k\ell\sigma}\Big(x, a\frac{\xi}{|\xi|}\Big)\, |\xi|^{-(n-1)/2-\ell}, \quad |\xi| \to \infty.$$

Note that locally uniformly on $B_b \setminus 0$ we have a result that agrees with (3.4) (as it must), but there is a spike concentrated near $x = 0$. In particular,

$$e^{it\Delta} f(0) \sim e^{ia^2/4t} \sum_{k\ge 0} \hat{\alpha}_k(0, 0) t^{1+k-n/2}.$$

We emphasize two cases:

$$n = 2: \quad e^{it\Delta} f(0) \sim e^{ia^2/4t}\big(\hat{\alpha}_0(0, 0) + O(t)\big),$$
$$n = 3: \quad e^{it\Delta} f(0) \sim e^{ia^2/4t}\big(\hat{\alpha}_0(0, 0)t^{-1/2} + O(t^{1/2})\big).$$

In case $n = 2$, $e^{it\Delta} f(0)$ has a bounded oscillatory divergence as $t \to 0$, reminiscent of the Pinsky phenomenon. (We recall from [24, Proposition 4.2] that $\|e^{it\Delta} f\|_{L^\infty}$ is bounded in this case.) In case $n = 3$, the best bound on $e^{it\Delta} f$ one typically has (for $|x| \le b < 1$) is

$$|e^{it\Delta} f(x)| \le C|t|^{1-n/2}\Big(1 + \Big|\frac{x}{t}\Big|\Big)^{-(n-1)/2}.$$

Using this one can readily verify the following.

Proposition 3.3 *Assume $n \ge 3$. Take $a > b > 0$. Then $\sup_{0<t\le 1} \|e^{it\Delta} f\|_{L^p(B_b)} < \infty$ for each $f = F\chi_{B_a}$, $F \in C^\infty(\mathbb{R}^n)$, if and only if $p \le \frac{2n}{n-2}$.*

The results discussed so far in this section have avoided the following situation. Suppose $p \in \partial\Omega$, $\mathcal{O}$ is a (sufficiently small) neighborhood of p, and $f = F\chi_\Omega$ with $F \in C_0^\infty(\mathcal{O})$. We need another technique to analyze $e^{it\Delta} f(x)$ uniformly on $\mathcal{O}$, as $t \to 0$. The fact that the critical point x of ψ_x is not bounded away from $\partial\Omega$ makes it undesirable to use the techniques of the earlier part of this section. The approach of the critical point to $\partial\Omega$ has a real consequence, the appearance of a Gibbs-like phenomenon at $\partial\Omega$.

To analyze this Gibbs-type phenomenon, we use a wave equation technique analogous to that used in [17, §11] to treat the Gibbs phenomenon for multi-dimensional Fourier inversion (see also [8]). In the current context, we have

$$e^{it\Delta} f(x) = (4\pi i t)^{-1/2} \int_{-\infty}^{\infty} e^{is^2/4t} u(s,x)\, ds, \tag{3.6}$$

where $u(s,x) = \cos s\sqrt{-\Delta}\, f(x)$. Equivalently, $u(s,x)$ solves the wave equation:

$$(\partial_s^2 - \Delta)u = 0, \quad u(0,x) = f(x), \quad \partial_s u(0,x) = 0.$$

Another way of writing (3.6) is $e^{it\Delta} f(x) = e^{it\partial_s^2} u(s,x)\big|_{s=0}$. We will make the hypothesis on $\mathcal{O}$ that $\psi \in C^\infty(\mathcal{O})$, where

$$\psi(x) = \begin{cases} \operatorname{dist}(x, \partial\Omega), & x \in \mathcal{O} \cap \Omega, \\ -\operatorname{dist}(x, \partial\Omega), & x \in \mathcal{O} \setminus \Omega. \end{cases}$$

We also will assume that $\mathcal{O}$ is convex, and that a line through a point in $\partial\Omega \cap \mathcal{O}$, normal to $\partial\Omega$, does not intersect $\partial\Omega \cap \mathcal{O}$ in any other point. Clearly, given $p \in \partial\Omega$, any sufficiently small ball centered at p satisfies these hypotheses. In such a case, $u(s,x)$ is given on $\mathbb{R} \times \mathcal{O}$ by a progressing wave expansion:

$$\begin{aligned} u(s,x) = {} & A_0(s,x)\chi_+(\psi(x) - s) + \sum_{j=1}^{k} A_j(s,x)\chi_+^j(\psi(x) - s) \\ & + A_0(-s,x)\chi_+(\psi(x) + s) + \sum_{j=1}^{k} A_j(-s,x)\chi_+^j(\psi(x) + s) + R_k(s,x). \end{aligned} \tag{3.7}$$

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Here $A_j \in C^\infty(\mathbb{R} \times \mathcal{O})$, $R_k \in C^k(\mathbb{R} \times \mathcal{O})$, and $\chi_+(x) = \chi_{\mathbb{R}^+}(x)$, $\chi_+^j(x) = x^j \chi_+(x)$. Discussion of this basic method of geometrical optics can be found in [20, Ch. 6, §6]. There exists $T_0 < \infty$ such that each A_j is supported on $|s| \le T_0$. If n is odd, $u(s,x)$ is supported on $|s| \le T_0$. If n is even, u is smooth on $(\mathbb{R} \setminus [-T_0, T_0]) \times \mathcal{O}$, with symbolic behavior as $|s| \to \infty$ of order $-n/2$, as can be read off from the fundamental solution to the wave equation. Without loss of generality we can assume each $A_j(s,x)$ is independent of s for s close to $\psi(x)$. The analysis of Section 2 applies to each piece $e^{it\partial_s^2} A_j(\pm s, x)\chi_+^j(\psi(x) \mp s)$. In particular, we have

$$e^{it\partial_s^2} A_0(s,x)\chi_+(\psi(x) - s) = A_0(\psi(x), x)\Big[\mathrm{Fr}\Big(\frac{\psi(x) - s}{\sqrt{4t}}\Big) + \frac{1}{2}\Big] + B_0(s,t,x),$$

where B_0 is a relatively tame remainder term. Bringing in its counterpart with s replaced by $-s$, we have

$$\begin{aligned} e^{it\partial_s^2}\Big[A_0(\psi(x), x)\chi_+(\psi(x) - s) + A_0(\psi(x), x)\chi_+(\psi(x) + s)\Big]\Big|_{s=0} \\ = 2A_0(\psi(x), x)\Big[\mathrm{Fr}\Big(\frac{\psi(x)}{\sqrt{4t}}\Big) + \frac{1}{2}\Big] + 2B_0(0,t,x). \end{aligned}$$

Note from (3.7) that

$$x \in \partial\Omega \implies 2A_0(\psi(x), x) = 2A_0(0,x) = F(x). \tag{3.8}$$

Taking into account how the results of Section 2 apply to the other terms in (3.7), we have the following.

Proposition 3.4 *Assume $p \in \partial\Omega$ and $\mathcal{O}$ is a sufficiently small neighborhood of p, as described above. Consider $f = F\chi_\Omega$, $F \in C_0^\infty(\mathcal{O})$. Then, for $x \in \mathcal{O}$, $t \in (-1,1)$,*

$$e^{it\Delta} f(x) = 2A_0(\psi(x), x)\Big[\mathrm{Fr}\Big(\frac{\psi(x)}{\sqrt{4t}}\Big) + \frac{1}{2}\Big] + R(t,x),$$

where $A_0 \in C^\infty(\mathbb{R} \times \mathcal{O})$, (3.8) holds, and, as $t \to 0$,

$$R(t,x) \to f(x) - 2A_0(\psi(x), x)\chi_\Omega(x), \quad \textit{uniformly on } \mathcal{O}. \tag{3.9}$$

Note that the right side of (3.9) is piecewise smooth and Lipschitz continuous.

