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On the Action of the Dual Group on the Cohomology of Perverse Sheaves on the Affine Grassmannian

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Abstract. It was proved by Ginzburg, Mirkovic and Vilonen that the $G(O)$-equivariant perverse sheaves on the affine Grassmannian of a connected reductive group G form a tensor category equivalent to the tensor category of finite dimensional representations of the dual group $G^{\vee}$. In this paper we construct explicitly the action of $G^{\vee}$ on the global cohomology of a perverse sheaf.

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0. Introduction

It was proved by Ginzburg, Mirkovic and Vilonen that the $G(O)$-equivariant perverse sheaves on the affine Grassmannian of a connected reductive group G form a tensor category equivalent to the tensor category of finite-dimensional representations of the Langlands dual group $G^{\vee}$ (see [G] and [MV]). The proof uses the Tannakian formalism. The purpose of this paper is to explicitly construct the action of $G^{\vee}$ on the global cohomology of a perverse sheaf. More precisely, we define the action of the Chevalley generators of the Lie algebra of the group $G^{\vee}$ and prove that they satisfy the Serre relations. In order to do so, we first prove that the Chevalley generators are primitive with respect to the coproduct in Lemma 2.3. Then we check the relations in the minuscule and quasi-minuscule cases. The formula for the action of the generators is not new in the sense that it is forced by the compatibility of the above-mentioned equivalence of categories with restrictions to Levi subgroups.

It would be interesting to find a q-analogue of this construction. It would give the global counterpart to [BG].

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1. Notations and Reminder on Affine Grassmanians

1.1. Let G be a connected reductive complex algebraic group. Let B, T, be a Borel and a Cartan subgroup of G. Let $U \subset B$ be the unipotent radical of B. Let B^- a Borel subgroup such that $B \cap B^- = T$. Set $X_T = \mathrm{Hom}(T, \mathbb{G}_m)$ and $X_T^\vee = \mathrm{Hom}(\mathbb{G}_m, T)$ be the weight an the coweight lattice of G. For simplicity, we write $X = X_T$ and $X^\vee = X_T^\vee$. Let $(\, , \,): X \times X^\vee \to \mathbb{Z}$ be the natural pairing. Let R be the set of roots, $R^\vee$ the set of coroots. Let $R_\pm \subset R$, $R_\pm^\vee \subset R^\vee$ be the subsets of positive and negative roots and coroots. Let $X_+ \subset X$, $X_+^\vee \subset X^\vee$ be the subsets of dominant weights and coweights. Let $\rho_G \in X$ be half the sum of all positive roots. If there is no ambiguity we simply write ρ instead of ρ_G. Let $G^\vee$ and $Z(G)$ be the dual group and the center of G. Let $\alpha_i, \alpha_i^\vee$, $i \in I$, be the simple roots and the simple coroots, and let $\omega_i, \omega_i^\vee$ be the fundamental weights and coweights. For any root $\alpha \in R$, let $U_\alpha \subset G$ be the corresponding root subgroup. If $\alpha = \alpha_i$, $i \in I$, we simply set $U_i = U_{\alpha_i}$ and $U_i^- = U_{-\alpha_i}$. Let W be the Weyl group of G. For any $i \in I$ let s_i be the simple reflexion corresponding to the simple root α_i.

1.2. Let $K = \mathbb{C}((t))$ be the field of Laurent formal series, and let $O = \mathbb{C}[[t]]$ be the subring of integers. Recall that $G(O)$ is a group scheme and that $G(K)$ is a group ind-scheme. The quotient set $\mathrm{Gr}^G = G(K)/G(O)$ is endowed with the structure of an ind-scheme. We may write Gr instead of Gr^G, hoping that it does not cause confusion. For any coweight $\lambda^\vee \in X^\vee$, let $t^{\lambda^\vee} \in T(K)$ be the image of t from the group homomorphism $\lambda^\vee: \mathbb{G}_m(K) \to T(K)$. If $\lambda^\vee$ is dominant, set $e_{\lambda^\vee} = t^{\lambda^\vee} G(O)/G(O) \in \mathrm{Gr}$. The $G(O)$-orbit $\mathrm{Gr}_{\lambda^\vee} = G(O) \cdot e_{\lambda^\vee}$ is connected and simply connected. Let $\overline{\mathrm{Gr}}_{\lambda^\vee}$ be its Zariski closure. Let $\mathcal{P}_G$ be the category of $G(O)$-equivariant perverse sheaves on Gr. For any $\lambda^\vee$, let $\mathcal{IC}_{\lambda^\vee}$ be the intersection cohomology complex on $\mathrm{Gr}_{\lambda^\vee}$ with coefficients in $\mathbb{C}$. Consider the fiber product $G(K) \times_{G(O)} \mathrm{Gr}$. It is the quotient of $G(K) \times \mathrm{Gr}$ by $G(O)$, where $u \in G(O)$ acts on $G(K) \times \mathrm{Gr}$ by $(g, x) \mapsto (gu^{-1}, ux)$. The map

