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WEIGHTED COMPOSITION OPERATORS ON $H^\infty \cap \mathcal{B}_o$

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Abstract. We will characterize the boundedness and compactness of weighted composition operators on the closed subalgebra $H^\infty \cap \mathcal{B}_o$ between the disk algebra and the space of bounded analytic functions on the open unit disk.

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1. Introduction. Throughout this paper let H^∞ denote the Banach algebra of all bounded analytic functions f on the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, with the supremum norm

$$\|f\|_\infty = \sup\{|f(z)| : z \in \mathbb{D}\}.$$

Let the little Bloch space $\mathcal{B}_o$ be the class of analytic functions f on $\mathbb{D}$ such that

$$\lim_{|z|\to 1}(1 - |z|^2)f'(z) = 0.$$

There exist unbounded analytic functions in $\mathcal{B}_o$. Also not every functions in H^∞ belong to $\mathcal{B}_o$. An example is $f(z) = \exp((z + 1)/(z - 1))$. Then $H^\infty \cap \mathcal{B}_o$ is a closed subalgebra of H^∞ with the supremum norm, properly containing the disk algebra defined as a Banach algebra of all complex-valued continuous functions on the closure $\overline{\mathbb{D}}$ of $\mathbb{D}$, which are analytic on $\mathbb{D}$. Such spaces between the disk algebra and H^∞ were studied as new classes in 1970s–1980s. See [**4**, **8**] and also refer to [**11**] for Bloch spaces. The following alternative description can be found in [**4**, p. 432] that $H^\infty \cap \mathcal{B}_o = \mathrm{COP}$ consists of all those $f \in H^\infty$ such that f is constant on all the non-trivial Gleason parts of the maximal ideal space of H^∞ other than the image of $\mathbb{D}$.

Let u be a fixed analytic function on $\mathbb{D}$ and φ an analytic self-map of $\mathbb{D}$. Then we define the multiplication operator M_u and the composition operator C_φ by $M_u f = u \cdot f$ and $C_\varphi f = f \circ \varphi$, respectively, where f is analytic on $\mathbb{D}$. Moreover, we consider the product $M_u C_\varphi$ of these operators, which is called the weighted composition operator. It is easily seen that $M_u C_\varphi$ is bounded on H^∞ if and only if $u \in H^\infty$. Also it is known that C_φ is bounded on $\mathcal{B}_o$ if and only if $\varphi \in \mathcal{B}_o$ and the boundedness of $M_u C_\varphi$ on $\mathcal{B}_o$ was characterized in [**7**]. There has been much work on composition and weighted composition operators on various spaces of analytic functions. See [**3**, **9**] for an overview of these results.

Recently, it is given the characterization of the sum of weighted composition operators mapping $H^\infty \cap \mathcal{B}_o$ to the disk algebra in [**5**]. Properties of each weighted composition operator acting from $H^\infty \cap \mathcal{B}_o$ to $H^\infty \cap \mathcal{B}_o$ are left behind. We here carry on studying this problem. This research on a new class $H^\infty \cap \mathcal{B}_o$ will be a new step in the study of composition operators.

In the next section, we first consider the boundedness and compactness of weighted composition operators acting on $H^\infty \cap \mathcal{B}_o$. In this case, the so-called "weak convergence theorem" does not necessarily hold, but we could note that this theorem would be true after we could characterize the compactness of weighted composition operators. In Section 3, we will consider the domain of weighted composition operators as H^∞ bigger than $H^\infty \cap \mathcal{B}_o$. We present some examples concerning with our results. As a corollary, we have that the boundedness of $C_\varphi : H^\infty \to H^\infty \cap \mathcal{B}_o$ is equivalent to the compactness of $C_\varphi : \mathcal{B}_o \to \mathcal{B}_o$. These results would be curious and interesting.

2. Boundedness and compactness on $H^\infty \cap \mathcal{B}_o$. At first we have the result on the boundedness of M_uC_φ on $H^\infty \cap \mathcal{B}_o$.

