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ERDÖS-TURÁN MEAN CONVERGENCE THEOREM FOR LAGRANGE INTERPOLATION AT LOBATTO POINTS

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Let $\{Q_n\}$ denote the orthogonal polynomials associated with the weight function ρ on $[-1,1]$ and let $\{x_{ni}\}_{i=0}^{n+1}$ denote the zeros of $(1-x^2)Q_n(x)$. Consider the Lagrange polynomials which interpolate a given continuous function at these points. It is shown that, as $n \to \infty$, the Lagrange polynomial converges to the function in the w weighted mean square sense, where $w(x) = \rho(x)/(1-x^2)$, provided that w is integrable. An application to numerical product integration is noted.

1. Introduction.

In a comprehensive review of Lagrangian interpolation (Nevai [6]) several interesting unsolved problems were noted; amongst them the weighted mean p convergence behaviour of Lagrangian interpolation on a finite interval at the zeros of orthogonal polynomials supplemented by the

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end points. For definiteness we regard the interval as $[-1,1]$ and $\{Q_n\}$ as the orthogonal polynomials associated with a weight function ρ. For the particular case $\rho(x) = (1-x^2)^{\frac{1}{2}}$ the polynomials are the second kind Chebyshev polynomials U_n, and the points are the familiar Clenshaw-Curtis or "practical Chebyshev" points. For these points a range of mean convergence results is available from trigonometrical interpolation (Zygmund [13]) and these have been exploited in applications to numerical integration (Sloan and Smith [8], [9]). For the case of Jacobi-like polynomials in general, extensive results have now been obtained ([5], [7], [10], [11], [12]) .

In the present note we regard ρ as having no specific form other than that

$$w(x) = \rho(x)/(1-x^2)$$

be integrable. We then prove the simplest mean (square) convergence result or Erdös-Turán theorem [2] for the weight w. The development uses the alternative characterization of the interpolation points as the nodes of the (Lobatto) rule of highest precision, namely $(2n+1)$, of the form

$$\int_{-1}^{1} w(x)f(x)dx = \sum_{i=1}^{n} w_{ni}f(x_{ni}) + w_{n0}f(1) + w_{n,n+1}f(-1) + E_n(f) , \tag{1.1}$$

having the constraint that the end points be nodes (Davis and Rabinowitz [1]; Krylov [4]). The $\{x_{ni}\}_{i=1}^{n}$ in equation (1.1) are zeros of the polynomial Q_n. From the theory of Lobatto rules it is also easily shown that all of the weights in (1.1) are positive. It is convenient to write equation (1.1) as

$$\int_{-1}^{1} w(x)f(x)dx = \sum_{i=0}^{n+1} w_{ni}f(x_{ni}) + E_n(f) , \tag{1.2}$$

$$\text{with} \quad x_{n0} = 1 , \quad x_{n,n+1} = -1 ,$$

and to regard $\{x_{ni}\}_{i=0}^{n+1}$ as Lobatto points for the weight w.

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2. Mean square convergence of Lagrange Lobatto interpolation.

THEOREM 1. (Erdös-Turán theorem). *Let* $w(x) = \rho(x)/(1-x^2)$ *be a weight function on* $[-1,1]$. *Let* $\{x_{ni}\}_{i=1}^{n}$ *be the zeros of the orthogonal polynomial of degree* n *associated with the weight* ρ *and let* $x_{n0} = 1$, $x_{n,n+1} = -1$. *For any* $f \in C[-1,1]$ *let* $L_{n+1}(f,x)$ *be the Lagrange polynomial interpolating* f *at the points* $\{x_{ni}\}_{i=0}^{n+1}$ *then*

$$\int_{-1}^{1} w(x)(L_{n+1}(f,x) - f(x))^2 dx \to 0$$

as $n \to \infty$.

Proof. The proof uses the Lobatto rule (1.1) or (1.2) to show that the norm of the interpolation operator from C to the w weighted L_2 space as defined by

$$||L_{n+1}|| = \sup_{f \in C[-1,1]} \frac{\left\{\int_{-1}^{1} w(x)(L_{n+1}(f,x))^2 dx\right\}^{\frac{1}{2}}}{||f||_\infty}$$

is bounded uniformly in n. However, the Lobatto rule does not have sufficient precision to integrate $(L_{n+1}(f,x))^2$ exactly and this requires modification of the elementary method often used to prove the usual Erdös-Turán theorem with Gaussian points (Johnson and Riess [3] Theorem 6.8). The method extends to the Radau case where one end point is included, but not to the Lobatto case where both end points are included.

Instead we first use a formula usually attributed to Aitken to write

$$L_{n+1}(f,x) = \left(\frac{1+x}{2}\right) L_n^{(1)}(f,x) + \left(\frac{1-x}{2}\right) L_n^{(-1)}(f,x) ,$$

where $L_n^{(1)}(f,x)$ is the unique Lagrange polynomial of degree n or less that interpolates f at $\{x_{ni}\}_{i=1}^{n}$ together with $x_{n,0} = 1$, that is at $\{x_{ni}\}_{i=0}^{n}$, and similarly $L_n^{(-1)}(f,x)$ interpolates f at $\{x_{ni}\}_{i=1}^{n}$ together with $x_{n,n+1} = -1$, that is at $\{x_{ni}\}_{i=1}^{n+1}$.