4 Data Singular and Oscillatory

In this section we analyze the behavior of (1.15), *i.e.,*

$$e^{it\Delta}(\nu_\nu \Phi_s^{jk} e^{i\nu x^2/4s}), \tag{4.1}$$

uniformly for $s, t \in (0, T_0]$. We recall that

$$\Phi_s^{jk}(x) = \Phi^{jk}\Big(\frac{x}{\sqrt{4s}}\Big), \quad \Phi^{jk}(x) = \Phi(x)^j\,\overline{\Phi(x)}^k,$$

with Φ as in (1.7)–(1.8). Note that Φ is odd, so Φ^{jk} is even or odd depending on the parity of $j+k$. The factor v_ν is a multiple of a power of v times a power of $\overline{v}$, with v as in (1.10), when $q(u)$ is given by (1.11). There is no loss of generality in taking $v_\nu = v_\nu(x)$, C^∞ on $\mathbb{R}\setminus 0$, rapidly decreasing as $|x|\to\infty$, with a jump discontinuity at $x=0$. As mentioned in the introduction, there is particularly interesting behavior near $t = s/(-\nu)$ when ν is a negative integer. Note that if $\nu \neq 0$, then by scaling we can take $\nu = \pm 1$. For now we take $\nu = -1$. Later in this section we indicate how things work for $\nu = +1$ and for $\nu = 0$. Thus we are investigating

$$
\begin{aligned}
(4.2)\qquad & e^{it\Delta}(v\Phi_s^{jk}e^{-ix^2/4s}) \\
&= (4\pi it)^{-1/2}\int_{-\infty}^{\infty} e^{i(x-y)^2/4t}v(y)e^{-iy^2/4s}\Phi^{jk}\Big(\frac{y}{\sqrt{4s}}\Big)\,dy \\
&= (4\pi it)^{-1/2}e^{ix^2/4t}\int_{-\infty}^{\infty} e^{-ixy/2t}e^{iy^2(1/4t-1/4s)}v(y)\Phi^{jk}\Big(\frac{y}{\sqrt{4s}}\Big)\,dy \\
&= A(s,t,x).
\end{aligned}
$$

Again v is piecewise smooth on $\mathbb{R}$, rapidly decreasing, with a jump at $x=0$. If we make the change of variable $\eta = y/\sqrt{4s}$ and also set

$$(4.3)\qquad z = \frac{x\sqrt{s}}{t},\quad \tau = \frac{s}{t}-1,\quad \delta = \sqrt{4s},$$

we get

$$(4.4)\qquad A(s,t,x) = Ce^{ix^2/4t}B(\tau,\delta,z),$$

where

$$(4.5)\qquad B(\tau,\delta,z) = (\tau+1)^{1/2}\int_{-\infty}^{\infty} e^{-iz\eta}e^{i\tau\eta^2}v(\delta\eta)\Phi^{jk}(\eta)\,d\eta.$$

Note that

$$(4.6)\qquad x\in\mathbb{R},\ s,t\in(0,T_0] \implies z\in\mathbb{R},\ \tau\in(-1,\infty),\ \delta\in(0,\sqrt{4T_0}],$$

so we want to analyze $B(\tau,\delta,z)$, uniformly in this range. Note that

$$(4.7)\qquad B(0,\delta,z) = \hat{w}_\delta^{jk}(z),$$

where

$$(4.8)\qquad w_\delta^{jk}(\eta) = v(\delta\eta)\Phi^{jk}(\eta),$$

and then

$$(4.9)\qquad B(\tau,\delta,z) = (\tau+1)^{1/2}e^{i\tau\Delta}\hat{w}_\delta^{jk}(z).$$

Let us make the hypothesis that

$$\lim_{x\to\pm 0} v(x) = a_\pm, \quad a_+ \neq 0, \ a_- \neq 0. \tag{4.10}$$

(If v happens not to satisfy (4.10), we can easily write it as a sum of two functions that do.) Set

$$u(\eta) = a_\pm^{-1} v(\eta), \quad \pm\eta > 0, \quad u_\delta(\eta) = u(\delta\eta), \tag{4.11}$$

and

$$\Psi^{jk}(\eta) = a_\pm \Phi^{jk}(\eta), \quad \pm\eta > 0, \tag{4.12}$$

so that

$$w_\delta^{jk}(\eta) = u_\delta(\eta)\Psi^{jk}(\eta). \tag{4.13}$$

Note that $u(\eta)$ is smooth on $\mathbb{R}\setminus 0$, rapidly decreasing as $|\eta| \to \infty$, and that u has no jump at $\eta = 0$, though its first derivative might jump. We have

$$\hat{w}_\delta^{jk}(z) = \hat{u}_\delta * \hat{\Psi}^{jk}(z), \tag{4.14}$$

with $\hat{u}_\delta(z) = \delta^{-1}\hat{u}(\delta^{-1}z)$, and we can rewrite (4.9) as

$$B(\tau, \delta, z) = (\tau + 1)^{1/2} \hat{u}_\delta * E_\tau(z), \tag{4.15}$$

where

$$E_\tau(z) = e^{i\tau\Delta}\hat{\Psi}^{jk}(z). \tag{4.16}$$

Note that $\hat{u}$ is smooth and

$$|\hat{u}(z)| \le C(1 + |z|)^{-2}, \tag{4.17}$$

so convolution by $\hat{u}_\delta$ is a standard sort of mollifier. We next give an analysis of $E_\tau(z)$, valid uniformly for τ in any bounded subset of $(-\infty, \infty)$. Once this is done, it remains to analyze $B(\tau, \delta, z)$ in the $\tau \to +\infty$ limit, which we will undertake in due course.

To analyze (4.16), recall that $\Phi^{jk}(\eta)$ has a jump discontinuity at $\eta = 0$ and is even (odd) provided $j + k$ is even (odd). Also, by (1.8),

$$\Phi^{jk}(\eta) \sim a_{jk}\eta^{-(j+k)} + \cdots, \quad |\eta| \to \infty. \tag{4.18}$$

Hence $\Psi^{jk}(\eta)$ has a jump discontinuity at $\eta = 0$ and

$$\Psi^{jk}(\eta) \sim a_{jk}^\pm \eta^{-(j+k)} + \cdots, \quad \pm\eta \to +\infty. \tag{4.19}$$

The jump discontinuity for $\Psi^{jk}(\eta)$ yields the asymptotic behavior

$$\hat{\Psi}^{jk}(z) \sim b_{jk} z^{-1} + \cdots, \quad |z| \to \infty. \tag{4.20}$$

Furthermore, $\hat{\Psi}^{jk}$ is C^∞ on $\mathbb{R} \setminus 0$, and the nature of its singularity at $z = 0$ is determined by the asymptotic expansion (4.19). We concentrate on the case $\hat{\Psi}^{01}$, since $\hat{\Psi}^{10}$ is similar and $\hat{\Psi}^{jk}$ for $j + k \geq 2$ (also having a similar sort of analysis) is less singular. We see that the odd part of $\hat{\Psi}^{01}$ has a jump and the even part has a logarithmic singularity (plus lower order singularities). Thus

$$\hat{\Psi}^{01}(z) = A \log |z| + B \operatorname{sgn} z + R(z), \tag{4.21}$$

where $R \in C^\infty(\mathbb{R} \setminus 0)$ is continuous, though its first derivative can have a jump and/or logarithmic singularity. Now we can write $\hat{\Psi}^{01} = \hat{\Psi}^{01}_0 + \hat{\Psi}^{01}_b$ where $\hat{\Psi}^{01}_0(z) = \hat{\Psi}^{01}(z)$ for small $|z|$ and $\hat{\Psi}^{01}_b(z) = \hat{\Psi}^{01}(z)$ for large $|z|$, $\hat{\Psi}^{01}_b \in S^{-1}_{\mathrm{cl}}(\mathbb{R}^n)$, and we can appeal to results of Section 2 to obtain

$$\begin{aligned} &e^{i\tau\Delta} \hat{\Psi}^{01}_b(z) \ \text{ smooth in } \ (\tau, z), \\ &e^{i\tau\Delta} \hat{\Psi}^{01}_b(z) \sim \sum_{\ell \geq 0} \frac{(i\tau)^\ell}{\ell!} \Delta^\ell \hat{\Psi}^{01}_b(z), \quad |z| \to \infty, \end{aligned} \tag{4.22}$$

and, for $\tau > 0$

$$\begin{aligned} e^{i\tau\Delta} \hat{\Psi}^{01}_0(z) = A \Big[e^{iz^2/4\tau} \hat{L}_2\Big(\frac{z}{\sqrt{4\tau}}\Big) - \frac{1}{2} \log \frac{1}{\tau} + e^{i\Delta} L_b(\tau^{-1/2} z) - e^{i\tau\Delta} L_b(z) \Big] \\ + 2B \Big[\operatorname{sgn} z + e^{iz^2/4\tau} \Phi\Big(\frac{z}{\sqrt{4\tau}}\Big) \Big] + r(\tau, z), \end{aligned} \tag{4.23}$$

where L_b, $\hat{L}_2$ are as in (2.12), (2.15), Φ is as in (1.7)–(1.8), and $r(\tau, z)$ has tamer behavior. There is a similar formula for $\tau < 0$.