$$\tilde{p}: G(K) \times_{G(O)} \mathrm{Gr} \to \mathrm{Gr}, \quad (g, x) \mapsto ge_0$$

is the locally trivial fibration with fiber Gr associated to the $G(O)$-bundle $p: G(K) \to \mathrm{Gr}$. Thus $G(K) \times_{G(O)} \mathrm{Gr}$ is an ind-scheme: it is the inductive limit of the subschemes $p^{-1}(\overline{\mathrm{Gr}}_{\lambda_1^\vee}) \times_{G(O)} \overline{\mathrm{Gr}}_{\lambda_2^\vee}$. Consider also the map

$$m: G(K) \times_{G(O)} \mathrm{Gr} \to \mathrm{Gr}, \quad (g, x) \mapsto gx.$$

For any $\lambda_1^\vee, \lambda_2^\vee \in X_+^\vee$ let $\mathcal{IC}_{\lambda_1^\vee} \star \mathcal{IC}_{\lambda_2^\vee}$ be the direct image by m of the intersection cohomology complex of the subvariety

$$p^{-1}(\overline{\mathrm{Gr}}_{\lambda_1^\vee}) \times_{G(O)} \overline{\mathrm{Gr}}_{\lambda_2^\vee} \subset G(K) \times_{G(O)} \mathrm{Gr}.$$

The complex $\mathcal{IC}_{\lambda_1^\vee} \star \mathcal{IC}_{\lambda_2^\vee}$ is perverse (see [MV], and [NP, Corollaire 9.7] for more details). It is known that the cohomology sheaves of the complex $\mathcal{IC}_{\lambda^\vee}$ are pure

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through an argument similar to [KT] (see also KL]). It is also known that any object in $\mathcal{P}_G$ is a direct sum of complexes $\mathcal{IC}_{\lambda^\vee}$ (see [BD, Proposition 5.3.3 (i)] for a proof). Thus we get a convolution product $\star : \mathcal{P}_G \times \mathcal{P}_G \to \mathcal{P}_G$. It is the convolution product defined by Mirkovic and Vilonen.

1.3. Let $P \subset G$ be a parabolic subgroup of G, $N \subset P$ be the unipotent radical, $M = P/N$ be the Levi factor. Let $M' = [M, M]$ be the semisimple part of M. Consider the diagram $\mathrm{Gr}^G \overset{\gamma}{\leftarrow} \mathrm{Gr}^P \overset{\pi}{\rightarrow} \mathrm{Gr}^M$,where the maps γ and π are induced by the embedding $P \subset G$ and the projection $P \to M$. The fibers of π are $N(K)$-orbits. Observe that Gr^M is not connected. The connected components of Gr^M are labelled by characters of the center of the dual group $M^\vee$. Let $\mathrm{Gr}^{M,\theta^\vee} \subset \mathrm{Gr}^M$ be the component associated to $\theta^\vee \in X_{Z(M^\vee)}$. By definition, $e_{\lambda^\vee} \in \mathrm{Gr}^{M,\theta^\vee}$ if and only if the restriction of $\lambda^\vee$ to $Z(M^\vee)$ coincides with $\theta^\vee$. The element $\rho - \rho_M$ belongs to $X^\vee_{Z(M^\vee)}$. Put

$$\mathrm{Gr}^{M,n} = \coprod_{2(\theta,\rho-\rho_M)=n} \mathrm{Gr}^{M,\theta^\vee}.$$

The following facts are proved in [BD, Section 5.3].

PROPOSITION. (a) *The functor* $\pi_! \gamma^*$ *gives a map* $\mathrm{res}^{GM} \colon \mathcal{P}_G \to \tilde{\mathcal{P}}_M = \bigoplus_n \mathcal{P}_{M,n}[-n]$, *where* $\mathcal{P}_{M,n}$ *is the subcategory of* $M(O)$*-equivariant perverse sheaves on* $\mathrm{Gr}^{M,n}$.