PROPOSITION 2.1. *For a fixed analytic function u on $\mathbb{D}$ and an analytic self-map φ of $\mathbb{D}$, M_uC_φ is bounded on $H^\infty \cap \mathcal{B}_o$ if and only if u and $u\varphi$ are in $H^\infty \cap \mathcal{B}_o$.*

Proof. The "only if" part is trivial. Suppose that u and $u\varphi$ are in $H^\infty \cap \mathcal{B}_o$. Then we notice that $(1-|z|^2)|u(z)\varphi'(z)|$ tends to 0 as $|z| \to 1$. It is sufficient to show that $M_uC_\varphi f \in \mathcal{B}_o$ for $f \in H^\infty \cap \mathcal{B}_o$. Then

$$\begin{aligned}
&(1-|z|^2)|(M_uC_\varphi f)'(z)| \\
&\leq (1-|z|^2)|u'(z)f(\varphi(z))| \\
&\quad + \frac{1-|z|^2}{1-|\varphi(z)|^2}|u(z)\varphi'(z)|(1-|\varphi(z)|^2)|f'(\varphi(z))|.
\end{aligned}$$

The first term above is bounded by $(1-|z|^2)|u'(z)|\|f\|_\infty$. The second one is bounded by $(1-|\varphi(z)|^2)|f'(\varphi(z))|\|u\|_\infty$ if $|\varphi(z)| \to 1$ and bounded by $(1-|z|^2)|u(z)\varphi'(z)|\|f\|_\infty$ if otherwise. Thus $M_uC_\varphi f \in H^\infty \cap \mathcal{B}_o$. □

In the proof of characterization of compact composition operators we usually need the so-called "weak convergence theorem", but we cannot show now. We here may prove only the following one-way part (see Proposition 3.11 in [**3**]).

PROPOSITION 2.2. *Let u be a fixed analytic function on $\mathbb{D}$ and φ an analytic self-map of $\mathbb{D}$. Suppose that M_uC_φ is compact on $H^\infty \cap \mathcal{B}_o$. Then $\|M_uC_\varphi f_n\|_\infty \to 0$ as $n \to \infty$ for every sequence $\{f_n\}_n$ in $H^\infty \cap \mathcal{B}_o$ with $\|f_n\|_\infty \leq 1$ satisfying $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$.*

THEOREM 2.3. *Let u be a fixed analytic function on $\mathbb{D}$ and φ an analytic self-map of $\mathbb{D}$ with $\|\varphi\|_\infty = 1$. Suppose that M_uC_φ is bounded on $H^\infty \cap \mathcal{B}_o$. Then the following are equivalent.*

(i) *M_uC_φ is compact on $H^\infty \cap \mathcal{B}_o$.*
(ii) $\lim_{|\varphi(z)| \to 1} |u(z)| = 0.$
(iii) *M_uC_φ is compact on H^∞.*

Proof. The equivalence of (ii) and (iii) is known (for instance, see [**2**]). Suppose that M_uC_φ is compact on $H^\infty \cap \mathcal{B}_o$. Let $f_n(z) = z^n$. Then $f_n \in H^\infty \cap \mathcal{B}_o$, $\|f_n\|_\infty \leq 1$ and $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. By Proposition 2.2, $\|M_uC_\varphi f_n\|_\infty = \|u\varphi^n\|_\infty \to 0$

as $n \to \infty$. So if $|u(z)|$ does not tend to 0 as $|\varphi(z)| \to 1$, a contradiction yields. Thus, we have condition (ii).

Conversely we suppose that condition (ii) holds. Let $\{f_n\}_n$ be a sequence in $H^\infty \cap \mathcal{B}_o$ with $\|f_n\|_\infty \leq 1$. As $M_u C_\varphi$ is compact on H^∞, there is a subsequence $\{f_{n_k}\}_{n_k}$ of $\{f_n\}_n$ in $H^\infty \cap \mathcal{B}_o$ and $f \in H^\infty$ such that

$$\|M_u C_\varphi f_{n_k} - f\|_\infty \to 0 \quad \text{as} \quad k \to \infty.$$

Here, as $H^\infty \cap \mathcal{B}_o$ is uniformly closed in H^∞, we obtain $f \in H^\infty \cap \mathcal{B}_o$ and so $M_u C_\varphi$ is compact on $H^\infty \cap \mathcal{B}_o$. □

Thus we can obtain a "weak convergence theorem".

Corollary 2.4. *Let u be a fixed analytic function on $\mathbb{D}$ and φ an analytic self-map of $\mathbb{D}$. Suppose that $M_u C_\varphi$ is bounded on $H^\infty \cap \mathcal{B}_o$. The following are equivalent.*

(i) *$M_u C_\varphi$ is compact on $H^\infty \cap \mathcal{B}_o$.*

(ii) *If every sequence $\{f_n\}_n$ is bounded in $H^\infty \cap \mathcal{B}_o$ and $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$, then $\|M_u C_\varphi f_n\|_\infty \to 0$ as $n \to \infty$.*

Also we have a result on the unweighted case.

