INVERSE PROBLEM : ACOUSTIC POTENTIAL VS ACOUSTIC LENGTH

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ABSTRACT. By using the quantization rule based on the WKB asymptotic method, we present an integral equation to infer the form of the acoustic potential of a fixed ℓ as a function of the acoustic length. Since we analyze the acoustic potential itself by taking account of some factors other than the sound velocity and we can analyze the radial modes by this scheme as well as nonradial modes, this method improves the accuracy and effectiveness of the inverse problem to infer the internal structure of the Sun, in particular, the deep interior of the Sun.

1. INTRODUCTION

The most important and unique aspect of solar oscillations is the possibility of a seismological approach, by which we can probe the solar deep interior by using oscillations. In this respect, there are two approaches which are complementary each other, that is, the forward problem and the inverse problem. In the former one, observed frequencies are compared with theoretical ones calculated with solar models of varying parameters, and parameters giving the best fit are then determined. Many works so far made in this approach show that the standard solar model yields a fairly good fit with observations, but there remains a small discrepancy between observed frequencies and theoretical ones. On the other hand, in the inverse problem, functional forms of certain physical quantities such as the sound velocity distribution $c=c(r)$ are determined directly by solving integral equations for eigenfrequencies. This approach would be, in principle, more useful as a seismological treatment and it may clarify the cause of the discrepancy between the observed frequencies and the theoretical ones of the standard solar model. Methods and techniques developed in the inverse problem of the terrestrial seismology (e.g., Parker 1977) are also useful in the solar case, and, in fact, some works have been made along this line (Ibrahim Denis and Denis 1984). However, unfortunately, those methods do not always give unique and convergent solutions. In order to solve these problems, Gough (1984) presented another method based on an asymptotic expression of eigenfrequencies. In his method, the integral

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equation is analytically inverted and then the solution is uniquely determined. Christensen-Dalsgaard et al. (1985) applied this method to obtain the sound velocity distribution $c(r)$ in the sun. Their numerical simulation using solar models reproduced quite well the sound speed in the outer parts of the sun for $r/R \gtrsim 0.5$. However, for the deep inside of $r/R \lesssim 0.5$, their solution of the integral equation significantly deviates from the original values. Thus, the more effective and mathematically accurate treatments of the inverse problem are desirable. In this paper, we present a new method to attack the inverse problem, which is also based on asymptotic expressions of eigenfrequencies, and compare it with Gough's method to clarify the cause of significant deviation of Christensen-Dalsgaard et al.'s solution.

2. PRINCIPLE

The wave equation for radial oscillations of stars is a Sturm-Liouville type differential equation, and that for nonradial oscillations also tends to be a Sturm-Liouville type in some limiting cases if the perturbation of gravitational potential is neglected. These equations are further reduced to a form similar to the Schrödinger equation in quantum mechanics, which is, in the case of p-mode oscillations, written as

$$d^2v/dr^2 + c^{-2}(r)[\omega^2 - \Phi_\ell(r)]\, v = 0 \,, \tag{1}$$

where v denotes an eigenfunction, ω is the eigenfrequency. Here, $\Phi_\ell(r)$ plays the role of potential and we call it 'acoustic potential'. It consists of the Lamb frequency, $\ell(\ell+1)c^2/r^2$, which depends on the degree of the mode ℓ, and the ℓ-independent term $\Psi(r)$, which is written in terms of the density scale height and so on:

$$\Phi_\ell(r) = \ell(\ell+1)\, c^2 \,/\, r^2 + \Psi(r) \,. \tag{2}$$

Based on the WKB asymptotic method, the wavenumber in the radial direction, k_r, is given by

$$k_r \simeq [\omega^2 - \Phi_\ell(r)]^{1/2} \,/\, c \,, \tag{3}$$

and the quantization rule leads to

$$(n + \varepsilon)\,\pi = \int_{r_1}^{r_2} [\omega^2 - \Phi_\ell(r)]^{1/2}\, c^{-1}\, dr \,. \tag{4}$$

Here, n is the radial order of the mode and r_1 and r_2 are the turning points at which

$$\Phi_\ell(r_i) = \omega^2 \qquad (i = 1,\, 2) \,. \tag{5}$$

Strictly speaking, the quantization rule, equation (4), gives only a relation between discrete eigenvalues ω and the corresponding integers n. But hereafter we extend this relation to non-integers n by interpolation and treat equation (4) as if it were a continuous function of ω

giving continuous numbers n. Since, for a fixed ℓ, the left hand side of equation (4) is a function of ω, equation (4) is regarded as an integral equation to give $c^{-1}\, dr/d\Phi_\ell$, that is,

$$f(\xi) = \int_{\xi_{min}}^{\xi} (\xi - \eta)^{1/2}\, \phi(\eta)\, d\eta\ . \tag{6}$$

Here, new variables are introduced as $\xi \equiv \omega^2$ and $\eta \equiv \Phi_\ell(r)$, and $\phi(\eta) \equiv c^{-1} \times dr/d\eta$ is an unknown function to be solved and $f(\xi) \equiv (n+\epsilon)\pi$ is a known function of ξ. Differentiation with respect to ξ reduces equation (6) to an Abel type integral equation. Then, the solution of equation (6) is given by

$$\int_{r_1}^{r_2} c^{-1}\, dr = 2/\pi \int_{\xi_{min}}^{\eta} df/d\xi (\eta - \xi)^{-1/2}\, d\xi \equiv s(\ell, \omega^2)\ . \tag{7}$$

The LHS of equation (7) gives the distance between the two turning points measured with the sound velocity, and we call it 'acoustic length'. Equation (7) then gives the acoustic potential of a given ℓ as a function of the acoustic length.