(b) *For any* $\mathcal{E}, \mathcal{F} \in \mathcal{P}_G$ *we have* $\mathrm{res}^{GM}(\mathcal{E} \star \mathcal{F}) = (\mathrm{res}^{GM}\mathcal{E}) \star (\mathrm{res}^{GM}\mathcal{F})$.

(c) *For any* $\mathcal{E} \in \mathcal{P}_G$ *we have* $H^*(\mathrm{Gr}, \mathcal{E}) = H^*(\mathrm{Gr}^M, \mathrm{res}^{GM}\mathcal{E})$.

(d) *If* $P_1 \subset P$ *is a parabolic subgroup and* M_1 *is its Levi factor then* res^{MM_1} *maps* $\tilde{\mathcal{P}}_M$ *to* $\tilde{\mathcal{P}}_{M_1}$, *and* $\mathrm{res}^{GM_1} = \mathrm{res}^{MM_1} \circ \mathrm{res}^{GM}$. □

1.4. Let $\tilde{\mathfrak{g}}$ be the affine Kac–Moody Lie algebra associated to G. Let $\tilde{\omega}_0$ be the fundamental weight of $\tilde{\mathfrak{g}}$ which is trivial on $\mathrm{Lie}(T)$. Let W_0 be the irreducible integrable highest-weight module of $\tilde{\mathfrak{g}}$ with higest weight $\tilde{\omega}_0$. Let π be the corresponding group homomorphism $G(K) \to \mathrm{PGL}(W_0)$ (see [Ku, Appendix C], for instance). The central extension $\tilde{G}(K)$ of $G(K)$ is the pull-back $\pi^* GL(W_0)$, where $GL(W_0)$ must be viewed as a $\mathbb{C}^\times$-principal bundle on $\mathrm{PGL}(W_0)$. The restriction of the central extension to $G(O)$, denoted by $\tilde{G}(O)$, splits, i.e. $\tilde{G}(O) = G(O) \times \mathbb{C}^\times$. Fix a highest-weight vector $w_0 \in W_0$. Let $\mathcal{L}_G$ be the pull-back of $O_{\mathbb{P}}(1)$ by the embedding of ind-schemes $\imath \colon \mathrm{Gr}^G \hookrightarrow \mathbb{P}(W_0)$ induced by the map

$$G(K) \to \mathbb{P}(W_0),\ g \mapsto [\mathbb{C} \cdot g w_0].$$

The sheaf $\mathcal{L}_G$ is obviously algebraic.

1.5. For any $i \in I$ let P_i be the corresponding subminimal parabolic subgroup of G. Let $N_i \subset P_i$ be the unipotent radical and put $M_i = P_i/N_i$. Hereafter we set

$$i\mathrm{res} = \mathrm{res}^{GM_i}, \quad \pi_i = \pi, \quad \gamma_i = \gamma, \quad Z_i = Z(M_i) \quad \text{and} \quad \mathcal{L}_i = \mathcal{L}_{M_i}.$$

The product by the first Chern class of $\mathcal{L}_i$ gives a map

$$l_i\colon H^*(\mathrm{Gr}^{M_i}, \mathcal{E}) \to H^{*+2}(\mathrm{Gr}^{M_i}, \mathcal{E}),$$

for any $\mathcal{E} \in \mathcal{P}_{M_i}$.

1.6. For any $\mu^\vee \in X^\vee$ set $S_{\mu^\vee} = U(K) \cdot e_{\mu^\vee}$. It was proved by Mirkovic and Vilonen that if $\mathcal{E} \in \mathcal{P}_G$, then

$$H^*(\mathrm{Gr}, \mathcal{E}) = \bigoplus_{\mu^\vee \in X^\vee} H_c^{2(\rho,\mu^\vee)}(S_{\mu^\vee}, \mathcal{E}), \tag{a}$$

(see [MV], and [NP] for more details). For any $i \in I$ and any $\mu^\vee \in X^\vee$ set also $S_{\mu^\vee}^{M_i} = U_i(K) \cdot e_{\mu^\vee} \subset \mathrm{Gr}^{M_i}$. The Grassmanian Gr^{M_i} may be viewed as the set of points of Gr which are fixed by the action of the group Z_i by left translations. This fixpoints subset is denoted by ${}^{Z_i}\mathrm{Gr}$. In particular, $S_{\mu^\vee}^{M_i}$ may be viewed as a subset of Gr.