Thus the behavior of $A(s, t, x)$ for $s > t > 0$ is given by (4.4), (4.15)–(4.17), and (4.23), with a similar behavior for $t > s > 0$. For $s = t > 0$ we have $\tau = 0$ and hence

$$A(t, t, x) = C e^{ix^2/4t} \, \hat{u}_\delta * \hat{\Psi}^{01}(t^{-1/2} x), \quad \delta = \sqrt{4t}, \tag{4.24}$$

in case $(j, k) = (0, 1)$.

These formulas give a good hold on the behavior of $A(s, t, x)$, uniformly for $x \in \mathbb{R}$ and $s/t = \tau + 1 \in (0, K]$, for any finite K. One striking aspect is the nature of the logarithmic blowup.

To study $B(\tau, \delta, z)$ as $\tau \to +\infty$, set

$$\varepsilon = \frac{1}{4\tau}, \tag{4.25}$$

so

$$B(\tau,\delta,z) = \Big(\frac{1+4\varepsilon}{4\varepsilon}\Big)^{1/2}\int_{-\infty}^{\infty} e^{i\eta^2/4\varepsilon}e^{-iz\eta}\nu(\delta\eta)\Phi^{jk}(\eta)\,d\eta \tag{4.26}$$
$$= \Big(\frac{1+4\varepsilon}{4\varepsilon}\Big)^{1/2}e^{-iz^2}\int_{-\infty}^{\infty} e^{i(\eta-2\varepsilon z)^2/4\varepsilon}\nu(\delta\eta)\Phi^{jk}(\eta)\,d\eta$$
$$= C(1+4\varepsilon)^{1/2}e^{-iz^2}\,e^{i\varepsilon\Delta}w_\delta^{jk}(2\varepsilon z),$$

where $w_\delta^{jk}(\eta)$ is given by (4.8). Note that

$$2\varepsilon z = \frac{1}{2}\,\frac{x\sqrt{s}}{s-t}.$$

Note that $\varepsilon \approx 0 \Leftrightarrow \tau \gg 1 \Leftrightarrow t \ll s$, so $\varepsilon \approx 0 \Rightarrow 2\varepsilon z \approx x/\sqrt{4s}$. It remains to examine

$$F(\varepsilon,\delta,x) = e^{i\varepsilon\Delta}w_\delta^{jk}(x), \tag{4.27}$$

where w_δ^{jk} is given by (4.8).

Let us pick $\chi \in C_0^\infty(\mathbb{R})$ such that $\chi(x) = 1$ for $|x| \le 1$, and write

$$F(\varepsilon,\delta,x) = F_0(\varepsilon,\delta,x) + F_1(\varepsilon,\delta,x) = e^{i\varepsilon\Delta}(\chi w_\delta^{jk})(x) + e^{i\varepsilon\Delta}((1-\chi)w_\delta^{jk})(x). \tag{4.28}$$

Note that

$$\chi(x)w_\delta^{jk}(x) = \chi(x)\nu(\delta x)\Phi^{jk}(x) \tag{4.29}$$

is a family of functions with fixed compact support, piecewise smooth with simple jump at $x = 0$, varying smoothly in $\delta \in \mathbb{R}$, so $F_0(\varepsilon,\delta,x)$ is simply a δ-smooth family of functions of the form $e^{i\varepsilon\Delta}\nu$ analyzed in Section 2, with ν a compactly supported piecewise smooth function with a jump at $x = 0$.

To analyze $F_1(\varepsilon,\delta,x)$, note that $(1-\chi)w_\delta^{jk} = \nu_\delta\Phi_b^{jk}$, where $\nu_\delta(x) = \nu(\delta x)$, $\Phi_b^{jk} = (1-\chi)\Phi^{jk} \in S^{-1}_{\mathrm{cl}}(\mathbb{R})$. Of course $\Phi_b^{jk}(x) = 0$ for $|x| \le 1$, so $\nu_\delta\Phi_b^{jk} \in \mathcal{S}(\mathbb{R})$ for each $\delta \in (0,1]$. Furthermore, $\{\nu_\delta\Phi_b^{jk} : \delta \in (0,1]\}$ is bounded in $S^{-1}(\mathbb{R})$, and if we define $a_\pm$, Ψ^{jk} as in (4.10)–(4.12) and set $\Psi_b^{jk} = (1-\chi)\Psi^{jk} \in S^{-1}_{\mathrm{cl}}(\mathbb{R})$, we have, as $\delta \to 0$, $\nu_\delta\Phi_b^{jk} \to \Psi_b^{jk}$ in $S^{-1+\eta}(\mathbb{R})$, for each $\eta > 0$. A special case of results in [14, §3] is that $e^{i\varepsilon\Delta}\colon S^m(\mathbb{R}) \to S^m(\mathbb{R})$ for each $m \in \mathbb{R}$, with continuous dependence on ε. In particular,

$$|F_1(\varepsilon,\delta,x)| \le C(1+|x|)^{-1}, \tag{4.30}$$

with C independent of $\varepsilon \in [-1,1]$, $\delta \in (0,1]$.

Having treated $A(s,t,x)$ given by (4.2) for $(j,k) = (0,1)$, we indicate the treatment of variants. The case $(j,k) = (1,0)$ has an identical analysis (though this does not actually arise in the analysis of (1.15), since $\nu < 0 \Leftrightarrow k > j$). Now if $j+k \geq 2$, we still have (4.3)–(4.20), but now $\hat{\Psi}^{jk}(z)$ is continuous (in fact, smooth almost of order C^{j+k-1}) near $z = 0$, rather than singular as in (4.21). The nature of $\hat{\Psi}^{jk}(z)$ as $|z| \to \infty$ is still given by (4.20). Hence $e^{i\tau\Delta}\hat{\Psi}_b^{jk}(z)$ is given as in (4.22), but $e^{i\tau\Delta}\hat{\Psi}_0^{jk}(z)$ is tamer than (4.23). The behavior of $B(\tau, \delta, z)$ for $\tau \geq 1$, given by (4.26) with $\varepsilon \in (0, 1/4]$, is much like that described above for $(j, k) = (0, 1)$. To summarize, $A(s, t, x)$ has behavior parallel to but tamer than the $(0, 1)$ case when $j+k \geq 2$. In particular, there is a bound

$$|A(s, t, x)| \leq C, \quad s, t \in (0, 1], \ x \in \mathbb{R}. \tag{4.31}$$

This provides an analysis of $A^{jk}(s, t, x)$ in (1.15) in case $\nu < 0$.

To treat $\nu > 0$ in (4.1), we consider the following variant of (4.2):

$$e^{it\Delta}(\nu\Phi_s^{jk}e^{ix^2/4s}) = A_+(s, t, x). \tag{4.32}$$

Thus one has

$$e^{iy^2(1/4t+1/4s)} \tag{4.33}$$

in place of $e^{iy^2(1/4t-1/4s)}$ in the second integral in (4.2). Hence

$$A_+(s, t, x) = Ce^{ix^2/4t}B_+(\tau, \delta, z), \tag{4.34}$$

with $B_+(\tau, \delta, z)$ given as in (4.5), but with

$$\tau = \frac{s}{t} + 1 \tag{4.35}$$

in place of $\tau = s/t - 1$ as in (4.3), and also $(\tau-1)^{1/2}$ instead of $(\tau+1)^{1/2}$ in front of the integral. Thus instead of $\tau \in (-1, \infty)$ one has $\tau \in (1, \infty)$. The analysis of $B_+(\tau, \delta, z)$ is done as before, but since τ is now bounded away from zero, the singularities that appear in (4.23) do not arise. In particular, there is again a bound:

$$|A_+(s, t, x)| \leq C, \quad s, t \in (0, 1], \ x \in \mathbb{R}. \tag{4.36}$$

To treat (1.15) when $\nu = 0$, we are looking at

$$e^{it\Delta}(\nu\Phi_s^{jj}) = A_0(s, t, x). \tag{4.37}$$

In such a case one has

$$e^{iy^2/4t} \tag{4.38}$$

in place of $e^{iy^2(1/4t-1/4s)}$ in the second integral in (4.2). Hence

$$A_0(s, t, x) = Ce^{ix^2/4t}B_0(\tau, \delta, z), \tag{4.39}$$

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with $B_0(\tau,\delta,z)$ given as in (4.5), but with $\tau = s/t$ in place of $\tau = s/t - 1$ as in (4.3), and also $\tau^{1/2}$ instead of $(\tau+1)^{1/2}$ in front of the integral. Thus instead of $\tau \in (-1,\infty)$ one has $\tau \in (0,\infty)$. The analysis of $B_0(\tau,\delta,z)$ is then much as that of $B(\tau,\delta,z)$ in (4.5)–(4.17). Note however that as long as $j \geq 1$, $j + j \geq 2$, so one has the sort of relatively tame behavior as observed for $B(\tau,\delta,z)$ in case $j+k \geq 2$. In particular there is a bound:

$$|A_0(s,t,x)| \leq C. \tag{4.40}$$

Finally, when $\nu = j = k = 0$, we are looking at

$$e^{it\Delta}\nu, \tag{4.41}$$

whose analysis was done in Section 2.