Once we obtain the acoustic potentials for various values of ℓ, we can separate the sound velocity term c^2/r^2 and the remaining term Ψ. In practice, the upper turning point of the acoustic cavity r_2 is almost independent of ℓ. On the other hand, the inner turning point r_1 is approximately given by

$$c^2(r_1)\ /\ r_1{}^2 = \omega^2\ /\ \ell(\ell + 1)\ , \tag{8}$$

for high order or high degree modes (i.e., $n \gg 1$ or $\ell \gg 1$). Then, by differentiating the acoustic length by ℓ, we obtain

$$d[c(r_1)/r_1]/d\ln r_1 = (2\ell+1)/[2\ell(\ell+1)]\cdot(\partial s/\partial \ell)^{-1}\ . \tag{9}$$

From equation (9), we obtain the sound speed structure in the solar interior. Once we know the sound speed structure c(r), we can get the other part of the potential Ψ. Numerical simulation based on this principle is now in progress.

3. COMPARISON WITH OTHER WORKS

Gough (1984) and Christensen-Dalsgaard et al. (1985) disregarded the term of $\Psi(r)$ as a first approximation, and, instead, dealt with the data of whole ℓ's together. Then they derived a relation

$$(n + \alpha)\ \pi/\ \omega = \int_{r_1}^{R} [r^2/c^2 - \ell(\ell + 1)\ /\ \omega^2]^{1/2}\, d\ln r\ , \tag{10}$$

which was obtained by dividing the quantization condition by ω, to interpret Duvall's (1982) relation which showed $(n+\alpha)/\omega$ is almost a function of the single variable $\omega/\sqrt{\ell(\ell+1)}$. They regarded equation (10) as an integral equation with an unknown r^2/c^2, and inverted this integral equation. However, disregard of $\Psi(r)$ makes the results inaccurate for low ℓ modes. Since it is such low ℓ modes that penetrate to the deep inside of the Sun and provide information from there, this approximation leads to a large discrepancy between the value of c obtained by their inverse procedure and the true value in $r/R<0.5$. On the other hand, since we take account of $\Psi(r)$ in the method presented here, we can accurately obtain the sound velocity structure in such deep interior. The present method can in principle deal with even the radial modes ($\ell=0$), whose frequencies have been accurately observed, and it can therefore improve the effectiveness of the inverse problem based on the asymptotic method to infer the internal structure of the Sun, in particular, the deep interior of the Sun.

4. APPLICATION TO g-MODE OSCILLATIONS

In the case of g-mode oscillations, the wave equation is reduced to

$$d^2v / dr^2 + \ell(\ell + 1) N^2(r) r^{-2} [\omega^{-2} - \Xi_\ell(r)] v = 0 , \qquad (11)$$

where $\Xi_\ell(r)$ is the 'gravity wave potential' which is given by

$$\Xi_\ell(r) = N^{-2} (r) + \Theta_\ell (r) , \qquad (12)$$

and N denotes the Brunt-Väisälä frequency. The potential consists of the Brunt-Väisälä frequency and the ℓ-dependent part of $\Theta_\ell(r)$. The quantization condition leads to

$$(n+\varepsilon)\pi/[\ell(\ell+1)]^{1/2} = \int_{r_1}^{r_2} [\omega^{-2} - \Xi_\ell(r)]^{1/2} N/r \, dr , \qquad (13)$$

which is, for a fixed ℓ, regarded as an integral equation of the same form as equation (6) with $\xi \equiv \omega^{-2}$, $\eta \equiv \Xi_\ell(r)$, $\phi(\eta) \equiv N/r \, dr/d\eta$, and $f(\xi) \equiv (n+\varepsilon)\pi/[\ell(\ell+1)]^{1/2}$. The solution is given by

$$\int_{r_1}^{r_2} N / r \, dr = 2 / \pi \int_{\eta}^{\xi_{max}} df / d\xi \, (\eta - \xi)^{-1/2} d\xi . \qquad (14)$$

The LHS is the propagating time of the gravity wave in its cavity, while the RHS gives the potential for the g-modes. That is, equation (14) determines the gravity wave potential as a function of the wave traveling time. This inverse method will be useful for g-modes of the Sun, the ZZ Ceti and other pulsating white dwarfs.

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