2. Construction of the Operators $\mathbf{e}_i$, $\mathbf{f}_i$, $\mathbf{h}_i$

2.1. To avoid useless complications, hereafter we assume that G is semi-simple. The generalization to the reductive case is immediate. For any $i \in I$ and $\mathcal{E} \in \mathcal{P}_G$, let $\mathbf{e}_i$ be the composition of the chain of maps

$$H^*(\mathrm{Gr}, \mathcal{E}) = H^*(\mathrm{Gr}^{M_i}, i\mathrm{res}\,\mathcal{E}) \xrightarrow{l_i} H^{*+2}(\mathrm{Gr}^{M_i}, i\mathrm{res}\,\mathcal{E}) = H^{*+2}(\mathrm{Gr}, \mathcal{E}).$$

Moreover, set

$$\mathbf{h}_i = \bigoplus_{\lambda^\vee \in X^\vee} (\alpha_i, \lambda^\vee)\, id_{H_c^*(S_{\lambda^\vee}, \mathcal{E})}\colon H^*(\mathrm{Gr}, \mathcal{E}) \to H^*(\mathrm{Gr}, \mathcal{E}).$$

By the hard Lefschetz theorem there is a unique linear operator $\mathbf{f}_i\colon H^*(\mathrm{Gr}, \mathcal{E}) \to H^{*-2}(\mathrm{Gr}, \mathcal{E})$ such that $(\mathbf{e}_i, \mathbf{h}_i, \mathbf{f}_i)$ is a $\mathfrak{sl}(2)$-triple.

THEOREM. *For any $\mathcal{E} \in \mathcal{P}_G$, the operators $\mathbf{e}_i, \mathbf{f}_i, \mathbf{h}_i$, with $i \in I$, give an action of the dual group $G^\vee$ on the cohomology $H^*(\mathrm{Gr}, \mathcal{E})$.* □

2.2. The rest of the paper is devoted to the proof of the theorem.

LEMMA. *For all $\lambda^\vee \in X^\vee$ we have*

$$\mathbf{e}_i(H_c^*(S_{\lambda^\vee}, \mathcal{E})) \subset H_c^*(S_{\lambda^\vee+\alpha_i^\vee}, \mathcal{E}) \quad \textit{and} \quad \mathbf{f}_i(H_c^*(S_{\lambda^\vee}, \mathcal{E})) \subset H_c^*(S_{\lambda^\vee-\alpha_i^\vee}, \mathcal{E}).$$

Proof. It is sufficient to check the first claim. Since

$$S_{\lambda^\vee} = N_i(K) U_i(K) \cdot e_{\lambda^\vee} = \pi_i^{-1}(S_{\lambda^\vee}^{M_i}),$$

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we get, for any $\mathcal{E} \in \mathcal{P}_G$,

$$H_c^*(S_{\lambda^\vee}, \mathcal{E}) = H_c^*(S_{\lambda^\vee}^{M_i}, i\mathrm{res}\,\mathcal{E}). \tag{a}$$

Now, if $\mathcal{E} \in \mathcal{P}_{M_i}$ then

$$l_i(H_c^*(S_{\lambda^\vee}^{M_i}, \mathcal{E})) = l_i(H_c^{(\alpha_i, \lambda^\vee)}(S_{\lambda^\vee}^{M_i}, \mathcal{E})) \subset H_c^{2+(\alpha_i,\lambda^\vee)}(\mathrm{Gr}^{M_i}, \mathcal{E}).$$

Moreover, for all $\mu^\vee \in X^\vee \simeq X_{T^\vee}$ we have

$$S_{\mu^\vee} \cap \mathrm{Gr}^{M_i, \theta^\vee} \neq \emptyset \iff \mu^\vee|_{Z(M_i^\vee)} = \theta^\vee.$$

Thus, if $\mathcal{E} \in \mathcal{P}_{M_i}$ then

$$l_i(H_c^*(S_{\lambda^\vee}^{M_i}, \mathcal{E})) = \bigoplus_{\mu^\vee} H_c^{(\alpha_i, \mu^\vee)}(S_{\mu^\vee}^{M_i}, \mathcal{E}),$$

where the sum is over all $\mu^\vee \in X^\vee \simeq X_{T^\vee}$ such that

$$\mu^\vee|_{Z(M_i^\vee)} = \lambda^\vee|_{Z(M_i^\vee)} \quad \text{and} \quad (\alpha_i, \mu^\vee) = (\alpha_i, \lambda^\vee + \alpha_i^\vee).$$