Using the results obtained so far in this section, we establish the following estimate, which will be useful in Section 5.

Proposition 4.1 *Let $A(s,t,x)$ be given by* (4.2). *In case* $(j,k) = (0,1)$ *or* $(1,0)$, *we have*

$$|A(s,t,x)| \leq C + C\left|\log\left|\frac{t}{s-t}\right|\,\right|, \tag{4.42}$$

for $x \in \mathbb{R}$, $s,t \in (0,1]$, $\tau = (s-t)/t \in (-1,1]$, and

$$|A(s,t,x)| \leq C, \tag{4.43}$$

for $x \in \mathbb{R}$, $s,t \in (0,1]$, $\tau = (s-t)/t \in [1,\infty)$. In case $j+k \geq 2$, we have (4.42) *for all $x \in \mathbb{R}$, $s,t \in (0,1]$.*

Proof Take $(j,k) = (0,1)$. First consider the case $|\tau| \leq 1$. Using (4.24) for $\tau \in (0,1]$, with a similar result for $\tau \in [-1,0)$, we have

$$|e^{i\tau\Delta}\hat{\Psi}_0^{01}(z)| \leq C + A\left|-\frac{1}{2}\log\frac{1}{\tau} + e^{i\Delta}L_b(\tau^{-1/2}z) - e^{i\tau\Delta}L_b(z)\right|,$$

for $z \in \mathbb{R}$. In light of (2.16)–(2.17), the right side is clearly bounded for $|z| \geq 1$, $|\tau| \leq 1$. To check it for $|z| \leq 1$, note that $|e^{i\tau\Delta}L_b(z)|$ is bounded for $|z| \leq 1$, and that $e^{i\Delta}L_b - L_b$ is bounded, so for $|z| \leq 1$, $\tau \in (0,1]$,

$$|e^{i\tau\Delta}\hat{\Psi}_0^{01}(z)| \leq C + A\left|-\frac{1}{2}\log\frac{1}{\tau} + L_b(\tau^{-1/2}z)\right|.$$

We can assume $L_b(z) = (1-\chi(z))\log|z|$, with $\chi \in C_0^\infty(\mathbb{R})$, $\operatorname{supp}\chi \subset [-2,2]$, $\chi = 1$ on $[-1,1]$, and write

$$-\frac{1}{2}\log\frac{1}{\tau} + L_b(\tau^{-1/2}z) = -\frac{1}{2}\chi(\tau^{-1/2}z)\log\frac{1}{\tau} + \big(1-\chi(\tau^{-1/2}z)\big)\log|z|.$$

We deduce that

$$|e^{i\tau\Delta}\hat{\Psi}_0^{01}(z)| \le C + C\log\frac{1}{\tau}, \tag{4.44}$$

for $z \in \mathbb{R}$, $\tau \in (0,1]$.

Now (4.22) plus the fact that $\hat{\Psi}_b^{01} \in S_{\rm cl}^{-1}(\mathbb{R})$, which follows from (4.20), implies $|e^{i\tau\Delta}\hat{\Psi}_b^{01}(z)| \le C$, for $z \in \mathbb{R}$, $|\tau| \le 1$. This together with (4.44) and its analogue for $\tau \in [-1,0)$ gives

$$|e^{i\tau\Delta}\hat{\Psi}^{01}(z)| \le C + C\log\Big|\frac{1}{\tau}\Big|, \quad z \in \mathbb{R},\ |\tau| \le 1.$$

From here, we apply (4.15)–(4.17) to get

$$|B(\tau,\delta,z)| \le C + C\log\Big|\frac{1}{\tau}\Big|, \quad z \in \mathbb{R},\ |\tau| \le 1,\ \delta \in (0,\infty),$$

which in turn, by (4.4), gives (4.42) for $x \in \mathbb{R}$, $\tau \in (-1,1]$.

On the other hand, it follows from (4.26) and (4.27) that

$$|B(\tau,\delta,z)| \le C, \quad z \in \mathbb{R},\ \tau \ge 1,\ \delta \in (0,2],$$

which gives (4.43) for $x \in \mathbb{R}$, $\tau \in [1,\infty)$. This treats $(j,k) = (0,1)$. The case $(j,k) = (1,0)$ is similar, and the cases where $j+k \ge 2$ are parallel, and a little simpler. ■

The result (4.42) carries no information when $s = t$. On the other hand, (4.24) together with (4.21) readily yield

$$C_1 \log\frac{1}{t} \le \|A(t,t,\cdot)\|_{L^\infty} \le C_2 \log\frac{1}{t}, \quad 0 < t \le \frac{1}{2}.$$

This will lead to an analogous result for $v_0(t,t,x) = e^{it\Delta}q(e^{it\Delta}f)$, when f is compactly supported and piecewise smooth, with a jump. By other methods, the upper bound

$$\|v_0(s,t,\cdot)\|_{L^\infty} \le C \log\frac{1}{t}, \quad 0 < t \le \frac{1}{2} \tag{4.45}$$

will be established in Section 5, for a somewhat more general family of functions f on $\mathbb{R}$. This result complements what one obtains from Proposition 4.1. In Section 5 it will be shown that Proposition 4.1 leads to the estimate

$$\|v(t,\cdot)\|_{L^\infty} \le C, \quad 0 < t \le \frac{1}{2}, \tag{4.46}$$

on $v(t,x) = t^{-1}\int_0^t v_0(s,t-s,x)\,ds$. Note that (4.46) does not follow from (4.45).

So far in this section we have investigated oscillatory integrals of the form (4.2), which came from (1.15). Such functions arose to describe $e^{it\Delta}q(e^{is\Delta}f)$, via (1.12)–(1.14), when f is piecewise smooth on $\mathbb{R}$, with a single jump, at $x = 0$, in which case $e^{is\Delta}f$ is given by (1.9)–(1.10) (with t replaced by s). Now we are also interested in cases when f has two or more jumps. Then $e^{it\Delta}f$ would be a sum of several terms such as appear on the right side of (1.9), with x replaced by $x - a_j$, if the jumps occur at $x = a_j$. This leads to oscillatory integrals like the first integral in (4.2), with the factor $v(y)e^{-iy^2/4s}\Phi^{jk}(y/\sqrt{4s})$ replaced by products of several such factors, with arguments $y - a_j$. For example, if $f(x)$ has two jumps, say at $x = 0$ and $x = a$, (4.2) is replaced by

$$\begin{aligned}(4.47)\quad A(s,t,x) = (4\pi it)^{-1/2}\int_{-\infty}^{\infty} e^{i(x-y)^2/4t} v_1(y) e^{-iy^2/4s}\Phi^{jk}\Big(\frac{y}{\sqrt{4s}}\Big)\\ \times\ v_2(y-a)e^{-i(y-a)^2/4s}\Phi^{\ell m}\Big(\frac{y-a}{\sqrt{4s}}\Big)\,dy.\end{aligned}$$

Manipulations parallel to those done in (4.3)–(4.5) yield

$$(4.48)\qquad A(s,t,x) = Ce^{i(x^2/4t - a^2/4s)}B(\tau,\delta,z),$$

where

$$(4.49)\qquad z = \frac{x\sqrt{s}}{t},\quad \tau = \frac{s}{t} - 2,\quad \delta = \sqrt{4s},$$

and

$$\begin{aligned}(4.50)\quad B(\tau,\delta,z) = (\tau+2)^{1/2}\int_{-\infty}^{\infty} e^{-i(z+a/\sqrt{s})\eta}e^{i\tau\eta^2}v_1(\delta\eta)\Phi^{jk}(\eta)\\ \times\ v_2(\delta\eta - a)\Phi^{\ell m}\Big(\eta - \frac{a}{\delta}\Big)\,d\eta.\end{aligned}$$

We assume $j + k \geq 1$ and $\ell + m \geq 1$. Without getting into an analysis of such integrals as detailed as that reported in (4.15)–(4.23), for example, we will establish the (intuitively reasonable) fact that $A(s,t,x)$ vanishes in the limit $s \to 0$. In fact, we will show that $B(\tau,\delta,z)$ in (4.50) vanishes as $\delta \to 0$, uniformly in (τ, z). Hence the nonlinear interactions of different jump singularities play a minor role in the behavior of $e^{it\Delta}q(e^{is\Delta}f)$. We present the details for two jumps, which involve an analysis of (4.50), but it will be clear how a similar analysis works for a larger number of jumps.