Corollary 2.5. *Let φ be an analytic self-map of $\mathbb{D}$. Suppose that C_φ is bounded on $H^\infty \cap \mathcal{B}_o$. The following are equivalent.*

(i) *C_φ is compact on $H^\infty \cap \mathcal{B}_o$.*

(ii) $\|\varphi\|_\infty < 1$.

(iii) *C_φ is compact on H^∞.*

3. $M_u C_\varphi : H^\infty \to H^\infty \cap \mathcal{B}_o$. In this section, we consider the domain of weighted composition operator as a lager space H^∞ than $H^\infty \cap \mathcal{B}_o$. That is, we will study properties of weighted composition operators mapping H^∞ to $H^\infty \cap \mathcal{B}_o$.

At first we consider the boundedness.

Theorem 3.1. *Let u be a fixed analytic function on $\mathbb{D}$ and φ an analytic self-map of $\mathbb{D}$. Then the following are equivalent.*

(i) *$M_u C_\varphi : H^\infty \to H^\infty \cap \mathcal{B}_o$ is bounded.*

(ii) *$u \in H^\infty \cap \mathcal{B}_o$ and*
$$\lim_{|z|\to 1} \frac{1-|z|^2}{1-|\varphi(z)|^2}|u(z)\varphi'(z)| = 0.$$

Proof. Suppose that $M_u C_\varphi : H^\infty \to H^\infty \cap \mathcal{B}_o$ is bounded. We note that $M_u C_\varphi 1 = u$, $M_u C_\varphi z = u\varphi \in H^\infty \cap \mathcal{B}_o$.

Then

$$\begin{aligned}&(1-|z|^2)|(M_u C_\varphi z)'(z)| \\ &\geq (1-|z|^2)|u(z)\varphi'(z)| - (1-|z|^2)|u'(z)\varphi(z)|.\end{aligned}$$

So $(1-|z|^2)|u(z)\varphi'(z)| \to 0$ as $|z| \to 1$.

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In the case that $|z_n| \to 1$ and $|\varphi(z_n)| \not\to 1$, it is trivial that

$$\lim_{|z_n|\to 1} \frac{1-|z_n|^2}{1-|\varphi(z_n)|^2}|u(z_n)\varphi'(z_n)| = 0.$$

In the case that $|z_n| \to 1$ and $|\varphi(z_n)| \to 1$, we could take a subsequence $\{\varphi(z_{n_k})\}$ of $\{\varphi(z_n)\}$. Moreover, we can choose a subsequence satisfying the thin condition, that is,

$$\prod_{j, j\neq k}^{\infty} \left| \frac{\varphi(z_{n_k}) - \varphi(z_{n_j})}{1-\overline{\varphi(z_{n_k})}\varphi(z_{n_j})} \right| \geq \delta$$

for some $\delta > 0$. Let f be a thin Blaschke product with zeros $\{\varphi(z_{n_k})\}$.

Then

$$\begin{aligned}
&(1-|z|^2)|(M_uC_\varphi f)'(z)| \\
&\geq (1-|z|^2)|u(z)\varphi'(z)f'(\varphi(z))| - (1-|z|^2)|u'(z)|\|f\|_\infty.
\end{aligned}$$

Here

$$\begin{aligned}
&(1-|z_{n_k}|^2)|u(z_{n_k})\varphi'(z_{n_k})f'(\varphi(z_{n_k}))| \\
&= \frac{1-|z_{n_k}|^2}{1-|\varphi(z_{n_k})|^2}|u(z_{n_k})\varphi'(z_{n_k})| \prod_{j, j\neq k}^{\infty} \left| \frac{\varphi(z_{n_k}) - \varphi(z_{n_j})}{1-\overline{\varphi(z_{n_k})}\varphi(z_{n_j})} \right| \\
&\geq \delta \frac{1-|z_{n_k}|^2}{1-|\varphi(z_{n_k})|^2}|u(z_{n_k})\varphi'(z_{n_k})|.
\end{aligned}$$

Since $M_uC_\varphi f$ and $u \in H^\infty \cap \mathcal{B}_o$, it holds that if $|z_{n_k}| \to 1$,

$$(1-|z_{n_k}|^2)|u(z_{n_k})\varphi'(z_{n_k})f'(\varphi(z_{n_k}))| \to 0.$$

By the thin condition, we obtain

$$\lim_{|z_{n_k}|\to 1} \frac{1-|z_{n_k}|^2}{1-|\varphi(z_{n_k})|^2}|u(z_{n_k})\varphi'(z_{n_k})| = 0.$$

Conversely we may directly get that condition (ii) implies the boundedness. □

We here present examples related to the condition (ii) in Theorem 3.1.