The only possibility is $\mu^\vee = \lambda^\vee + \alpha_i^\vee$. □

The lemma implies that $[\mathbf{h}_i, \mathbf{e}_j] = (\alpha_i, \alpha_j^\vee)\,\mathbf{e}_j$ for all $i, j \in I$. Since $\mathbf{e}_i, \mathbf{f}_i$, are locally nilpotent and since $[\mathbf{e}_i, \mathbf{f}_i] = \mathbf{h}_i$ by construction, if $[\mathbf{e}_i, \mathbf{f}_j] = 0$ for any $i \neq j$, then the operators $\mathbf{e}_i, \mathbf{f}_i, \mathbf{h}_i$, give a representation of the Lie algebra $\mathfrak{g}^\vee$ of $G^\vee$ on the cohomology group $H^*(\mathrm{Gr}, \mathcal{E})$ for any $\mathcal{E} \in \mathcal{P}_G$ (see [Ka, Section 3.3]). The action of the operators $\mathbf{h}_i$ lifts to an action of the torus of $G^\vee$. Thus, the representation of the Lie algebra $\mathfrak{g}^\vee$ lifts to a representation of the group $G^\vee$. By (1.3.d), in order to check the relation $[\mathbf{e}_i, \mathbf{f}_j] = 0$ for $i \neq j$ we can assume that the group G has rank 2.

2.3. Recall that any complex $\mathcal{IC}_{\lambda^\vee}$ is a direct factor of a product $\mathcal{IC}_{\lambda_1^\vee} \star \mathcal{IC}_{\lambda_2^\vee} \star \cdots \star \mathcal{IC}_{\lambda_n^\vee}$ such that the coweights $\lambda_i^\vee$ are either minuscule or quasi-minuscule (see [NP, Proposition 9.6]). Observe that [NP, Lemmes 10.2, 10.3] imply indeed that if the set of minuscule coweights is nonempty, then we can find such a product with all the $\lambda_i^\vee$'s beeing minuscule. Recall also that for any $\mathcal{E}, \mathcal{F} \in \mathcal{P}_G$ there is a canonical isomorphism of graded vector spaces

$$H_c^*(S_{\lambda^\vee}, \mathcal{E} \star \mathcal{F}) \simeq \bigoplus_{\mu^\vee + \nu^\vee = \lambda^\vee} H_c^*(S_{\mu^\vee}, \mathcal{E}) \otimes H_c^*(S_{\nu^\vee}, \mathcal{F}), \tag{a}$$

(see [MV], and [NP, Proof of Theorem 3.1] for more details). Let $\Delta(\mathbf{e}_i), \Delta(\mathbf{f}_i), \Delta(\mathbf{h}_i)$, be the composition

$$\begin{aligned} H^*(\mathrm{Gr}, \mathcal{E}) \otimes H^*(\mathrm{Gr}, \mathcal{F}) = H^*(\mathrm{Gr}, \mathcal{E} \star \mathcal{F}) &\xrightarrow{\mathbf{e}_i, \mathbf{f}_i, \mathbf{h}_i} H^*(\mathrm{Gr}, \mathcal{E} \star \mathcal{F}) \\ &= H^*(\mathrm{Gr}, \mathcal{E}) \otimes H^*(\mathrm{Gr}, \mathcal{F}), \end{aligned}$$

where the equalities are given by (1.6.a) and (a).

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LEMMA. *If* $x = \mathbf{e}_i, \mathbf{f}_i, \mathbf{h}_i$, *then* $\Delta(x) = x \otimes 1 + 1 \otimes x$.