For one estimate on (4.50), note that v_1 and v_2 are bounded and

$$\begin{aligned}(4.51)\qquad &\int_{-\infty}^{\infty}|\Phi^{jk}(\eta)\Phi^{\ell m}(\eta - \delta^{-1}a)|\,d\eta\\ &\leq C\int_{-\infty}^{\infty}(1+|\eta|)^{-1}(1+|\eta-\delta^{-1}a|)^{-1}\,d\eta\\ &\leq C\,\frac{\delta}{a}\log\frac{a}{\delta}\end{aligned}$$

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for $\delta < a/2$. The first estimate can be improved for $j+k$ or $\ell+m \geq 2$. So we have

$$|B(\tau,\delta,z)| \leq C(\tau+2)^{1/2}\,\frac{\delta}{a}\log\frac{a}{\delta}, \tag{4.52}$$

which is a good bound for τ bounded.

To get a good bound for $\tau \in [1,\infty)$, we take $\varepsilon = 1/4\tau$ and, parallel to (4.26), obtain

$$B(\tau,\delta,z) = C(1+2\varepsilon)^{1/2}e^{-i(z+a/\sqrt{s})^2}e^{i\varepsilon\Delta}(w_\delta^{jk}u_{a,\delta}^{\ell m})\bigl(-2\varepsilon(z+s^{-1/2}a)\bigr), \tag{4.53}$$

where

$$w_\delta^{jk}(\eta) = v_1(\delta\eta)\Phi^{jk}(\eta), \quad u_{a,\delta}^{\ell m}(\eta) = v_2(\delta\eta - a)\Phi^{\ell m}\Bigl(\eta - \frac{a}{\delta}\Bigr). \tag{4.54}$$

To estimate this, we note that if $u_0 \in L^2(\mathbb{R})$ has finite total variation, then

$$|e^{it\Delta}u_0(x)| \leq \|u_0\|_{TV}\sup_{x\in\mathbb{R}}|\operatorname{Fr}(x)|, \tag{4.55}$$

where $\|u_0\|_{TV}$ is the total variation of u_0, and $\operatorname{Fr}(x)$ is given by (1.6). This follows from the fact that if $u_0 \in L^2(\mathbb{R})$ and $u_0' = \mu$ is a finite measure on $\mathbb{R}$, then

$$e^{it\Delta}u_0(x) = \int \operatorname{Fr}\Bigl(\frac{x-y}{\sqrt{4t}}\Bigr)\,d\mu(y), \tag{4.56}$$

cf. [24, Proposition 4.1]. Consequently,

$$|B(\tau,\delta,z)| \leq C(1+2\varepsilon)^{1/2}\|w_\delta^{jk}u_{a,\delta}^{\ell m}\|_{TV} \leq C(1+2\varepsilon)^{1/2}\delta, \tag{4.57}$$

the latter inequality by inspection from (4.54), with stronger estimates holding if $j+k \geq 2$ and $\ell+m \geq 2$. In summary, when $A(s,t,x)$ is given by (4.47) and $a \neq 0$, we have

$$|A(s,t,x)| \leq Cs^{1/2}\log\frac{2}{s}, \quad s,t \in (0,1],\ x \in \mathbb{R}. \tag{4.58}$$

5 Convergence of v_0 and v to $q(f)$

Here we discuss various ways in which

$$v_0(s,t,x) = e^{it\Delta}q(e^{is\Delta}f) \quad \text{and} \quad v(t,x) = \frac{1}{t}\int_0^t v_0(s,t-s,x)\,ds$$

converge to $q(f)$. We begin with some general results before specializing to results that use the material developed in Sections 1–4.

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Proposition 5.1 *Assume $f \in H^{\sigma,2}(\mathbb{R}^n)$ for some $\sigma \geq 0$ and*

$$\|e^{it\Delta} f\|_{L^\infty(\mathbb{R}^n)} \leq A < \infty, \quad \forall t \in \mathbb{R}. \tag{5.1}$$

Also assume

$$q\colon \mathbb{R}^2 \to \mathbb{R}^2 \text{ is smooth and } q(0) = 0. \tag{5.2}$$

Then

$$\lim_{s,t\to 0} v_0(s,t,\,\cdot\,) = q(f) \tag{5.3}$$

in L^2-norm and weak* in $H^{\sigma,2}(\mathbb{R}^n)$, and

$$\lim_{t\to 0} v(t,\,\cdot\,) = q(f) \tag{5.4}$$

in L^2-norm and weak* in $H^{\sigma,2}(\mathbb{R}^n)$.

Proof To begin, we have the well-known Moser-type estimate

$$\|q(e^{is\Delta} f)\|_{H^{\sigma,2}} \leq C(\|e^{is\Delta} f\|_{L^\infty})\|e^{is\Delta} f\|_{H^{\sigma,2}}, \tag{5.5}$$

for $\sigma \geq 0$ (*cf.* [20, Ch. 13, Proposition 10.2]), so the hypotheses on f yield a bound on the $H^{\sigma,2}$ norm of $q(e^{is\Delta} f)$. Hence

$$\|v_0(s,t,\,\cdot\,)\|_{H^{\sigma,2}} \leq A_1, \quad \|v(t,\,\cdot\,)\|_{H^{\sigma,2}} \leq A_2. \tag{5.6}$$

Next, we have $v_1(s) = q(e^{it\Delta} f) - q(f) = G(s,x)(e^{is\Delta} f - f)$, where

$$G(s,x) = \int_0^1 Dq\big(\sigma e^{is\Delta} f + (1-\sigma) f\big)\, d\sigma.$$

Hence $\|G(s,\,\cdot\,)\|_{L^\infty} \leq A_3$, and we have

$$\begin{aligned}\|v_0(s,t,\,\cdot\,) - q(f)\|_{L^2} &\leq \|v_1(s)\|_{L^2} + \|(e^{it\Delta} - I)q(f)\|_{L^2} \\ &\leq A_3\|e^{is\Delta} f - f\|_{L^2} + \|(e^{it\Delta} - I)q(f)\|_{L^2}.\end{aligned}$$

It is clear that this tends to 0 as $s, t \to 0$, so we have L^2 norm convergence in (5.2). The weak* convergence follows from this plus the uniform bound in (5.6), and the asserted convergence in (5.4) follows from this. ∎

If f is compactly supported in $\mathbb{R}^n$ and piecewise smooth, with jump across a smooth hypersurface, then $f \in H^{\sigma,2}(\mathbb{R}^n)$ for all $\sigma < 1/2$. As shown in [24, §4], the hypothesis (5.1) holds for all such f if $n \leq 2$ (though it can fail when $n \geq 3$). Consequently for such f and $n \leq 2$, if q satisfies (5.2), then, as $s, t \to 0$,

$$v_0(s,t,\,\cdot\,),\ v(t,\,\cdot\,) \longrightarrow q(f) \text{ weak}^* \text{ in } H^{\sigma,2}, \quad \forall\, \sigma < \frac{1}{2}. \tag{5.7}$$

One can improve (5.7) by working with Besov spaces. A compactly supported piecewise smooth function belongs to $B^{1/2}_{2,\infty}(\mathbb{R}^n)$. Now there is the following analogue of Proposition 5.1.

Proposition 5.2 *Assume* $f \in B^{\sigma}_{2,\infty}(\mathbb{R}^n)$ *for some* $\sigma > 0$, *and assume* (5.1) *holds. Also assume* q *satisfies* (5.2). *Then the limits* (5.3)–(5.4) *hold in the weak** *topology of* $B^{\sigma}_{2,\infty}(\mathbb{R}^n)$.