Then

$$\left\{\int_{-1}^{1} w(x)(L_{n+1}(f,x))^2\, dx\right\}^{\frac{1}{2}}$$

$$= \left\{\int_{-1}^{1} w(x)\left[\left(\tfrac{1+x}{2}\right)L_n^{(1)}(f,x) + \left(\tfrac{1-x}{2}\right)L_n^{(-1)}(f,x)\right]^2 dx\right\}^{\frac{1}{2}} ,$$

$$\le \left\{\int_{-1}^{1} w(x)\left(\tfrac{1+x}{2}\right)^2(L_n^{(1)}(f,x))^2 dx\right\}^{\frac{1}{2}} + \left\{\int_{-1}^{1} w(x)\left(\tfrac{1-x}{2}\right)^2(L_n^{(-1)}(f,x))^2 dx\right\}^{\frac{1}{2}} ,$$

$$\le \left\{\int_{-1}^{1} w(x)\left(\tfrac{1+x}{2}\right)(L_n^{(1)}(f,x))^2 dx\right\}^{\frac{1}{2}} + \left\{\int_{-1}^{1} w(x)\left(\tfrac{1-x}{2}\right)(L_n^{(-1)}(f,x))^2 dx\right\}^{\frac{1}{2}} ,$$

$$= \left\{\sum_{i=0}^{n} w_{ni}\left(\frac{1+x_{ni}}{2}\right)(L_n^{(1)}(f,x_{ni}))^2\right\}^{\frac{1}{2}} + \left\{\sum_{i=1}^{n+1} w_{ni}\left(\frac{1-x_{ni}}{2}\right)(L_n^{(-1)}(f,x_{ni}))^2\right\}^{\frac{1}{2}} ,$$

using the Lobatto rule (1.2) which has precision $(2n+1)$,

$$= \left\{\sum_{i=0}^{n} w_{ni}\left(\frac{1+x_{ni}}{2}\right)(f(x_{ni}))^2\right\}^{\frac{1}{2}} + \left\{\sum_{i=1}^{n+1} w_{ni}\left(\frac{1-x_{ni}}{2}\right)(f(x_{ni}))^2\right\}^{\frac{1}{2}} ,$$

$$\le ||f||_\infty\left\{\left[\sum_{i=0}^{n+1} w_{ni}\left(\frac{1+x_{ni}}{2}\right)\right]^{\frac{1}{2}} + \left[\sum_{i=0}^{n+1} w_{ni}\left(\frac{1-x_{ni}}{2}\right)\right]^{\frac{1}{2}}\right\}$$

since all $w_{ni} > 0$,

$$= ||f||_\infty \left\{\left(\int_{-1}^{1} w(x)\left(\frac{1+x}{2}\right)dx\right)^{\frac{1}{2}} + \left(\int_{-1}^{1} w(x)\left(\frac{1-x}{2}\right)dx\right)^{\frac{1}{2}}\right. ,$$

using (1.2) again ,

$$= ||f||_\infty M ,$$

where M is a constant independent of n and f .

Therefore $||L_{n+1}|| \le M = \left(\int_{-1}^{1} w(x)\left(\frac{1+x}{2}\right)dx\right)^{\frac{1}{2}} + \left(\int_{-1}^{1} w(x)\left(\frac{1-x}{2}\right)dx\right)^{\frac{1}{2}}$

and since polynomials are dense in $C[-1,1]$ and are replicated by sufficiently high degree Lagrange interpolation we conclude, in the usual way that,

$$\int_{-1}^{1} w(x)(L_{n+1}(f,x) - f(x))^2 dx \to 0$$

as $n \to \infty$ for every $f \in C[-1,1]$. □

The application of Theorem 1 to Jacobi weights gives the following corollary

COROLLARY 1. *If we interpolate* $f \in C[-1,1]$ *at the zeros of the Jacobi polynomial* $P_n^{(\alpha,\beta)}(x)$ $(\alpha,\beta > 0)$ *, together with the end points* $x = -1$ *and* $x = 1$ *, the resulting Lagrange interpolation polynomial* $L_{n+1}(f,x)$ *satisfies*

$$\int_{-1}^{1} (1-x)^{\alpha-1}(1+x)^{\beta-1}(L_{n+1}(f,x) - f(x))^2 dx \to 0$$

as $n \to \infty$.

The special case of $\alpha = \beta = \frac{1}{2}$ in Corollary 1 above corresponds to the use of the Clenshaw-Curtis or "practical Chebyshev" points and

$$\int_{-1}^{1} (1-x^2)^{-\frac{1}{2}}(L_{n+1}(f,x) - f(x))^2 dx \to 0$$

as $n \to \infty$. Stronger results are obtainable from the specifically Jacobi results [5], [7].

3. An application to numerical product integration.

Using the results of Sloan and Smith [9] the following convergence proposition for product integration is easily proved.

COROLLARY 2. Suppose an interpolatory product integration rule of the form

$$\int_{-1}^{1} k(x)f(x)dx \approx \sum_{i=0}^{n+1} \omega_{ni} f(x_{ni})$$

is constructed using the nodes $\{x_{ni}\}_{i=0}^{n+1}$ of Theorem 1, then the rule converges to the true integral as $n \to \infty$ for any $f \in C[-1,1]$ provided only that

$$\int_{-1}^{1} \frac{(k(x))^2}{w(x)}\, dx < \infty .$$

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