Proof. If $x = \mathbf{h}_i$ the equation is obvious. If $x = \mathbf{f}_i$ it is a direct consequence of the two others since a $\mathfrak{sl}(2)$-triple $(\mathbf{e}_i, \mathbf{h}_i, \mathbf{f}_i)$ is completely determined by $\mathbf{e}_i$ and $\mathbf{h}_i$. Thus, from (2.3a), (1.3b) and (1.3c), it suffices to check the equality when $G = \mathrm{SL}(2)$ and $x = \mathbf{e}_i$. Then, the operator $\mathbf{e}_i$ is the product by the first Chern class of the line bundle $\mathcal{L}_{\mathrm{SL}(2)}$ on $\mathrm{Gr}^{\mathrm{SL}(2)}$. More generally, for any simply connected group G, the $G(O)$-equivariant line bundle $\mathcal{L}_G$ on the Grassmannian Gr lifts uniquely to a $G(K)$-equivariant line bundle on the ind-scheme $G(K) \times_{G(O)} \mathrm{Gr}$. Let denote it by $\mathcal{L}_2$. The group $G(O)$ acts on the pull-back of $\mathcal{L}_G$ by the projection $G(K) \times \mathrm{Gr} \to \mathrm{Gr}$. The quotient is the bundle $\mathcal{L}_2$. The vector bundle $\mathcal{L}_2$ is algebraic, i.e. its restriction to the subscheme $p^{-1}(\overline{\mathrm{Gr}}_{\lambda_1^\vee}) \times_{G(O)} \overline{\mathrm{Gr}}_{\lambda_2^\vee}$ is an algebraic vector bundle for any $\lambda_1^\vee$, $\lambda_2^\vee$. Indeed, there is a normal pro-unipotent closed subgroup H of $G(O)$ such that $G(O)/H$ is finite-dimensional and H acts trivialy on $\overline{\mathrm{Gr}}_{\lambda_1^\vee}, \overline{\mathrm{Gr}}_{\lambda_2^\vee}$. Since H is pro-unipotent, the restriction of $\mathcal{L}_G$ to $\overline{\mathrm{Gr}}_{\lambda_2^\vee}$ is $G(O)/H$-equivariant. Thus the restriction of $\mathcal{L}_2$ to $p^{-1}(\overline{\mathrm{Gr}}_{\lambda_1^\vee}) \times_{G(O)} \overline{\mathrm{Gr}}_{\lambda_2^\vee}$ is identified with the algebraic sheaf on

$$(p^{-1}(\overline{\mathrm{Gr}}_{\lambda_1^\vee})/H) \times_{G(O)/H} \overline{\mathrm{Gr}}_{\lambda_2^\vee}$$

induced by the restriction of $\mathcal{L}_G$ to $\overline{\mathrm{Gr}}_{\lambda_2^\vee}$. Consider also the pull-back $\mathcal{L}_1$ of the line bundle $\mathcal{L}_G$ by the 1st projection $\tilde{p}\colon G(K) \times_{G(O)} \mathrm{Gr} \to \mathrm{Gr}$. We claim that

$$m^*\mathcal{L}_G = \mathcal{L}_1 \otimes \mathcal{L}_2. \tag{b}$$

Let $\mu\colon G(K) \times G(K) \to G(K)$ be the multiplication map. The product in the group $\tilde{G}(K)$ gives an isomorphism of bundles

$$\mu^* p^* \mathcal{L}_G \simeq p^*\mathcal{L}_G \boxtimes p^*\mathcal{L}_G$$

on $G(K) \times G(K)$. This isomorphism descends to the fiber product $G(K) \times_{G(O)} \mathrm{Gr}$ and implies (b). Observe now that (a) is induced by the canonical isomorphism

$$\left(p^{-1}(\overline{\mathrm{Gr}}_{\lambda_1^\vee}) \times_{G(O)} \overline{\mathrm{Gr}}_{\lambda_2^\vee}\right) \cap m^{-1}(S_{\lambda^\vee}) \simeq \bigsqcup_{\mu^\vee + \nu^\vee = \lambda^\vee} (S_{\mu^\vee} \cap \overline{\mathrm{Gr}}_{\lambda_1^\vee}) \times (S_{\nu^\vee} \cap \overline{\mathrm{Gr}}_{\lambda_2^\vee})$$

resulting from the local triviality of p (see [NP, Lemme 9.1]). By (*b*) this isomorphism identifies the restrictions of the line bundles $m^*\mathcal{L}_G$ and $\mathcal{L}_G \boxtimes \mathcal{L}_G$. Let $l_\mathcal{E}$ be the operator of product by the first Chern class of $\mathcal{L}_G$ on the global cohomology of the perverse sheaf $\mathcal{E} \in \mathcal{P}_G$. Then (a) gives $l_{\mathcal{E}\star\mathcal{F}} = l_\mathcal{E} \otimes 1 + 1 \otimes l_\mathcal{F}$. □

2.4. From Section 2.3 and (1.3.d) we are reduced to check the relation $[\mathbf{e}_i, \mathbf{f}_j] = 0$, $i \neq j$, on the cohomology group $H^*(\mathrm{Gr}, \mathcal{E})$ when G is adjoint, has rank 2, and $\mathcal{E} = \mathcal{IC}_{\lambda^\vee}$, with $\lambda^\vee$ minuscule or quasi-minuscule. For any dominant coroot $\lambda^\vee$ let $\Omega(\lambda^\vee) \subset X^\vee$ be the set of weights of the simple $G^\vee$-module with highest weight $\lambda^\vee$. Recall that

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(a) the coweight $\lambda^\vee \in X_+^\vee - \{0\}$ is minuscule if and only if $\Omega(\lambda^\vee) = W \cdot \lambda^\vee$, if and only if $(\alpha, \lambda^\vee) = 0, \pm 1$, for all $\alpha \in R$,

(b) the coweight $\lambda^\vee \in X_+^\vee - \{0\}$ is quasi-minuscule if and only if $\Omega(\lambda^\vee) = W \cdot \lambda^\vee \cup \{0\}$, if and only if $\lambda^\vee$ is a maximal short coroot. Moreover if $\lambda^\vee$ is quasi-minuscule then $(\alpha, \lambda^\vee) = 0, \pm 1$, for all $\alpha \in R - \{\pm\lambda\}$.