Proof First we note that the weak* topology is given via the duality (*cf.* [25, p. 178])

$$B^{\sigma}_{2,\infty}(\mathbb{R}^n) = B^{-\sigma}_{2,1}(\mathbb{R}^n)'; \tag{5.8}$$

To proceed, results of [18] imply

$$\|q(e^{is\Delta} f)\|_{B^{\sigma}_{2,\infty}} \le C(\|e^{is\Delta} f\|_{L^\infty})\|e^{is\Delta} f\|_{B^{\sigma}_{2,\infty}},$$

for $\sigma > 0$. (In fact, $q(f) = M_q(f; x, D)f$, as in [19, (3.1.15)], and $M_q(f; x, D)$ is a zero-order paradifferential operator with symbol in $S^0_{1,1}$, which is hence bounded on $B^{\sigma}_{2,\infty}$ whenever $\sigma > 0$, with operator bound depending on $\|f\|_{L^\infty}$.) Also $e^{is\Delta}$ is a bounded semigroup on $B^{\sigma}_{2,\infty}(\mathbb{R}^n)$, so we have

$$\|v_0(s, t, \cdot)\|_{B^{\sigma}_{2,\infty}} \le A_4. \tag{5.9}$$

Since $B^{\sigma}_{2,\infty}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$ for $\sigma > 0$, we can apply Proposition 5.1 to obtain L^2-norm convergence in (5.3). In light of (5.8) such convergence plus the bound (5.9) implies that convergence in (5.3) holds weak* in $B^{\sigma}_{2,\infty}(\mathbb{R}^n)$, and hence so does convergence in (5.4). ■

Corollary 5.3 *Assume* f *is compactly supported and piecewise smooth on* $\mathbb{R}^n$, *with* $n \le 2$. *Assume* q *satisfies* (5.2). *Then the limits* (5.3)–(5.4) *hold in the weak** *topology of* $B^{1/2}_{2,\infty}(\mathbb{R}^n)$.

In case $n = 1$, $B^{1/2}_{2,\infty}(\mathbb{R})$ just fails to be contained in $L^\infty(\mathbb{R})$. One has the following result, along lines pioneered in [3]. (See also [4, 5] for related results.) First, for general n, if $\sigma > n/p + \delta$, there exists $C < \infty$ such that for all $\varepsilon \in (0, 1]$, (*cf.* [19, (B.1.8)]),

$$\|u\|_{L^\infty} \le C\varepsilon^{\delta}\|u\|_{H^{\sigma,p}} + C\Big(\log\frac{1}{\varepsilon}\Big)\|u\|_{B^0_{\infty,\infty}}.$$

It is standard that $B^{n/2}_{2,\infty}(\mathbb{R}^n) \subset B^0_{\infty,\infty}(\mathbb{R}^n)$, so

$$\|u\|_{L^\infty} \le C\varepsilon^{\delta}\|u\|_{H^{\sigma,p}} + C\Big(\log\frac{1}{\varepsilon}\Big)\|u\|_{B^{n/2}_{2,\infty}}.$$

Now if we arrange that the norms satisfy $\|u\|_{B^{n/2}_{2,\infty}} \le \|u\|_{H^{\sigma,p}}$, which can be done, and pick ε optimally, this gives the estimate

$$\|u\|_{L^\infty} \le C\|u\|_{B^{n/2}_{2,\infty}}\Big[1 + \log\Big(\frac{\|u\|_{H^{\sigma,p}}}{\|u\|_{B^{n/2}_{2,\infty}}}\Big)\Big] \tag{5.10}$$

for functions on $\mathbb{R}^n$. We will apply this to prove the following.

Proposition 5.4 *Assume* $f \in B^{1/2}_{2,\infty}(\mathbb{R})$ *and assume* (5.1) *holds. Also assume* $q\colon \mathbb{R}^2 \to \mathbb{R}^2$ *is smooth and*

$$q(0) = 0, \quad Dq(0) = 0. \tag{5.11}$$

Then, for $s, t \in (0, 1/2]$,

$$\|v_0(s, t, \cdot)\|_{L^\infty} \leq C \log \frac{1}{t}. \tag{5.12}$$

Proof In this case, (5.9) holds with $\sigma = 1/2$. Hence the quantity

$$\|v_0(s, t, \cdot)\|_{B^{1/2}_{2,\infty}} \log \|v_0(s, t, \cdot)\|_{B^{1/2}_{2,\infty}}$$

is also bounded, and we have, via (5.10),

$$\|v_0(s, t, \cdot)\|_{L^\infty} \leq C + C\|v_0(s, t, \cdot)\|_{B^{1/2}_{2,\infty}} \log \|v_0(s, t, \cdot)\|_{H^{\sigma,p}}, \tag{5.13}$$

provided $\sigma p > 1$. We proceed to estimate the right side of (5.13), for judiciously chosen σ and p.

The hypothesis (5.11) on $q(u)$ allows us to write $q(u) = Q(u)u$, $Q \in C^\infty$, $Q(0) = 0$. The hypotheses on f imply $f \in H^{\sigma,2}$ for all $\sigma < 1/2$, as well as $e^{is\Delta} f \in L^\infty$. Then the obvious analogue of (5.5) yields

$$\|Q(e^{is\Delta} f)\|_{H^{\sigma,2}} \leq A_5, \tag{5.14}$$

and of course we also have an L^∞ estimate, hence L^q estimates for all $q \in [2, \infty]$. To estimate the product $Q(u)u$, we apply the product estimate of [7] to get

$$\|uv\|_{H^{\sigma,p'}} \leq C\|u\|_{L^q}\|v\|_{H^{\sigma,2}} + C\|u\|_{H^{\sigma,2}}\|v\|_{L^q}, \tag{5.15}$$

valid for

$$\frac{1}{p'} = \frac{1}{q} + \frac{1}{2}, \quad q \in (2, \infty]. \tag{5.16}$$

(See also [21, Ch. 2, Proposition 1.1] for a proof.) In particular, this works with

$$\sigma = \frac{1}{3}, \quad q = 4, \quad p' = \frac{4}{3}, \quad p = 4, \tag{5.17}$$

and we have

$$\|q(e^{is\Delta} f)\|_{H^{1/3,4/3}} \leq A_6. \tag{5.18}$$

We now use the *dispersive estimate*, which for $e^{it\Delta}$ acting on functions on $\mathbb{R}^n$ is

$$\|e^{it\Delta} g\|_{L^p} \leq C|t|^{-n(1/2-1/p)} \|g\|_{L^{p'}}, \tag{5.19}$$

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valid for $g \in L^{p'}(\mathbb{R}^n)$ with $p' \in [1,2]$ the dual exponent of $p \in [2,\infty]$, with $C = C(n,p)$. This well-known result follows by interpolation from its endpoint cases, $p = 2$ and $p = \infty$. The $p = 2$ case is obvious from the unitarity of $e^{it\Delta}$ on $L^2(\mathbb{R}^n)$, and the $p = \infty$ case follows readily from the formula (1.5). Since $e^{it\Delta}$ commutes with powers of $I - \Delta$, we also have

$$\|e^{it\Delta} g\|_{H^{\sigma,p}} \le C|t|^{-n(1/2-1/p)} \|g\|_{H^{\sigma,p'}}. \tag{5.20}$$

We apply this when $n = 1$ and σ, p and p' are as in (5.17), to deduce via (5.18) that $\|e^{it\Delta} q(e^{is\Delta} f)\|_{H^{1/3,4}(\mathbb{R})} \le C|t|^{-1/4}$. Thus we can take $\sigma = 1/3$, $p = 4$ in (5.13) and the desired estimate (5.12) follows. ■

Remark. The estimates (5.14)–(5.20) hold for f satisfying

$$\|f\|_{H^{\sigma,2}(\mathbb{R}^n)} \le A, \quad \|e^{it\Delta} f\|_{L^\infty(\mathbb{R}^n)} \le A, \tag{5.21}$$

with $\sigma \in (0,1)$, for any n. Take

$$\sigma = \frac{1}{2} - \varepsilon, \quad p = 2 + 6\varepsilon, \tag{5.22}$$

so

$$\sigma p = 1 + \varepsilon(3 - 5\varepsilon) > 1, \quad \text{for } 0 < \varepsilon < \frac{3}{5}, \tag{5.23}$$

to deduce that

$$\|e^{it\Delta} q(e^{is\Delta} f)\|_{H^{\sigma,p}(\mathbb{R}^n)} \le Ct^{-3\varepsilon n/(2+6\varepsilon)}. \tag{5.24}$$

We will find this useful in Section 6.

We now apply the results of Section 4 to obtain an estimate complementary to (5.13), for the smaller classes of functions f and q used in Section 4.