EXAMPLE 3.2. Suppose that φ has a finite angular derivative at ζ in the unit circle, where ζ is only one point attaining $|\varphi(\zeta)| = 1$. Then

$$\lim_{z\to\zeta} \frac{1-|z|^2}{1-|\varphi(z)|^2}|\varphi'(z)| \neq 0.$$

And so letting $u \in H^\infty \cap \mathcal{B}_o$ with $u(\zeta) = 0$, these u and φ satisfy the condition (ii). Explicitly, let $\varphi(z) = (1+z)/2$ and $u(z) = 1 - z$.

Or, let $\sigma(z) = (1+z)/(1-z)$ and

$$\varphi_s(z) = \frac{\sigma(z)^s - 1}{\sigma(z)^s + 1} \quad \text{for} \quad s \in (0, 1).$$

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This φ_s is called the lens map and

$$\lim_{z\to\pm 1} \frac{1-|z|^2}{1-|\varphi(z)|^2}|\varphi'(z)| = s.$$

Then we put $u(z) = 1 - z^2$.

We have the following as special cases of Theorem 3.1.

Corollary 3.3. *Let u be a fixed analytic function on $\mathbb{D}$ and φ an analytic self-map of $\mathbb{D}$. Then*

(i) *$M_u : H^\infty \to H^\infty \cap \mathcal{B}_o$ is bounded if and only if $u \equiv 0$.*

(ii) *The following are equivalent.*

(a) *$C_\varphi : H^\infty \to H^\infty \cap \mathcal{B}_o$ is bounded.*

(b) $\displaystyle\lim_{|z|\to 1} \frac{1-|z|^2}{1-|\varphi(z)|^2}|\varphi'(z)| = 0.$

(c) *C_φ is bounded on $\mathcal{B}_o$.*

The equivalence of (b) and (c) in (ii) is due to [**6**]. There exist inner functions satisfying condition in (ii) above. For example, see [**1**, **10**] and [**6**, Theorem 5].

Next to characterize compact composition operators we have the following "weak convergence theorem".

Proposition 3.4. *Let u be a fixed analytic function on $\mathbb{D}$ and φ an analytic self-map of $\mathbb{D}$. Suppose that $M_uC_\varphi : H^\infty \to H^\infty \cap \mathcal{B}_o$ is bounded. Then the following are equivalent.*

(i) *$M_uC_\varphi : H^\infty \to H^\infty \cap \mathcal{B}_o$ is compact.*

(ii) *If every sequence $\{f_n\}_n$ in H^∞ with $\|f_n\|_\infty \le 1$ satisfies $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$, then $\|M_uC_\varphi f_n\|_\infty \to 0$ as $n \to \infty$.*

Theorem 3.5. *Let u be a fixed analytic function on $\mathbb{D}$ and φ an analytic self-map of $\mathbb{D}$. Suppose that $M_uC_\varphi : H^\infty \to H^\infty \cap \mathcal{B}_o$ is bounded. Then the following are equivalent.*

(i) *$M_uC_\varphi : H^\infty \to H^\infty \cap \mathcal{B}_o$ is compact.*

(ii) *$M_uC_\varphi : H^\infty \to H^\infty$ is compact.*

(iii) $\|\varphi\|_\infty < 1$ *or* $\displaystyle\lim_{|\varphi(z)|\to 1} |u(z)| = 0.$

Proof. The equivalence of (i) and (ii) is trivial and the equivalence between (ii) and (iii) is well known. □

Also we have results on the unweighted case.

Corollary 3.6. *Let u be a fixed analytic function on $\mathbb{D}$ and φ an analytic self-map of $\mathbb{D}$. Then*

(i) *$M_u : H^\infty \to H^\infty \cap \mathcal{B}_o$ is compact if and only if $u \equiv 0$.*

(ii) *The following are equivalent.*

(a) *$C_\varphi : H^\infty \to H^\infty \cap \mathcal{B}_o$ is compact.*

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(b) $\|\varphi\|_\infty < 1$.

(c) *C_φ is compact on H^∞.*

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