For any coweight $\lambda^\vee$ we consider the isotropy subgroup $G_{\lambda^\vee}$ of $e_{\lambda^\vee}$ in G. Thus

$$G_{\lambda^\vee} = T \prod_{(\alpha,\lambda^\vee) \leqslant 0} U_\alpha.$$

In particular $B^- \subset G_{\lambda^\vee}$ if $\lambda^\vee$ is dominant, and we can consider the line bundle $\mathcal{L}(\lambda)$ on $G/G_{\lambda^\vee}$ associated to the weight λ. The structure of $\overline{\mathrm{Gr}}_{\lambda^\vee}$ for $\lambda^\vee$ minuscule or quasi-minuscule is described as follows in [NP].

PROPOSITION. (c) *If $S_{\mu^\vee} \cap \overline{\mathrm{Gr}}_{\lambda^\vee} \neq \emptyset$, then $\mu^\vee \in \Omega(\lambda^\vee)$.*

(d) *If $\mu^\vee \in W \cdot \lambda^\vee$, then $S_{\mu^\vee} \cap \overline{\mathrm{Gr}}_{\lambda^\vee} = S_{\mu^\vee} \cap \mathrm{Gr}_{\lambda^\vee}$.*

(e) *If $\lambda^\vee \in X_+^\vee$ is minuscule, then*

$$\overline{\mathrm{Gr}}_{\lambda^\vee} = \mathrm{Gr}_{\lambda^\vee} = G/G_{\lambda^\vee} \quad \text{and} \quad S_{w\cdot\lambda^\vee} \cap \mathrm{Gr}_{\lambda^\vee} \simeq UwG_{\lambda^\vee}/G_{\lambda^\vee} \quad \forall w \in W.$$

(f) *Assume that $\lambda^\vee \in X_+^\vee$ is quasi-minuscule. Then $\mathrm{Gr}_{\lambda^\vee} \simeq \mathcal{L}(\lambda)$ and $\overline{\mathrm{Gr}}_{\lambda^\vee} \simeq \mathcal{L}(\lambda) \cup \{e_0\}$ as a G-varieties. Moreover,*

$$S_{w\cdot\lambda^\vee} \cap \mathrm{Gr}_{\lambda^\vee} \simeq \begin{cases} UwG_{\lambda^\vee}/G_{\lambda^\vee}, & \text{if} \quad w\cdot\lambda \in R_-, \\ \mathcal{L}|_{UwG_{\lambda^\vee}/G_{\lambda^\vee}}, & \text{if} \quad w\cdot\lambda \in R_+. \end{cases} \qquad \square$$

3. Proof of the Relation $[\mathbf{e}_i, \mathbf{f}_j] = 0$

3.1. Assume that G is adjoint, has rank two, and set $I = \{1, 2\}$. The Bruhat decomposition for M_i implies that $M_i e_{\lambda^\vee} = U_i e_{s_i\cdot\lambda^\vee} \cup U_i U_{-\alpha_i^\vee} e_{\lambda^\vee}$. Thus,

(a) if $(\alpha_i, \lambda^\vee) > 0$ then $M_i e_{\lambda^\vee} = M_i e_{s_i\cdot\lambda^\vee} = U_i e_{\lambda^\vee} \cup \{e_{s_i\cdot\lambda^\vee}\}$,

(b) if $(\alpha_i, \lambda^\vee) = 0$ then $M_i e_{\lambda^\vee} = \{e_{\lambda^\vee}\}$.