Proposition 5.5 *Assume $q(u)$ is a finite linear combination of $u^\ell \overline{u}^m$ with $\ell, m \ge 1$. Assume f is piecewise smooth on $\mathbb{R}$ and rapidly decreasing at infinity, with a finite number of jump discontinuities. Then*

$$\|v_0(s,t,\cdot)\|_{L^\infty} \le C + C\left|\log\left|\frac{t}{s-t}\right|\right|, \tag{5.25}$$

for $s, t \in (0,1]$, $\tau = (s-t)/t \in (-1,1]$, and

$$\|v_0(s,t,\cdot)\|_{L^\infty} \le C, \tag{5.26}$$

for $s, t \in (0,1]$, $\tau = (s-t)/t \in [1,\infty)$.

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Proof As shown in (1.12)–(1.15), when f has one jump, at $x = 0$, $v_0(s,t,x)$ is a finite sum of functions of the form $A^{jk}(s,t,x)$ in (1.15), with $0 \le j \le \ell$, $0 \le k \le m$, $\nu = j - k$. By Proposition 4.1, the contribution from $(j,k) = (0,1)$ satisfies (5.25)–(5.26), and all the other contributions are uniformly bounded, in case $k \ge j \ge 0$ and $j + k \ge 2$. By the analysis in (4.31)–(4.41), all other contributions are uniformly bounded. This completes the proof, for the case of one jump. The extension to the case where f might have more than one jump follows from the analysis given in (4.47)–(4.58). ■

Proposition 5.5 has the following important corollary.

Proposition 5.6 *In the setting of Proposition* 5.5*,*

$$\|v(t,\,\cdot\,)\|_{L^\infty} \le C, \quad \forall\, t \in (0,1]. \tag{5.27}$$

Proof We have

$$\|v(t,\,\cdot\,)\|_{L^\infty} \le \frac{1}{t}\int_0^{2t/3} \|v_0(s,t-s,\,\cdot\,)\|_{L^\infty}\,ds + \frac{1}{t}\int_{2t/3}^{t} \|v_0(s,t-s,\,\cdot\,)\|_{L^\infty}\,ds. \tag{5.28}$$

For the first integral on the right, use (5.25), with t replaced by $t - s$. We have

$$\begin{aligned}
\frac{1}{t}\int_0^{2t/3} \|v_0(s,t-s,\,\cdot\,)\|_{L^\infty}\,ds &\le C + \frac{C}{t}\int_0^{2t/3}\left|\log\left|\frac{t-s}{2s-t}\right|\right|\,ds \\
&\le C + C\int_0^{2/3}\left|\log\left|\frac{1-\sigma}{2\sigma-1}\right|\right|\,d\sigma \\
&\le C_2,
\end{aligned}$$

since the logarithmic singularity at $\sigma = 1/2$ is integrable. For the second integral on the right side of (5.28), use (5.26), again with t replaced by $t - s$. Then (5.27) follows. ■

We end this section with the following useful complement to Proposition 5.6.

Proposition 5.7 *In the setting of Proposition* 5.5*,* $v_0(s,t,x) \to q(f(x))$*, locally uniformly on* $\mathbb{R} \setminus S$*, as* $s,t \to 0$*, where* $S \subset \mathbb{R}$ *is the singular set of* f*. Hence*

$$v(t,x) \longrightarrow q(f(x)), \quad \textit{locally uniformly on } \mathbb{R} \setminus S, \tag{5.29}$$

as $t \to 0$.

Proof As noted in the proof of Proposition 5.5, when f has one jump, at $x = 0$, $v_0(s,t,x)$ is a finite sum of functions of the form $A^{jk}(s,t,x)$ in (1.15), with $0 \le j \le \ell$, $0 \le k \le m$, $\nu = j - k$. In particular, we have $A^{00}(s,t,x) = \lambda e^{it\Delta}\big(v(s)^\ell\overline{v(s)}^m\big)(x)$, with $v(t,x)$ not as in (5.29) but rather as in (1.9)–(1.10), *i.e.*, $v(t,x) = f(x) + g(t,x)$, where $g(t,x)$ is smooth in (t,x), rapidly decreasing as $|x| \to \infty$, and of course vanishing at $t = 0$. Expanding $(f+g)^\ell(\overline{f}+\overline{g})^m$, we readily obtain from the analysis of Section

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2 that $A^{00}(s,t,x) - \lambda e^{it\Delta}(f^\ell \overline{f}^m)(x) \to 0$, uniformly in x, as $s,t \to 0$. Also the results of Section 2 imply $e^{it\Delta}(f^\ell \overline{f}^m) \to f^\ell \overline{f}^m$ locally uniformly on $\mathbb{R} \setminus 0$. Hence $A^{00}(s,t,x) \to \lambda f(x)^\ell \overline{f(x)}^m$, locally uniformly on $\mathbb{R} \setminus 0$, as $s,t \to 0$. Summing over the terms appearing in $q(u)$, we have the sum converging to $q(f(x))$, locally uniformly on $\mathbb{R} \setminus 0$.

We next claim that

$$A^{jk}(s,t,x) \longrightarrow 0, \quad \text{locally uniformly on } \mathbb{R} \setminus 0, \tag{5.30}$$

as $s,t \to 0$, for $j+k \geq 1$. Recall from (4.4) that when $(j,k) = (0,1)$,

$$A^{01}(s,t,x) = Ce^{ix^2/4t} B(\tau,\delta,z),$$

with (τ,δ,z) given by (4.3) and $B(\tau,\delta,z)$ by (4.5), and also by (4.15)–(4.16), *i.e.*,

$$B(\tau,\delta,z) = (\tau+1)^{1/2} \hat{u}_\delta * E_\tau(z), \quad E_\tau(z) = e^{i\tau\Delta} \hat{\Psi}^{01}(z). \tag{5.31}$$

We assume $|x| \geq b > 0$, and consider three cases.

Case 1: $s/t \leq s^{1/4}$. Then $\tau \approx -1$ and $(\tau+1)^{1/2} \leq s^{1/8}$. Now $\tau \approx -1 \Rightarrow |E_\tau(z)| \leq C$, for all z, giving $|B(\tau,\delta,z)| \leq Cs^{1/8}$.

Case 2: $s^{1/4} \leq s/t \leq 2$. Hence $-1 < \tau \leq 1$, and $|z| \geq |x|/s^{1/4} \geq b/s^{1/4}$. The behavior of $\hat{\Psi}^{01}(z)$ given in (4.20)–(4.21) implies that $E_\tau(z) \to 0$ as $|z| \to \infty$, locally uniformly in τ, and hence $B(\tau,\delta,z) \to 0$, by (5.31).

Case 3: $2 \leq s/t < \infty$. Then $1 \leq \tau < \infty$, so $0 < \varepsilon \leq 1/4$, with ε given by (4.25). We use (4.26) and note that $|2\varepsilon z| \geq |x|/2s^{1/2} \geq b/2s^{1/2}$. Now (4.26) gives

$$|B(\tau,\delta,z)| \leq C|F(\varepsilon,\delta,2\varepsilon z)|,$$

and (4.28) gives $F(\varepsilon,\delta,y) = F_0(\varepsilon,\delta,y) + F_1(\varepsilon,\delta,z)$. Furthermore, (4.29) gives $|F_0(\varepsilon,\delta,y)| \to 0$ as $|y| \to \infty$, locally uniformly in ε,δ, and (4.30) gives such a result for $|F_1(\varepsilon,\delta,y)|$.

This treats (5.30) for $(j,k) = (0,1)$. Treatments for other values of (j,k) are similar (and generally simpler). This gives Proposition 5.7 when f has just one singularity. As in the proof of Proposition 5.5, the extension to the case where f has more than one singularity follows from the analysis given in (4.47)–(4.58). ■

6 Implications for NLS

We now discuss implications of results of Sections 2–5 for solutions to

$$\frac{\partial u}{\partial t} = i\Delta u + q(u), \quad u(0,x) = f(x), \tag{6.1}$$

on $[0,T_0) \times \mathbb{R}^n$, with $n = 1$ or 2. As stated in the introduction, we assume $q\colon \mathbb{R}^2 \to \mathbb{R}^2$ is smooth and

$$q(0) = 0, \quad Dq(0) = 0. \tag{6.2}$$

We also assume

$$\|f\|_{H^{\sigma,2}} \le A, \quad \|e^{it\Delta} f\|_{L^\infty} \le A, \tag{6.3}$$

with $\sigma \ge 0$ for $n = 1$ and $\sigma \in (0,1)$ for $n = 2$. Then, as shown in [24], on some interval $t \in [0, T_0)$ we have

$$u(t) = e^{it\Delta} f + tv(t) + w(t), \tag{6.4}$$

where $v(t,x) = \frac{1}{t}\int_0^t v_0(s, t-s, x)\,ds$, $v_0(s,t,x) = e^{it\Delta} q(e^{is\Delta} f)$, and $w(t)$ is a remainder satisfying

$$\|w(t)\|_{L^\infty} \le Ct^\alpha, \tag{6.5}$$

where $\alpha = 3/2$ for $n = 1$ and one can take any $\alpha < 1+\sigma$ for $n = 2$.

Sections 2 and 3 were devoted to an analysis of $e^{it\Delta} f$ for special classes of functions f on $\mathbb{R}^n$, and Sections 4 and 5 to an analysis of $v(t)$, including some results for general n, some emphasizing the case where $n = 1$, and some further assuming that f is compactly supported and smooth on $\mathbb{R}$ and that $q(u)$ has the form

$$q(u) = \sum_{j=1}^{M} a_j u^{\ell_j} \overline{u}^{m_j}, \quad a_j \in \mathbb{C},\ \ell_j, m_j \ge 1. \tag{6.6}$$

We recall the estimate (5.12), which implies

$$\|tv(t)\|_{L^\infty} \le Ct \log \frac{1}{t}, \tag{6.7}$$

for $t \in (0, 1/2]$, provided $n = 1$, q satisfies (6.2), and $f \in B^{1/2}_{2,\infty}(\mathbb{R})$ satisfies (6.3). We also recall the stronger estimate (5.27), *i.e.*, $\|tv(t)\|_{L^\infty} \le Ct$, for $t \in (0, 1]$, provided $n = 1, q$ satisfies (6.6), and f is compactly supported and piecewise smooth on $\mathbb{R}$ (with jumps). As for corresponding estimates when $n = 2$, we recall the following result from [24, §3].

Proposition 6.1 *Take $\sigma \in (0,1)$ and assume a function f on $\mathbb{R}^2$ satisfies* (6.3). *Also assume $q(u)$ satisfies* (6.2). *Then*

$$\|e^{it\Delta} q(e^{is\Delta} f)\|_{H^{\sigma,p}} \le Ct^{-1+2/p}, \quad 2 < p < \infty.$$

In particular, taking $\delta > 0$ and $p = 2/(\sigma - \delta)$, we have

$$\|v_0(s,t,\cdot)\|_{L^\infty} \le C_\delta t^{-1+\sigma-\delta}, \tag{6.8}$$

and hence

$$t\|v(t)\|_{L^\infty} \le C_\delta t^{\sigma-\delta}. \tag{6.9}$$

If f is compactly supported on $\mathbb{R}^2$ and piecewise smooth with a jump, then (6.8)–(6.9) *hold for each $\sigma < 1/2$, and the estimate* (6.5) *on $w(t)$ holds for each $\alpha < 3/2$.*

Using these estimates on $tv(t)$ and $w(t)$, we can see that the solution $u(t)$ to (6.1) shares various special properties with $e^{it\Delta}f$. In detail, assume f is compactly supported and piecewise smooth on $\mathbb{R}^n$, with jump across $\partial\Omega$. The Gibbs-type phenomenon analyzed in Proposition 3.4 holds in the same form for $u(t,x)$ in place of $e^{it\Delta}f(x)$, for $n = 1$ and $n = 2$, as a consequence of (6.6) and (6.7) for $n = 1$, and (6.9) for $n = 2$.

Similarly when $n = 2$ and Ω is a disk, the Pinsky-type phenomenon analyzed in Proposition 3.2 holds with $e^{it\Delta}f(x)$ replaced by $u(t,x)$, at least mod $O(t^{1/2-\delta})$, so the spike at the center of the disk has the same form. Furthermore, for general compact smoothly bounded $\overline{\Omega} \subset \mathbb{R}^2$, the behavior of $e^{it\Delta}f(x)$ on and sufficiently near a fold caustic $\mathcal{C}_1$, given by (3.5), continues to hold for $u(t,x)$ (again mod $O(t^{1/2-\delta})$), since the behavior of $e^{it\Delta}f(x)$ on $\mathcal{C}_1$ is $\approx Ct^{1/3}$. Similarly, the behavior of $e^{it\Delta}f(x)$ on and sufficiently near higher order caustics persists for $u(t,x)$.

As shown in (3.4), the behavior of $e^{it\Delta}f(x)$ on the non-caustic set $\mathcal{C}_0$ includes oscillations of wave length $\approx \varphi(x)/t$ and amplitude $b_0(x)t^{1/2}$. In case $n = 1$, we can apply (6.7) and (6.5) to deduce that such behavior persists for $u(t,x)$. In case $n = 2$, the estimate (6.9) does not guarantee this conclusion. To see if the estimate (6.9) might be improved, one might try to extend the scope of Section 4 from one space dimension to two. We plan to take this up in a future paper.

Here we do improve (6.9) for a special class of functions f, namely those whose jump occurs on a circle. The argument uses a symmetry of Δ. It follows from the next proposition that the oscillations of $e^{it\Delta}f(x)$ in the non-caustic region $\mathcal{C}_0$ are present in the same form for $u(t,x)$, for such a class of functions.

Proposition 6.2 *Let D be a disk in $\mathbb{R}^2$ (centered at p), and let $f \in C^\infty(\mathbb{R}^2 \setminus \partial D)$ have compact support and a simple jump across ∂D. For $a > 0$, let $D_a = \{x \in \mathbb{R}^2 : |x - p| \le a\}$. Then for each $a > 0, \delta > 0$,*

$$\|e^{it\Delta}q(e^{is\Delta}f)\|_{L^\infty(\mathbb{R}^2\setminus D_a)} \le Ct^{-\delta}, \tag{6.10}$$

hence (6.4) *holds with $t\|v(t)\|_{L^\infty(\mathbb{R}^2\setminus D_a)} \le Ct^{1-\delta}$, for $t \in (0,1]$.*

Proof We may as well assume $p = 0$. Let $X = \partial/\partial\theta = x_1\partial_{x_2} - x_2\partial_{x_1}$, which generates rotation. Fix $\varepsilon > 0$, and set $\sigma = 1/2 - \varepsilon$. Using the fact that X^k commutes with $e^{it\Delta}$, we have $X^k e^{is\Delta}f \in H^{\sigma,2}(\mathbb{R}^2)$, $\|X^k e^{is\Delta}f\|_{L^\infty} \le A_k$. Hence $X^k q(e^{is\Delta}f) \in H^{\sigma,2}(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, with norm bounds, for each $k \in \mathbb{N}$. The arguments involving (5.21)–(5.24) then give

$$\|X^k e^{it\Delta}q(e^{is\Delta}f)\|_{H^{\sigma,p}(\mathbb{R}^2)} \le C_k t^{-3\varepsilon/(1+3\varepsilon)}, \tag{6.11}$$

for $\varepsilon \ll 1$, where $\sigma = \frac{1}{2} - \varepsilon$, $p = 2 + 6\varepsilon$. Now if we define

$$^K H^{\sigma,p}(\mathbb{R}^2) = \{f \in H^{\sigma,p}(\mathbb{R}^2) : X^k f \in H^{\sigma,p}(\mathbb{R}^2),\ \forall\, k \le K\},$$

then a variant of the Sobolev embedding theorem gives

$$\sigma p > 1,\ K \ge 2 \quad \Longrightarrow \quad {}^K H^{\sigma,p}(\mathbb{R}^2) \subset L^\infty(\mathbb{R}^2 \setminus D_a), \quad \forall\, a > 0.$$

Thus (6.10) follows from (6.11). ■

We have discussed results about details of the short time behavior of the solution $u(t,x)$ to (6.1) captured by $e^{it\Delta}f(x)$. In addition, results of Section 5 plus the estimate (6.5) apply to show that the term $tv(t)$ provides a correction to this, dominating the remainder $w(t)$. This is a consequence of the fact that the exponent α in (6.5) is greater than 1, together with the fact that

$$v(t) \to q(f) \tag{6.12}$$

in various ways, as $t \to 0$. In particular, if f is compactly supported and piecewise smooth on $\mathbb{R}^n$, $n = 1$ or 2, we have seen that (6.12) holds in L^2-norm, and weak* in $B^{1/2}_{2,\infty}(\mathbb{R}^n)$. It follows by Rellich's theorem that convergence holds in norm in $H^{\sigma,2}(B)$ for all $\sigma < \frac{1}{2}$, $B \subset \mathbb{R}^n$ bounded. In addition, if $n = 1$ and $q(u)$ has the form (6.6), we have, by Proposition 5.6, $v(t) \to q(f)$ boundedly, and, by Proposition 5.7, if S denotes the singular set of f, $v(t) \to q(f)$, locally uniformly on $\mathbb{R} \setminus S$.

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