3.2. Assume that $\lambda^\vee$ is a minuscule dominant coweight. Fix $\mu^\vee = w \cdot \lambda^\vee$ with $w \in W$, and fix $i \in I$. One of the following three cases holds

(a) we have $(\alpha_i, \mu^\vee) = 1$, and

$$S_{\mu^\vee}^{M_i} \cap \mathrm{Gr}_{\lambda^\vee} = U_i e_{\mu^\vee}, \quad S_{\mu^\vee-\alpha_i^\vee}^{M_i} \cap \mathrm{Gr}_{\lambda^\vee} = \{e_{\mu^\vee-\alpha_i^\vee}\},$$
$$\mathrm{Gr}_{\mu^\vee}^{M_i} = U_i e_{\mu^\vee} \cup \{e_{\mu^\vee-\alpha_i^\vee}\},$$

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(b) we have $(\alpha_i, \mu^\vee) = -1$, and

$$S^{M_i}_{\mu^\vee+\alpha_i^\vee} \cap \mathrm{Gr}_{\lambda^\vee} = U_i e_{\mu^\vee+\alpha_i^\vee}, \quad S^{M_i}_{\mu^\vee} \cap \mathrm{Gr}_{\lambda^\vee} = \{e_{\mu^\vee}\},$$
$$\mathrm{Gr}^{M_i}_{\mu^\vee} = U_i e_{\mu^\vee+\alpha_i^\vee} \cup \{e_{\mu^\vee}\},$$

(c) we have $(\alpha_i, \mu^\vee) = 0$, and $S^{M_i}_{\mu^\vee} \cap \mathrm{Gr}_{\lambda^\vee} = \mathrm{Gr}^{M_i}_{\mu^\vee} = \{e_{\mu^\vee}\}$.

Obviously, the sheaf $i\mathrm{res}\,\mathcal{IC}_{\lambda^\vee}$ is supported on $\mathrm{Gr}_{\lambda^\vee} \cap \mathrm{Gr}^{M_i}$. Thus, by (2.2.a) and Lemma 2.2, if $\mathbf{f}_2\mathbf{e}_1(H^*_c(S_{\mu^\vee}, \mathcal{IC}_{\lambda^\vee})) \neq \{0\}$ then

$$S^{M_1}_{\mu^\vee} \cap \mathrm{Gr}_{\lambda^\vee}, \quad S^{M_1}_{\mu^\vee+\alpha_1^\vee} \cap \mathrm{Gr}_{\lambda^\vee}, \quad S^{M_2}_{\mu^\vee+\alpha_1^\vee} \cap \mathrm{Gr}_{\lambda^\vee}, \quad S^{M_2}_{\mu^\vee+\alpha_1^\vee-\alpha_2^\vee} \cap \mathrm{Gr}_{\lambda^\vee}$$

are non empty. In particular, we get

$$(\alpha_1, \mu^\vee) = -1, \qquad (\alpha_2, \mu^\vee + \alpha_1^\vee) = 1.$$

Since $\lambda^\vee$ is minuscule, $\mu^\vee \in W \cdot \lambda^\vee$, and $(\alpha_2, \alpha_1^\vee) \leqslant 0$, we get

$$(\alpha_2, \mu^\vee) = 1, \quad (\alpha_1, \mu^\vee) = -1, \quad (\alpha_2, \alpha_1^\vee) = 0.$$

Similarly,

$$\mathbf{e}_1\mathbf{f}_2(H^*_c(S_{\mu^\vee}, \mathcal{IC}_{\lambda^\vee})) \neq \{0\} \Rightarrow (\alpha_2, \mu^\vee) = 1, \quad (\alpha_1, \mu^\vee) = -1, \quad (\alpha_1, \alpha_2^\vee) = 0.$$

Thus we are reduced to the case where $G = \mathrm{PGL}(2) \times \mathrm{PGL}(2)$, $M_1 \simeq \mathrm{PGL}(2) \times \{1\}$, $M_2 = \{1\} \times \mathrm{PGL}(2)$, $\lambda^\vee = \omega_1^\vee + \omega_2^\vee$, $\mu^\vee = -\omega_1^\vee + \omega_2^\vee$, and $\mathcal{IC}_{\lambda^\vee}$ is the constant sheaf on $\mathrm{Gr}_{\lambda^\vee}$. Then,

$$\mathrm{Gr}_{\lambda^\vee} \simeq \mathbb{P}^1 \times \mathbb{P}^1, \quad \mathrm{Gr}^{M_1} \cap \mathrm{Gr}_{\lambda^\vee} \simeq \mathbb{P}^1 \times \{0, \infty\}, \quad \mathrm{Gr}^{M_2} \cap \mathrm{Gr}_{\lambda^\vee} \simeq \{0, \infty\} \times \mathbb{P}^1.$$

Recall that, with the notations of Section 1.4, the fiber of $\mathcal{L}_G^{-1}$ at $e_{\lambda^\vee}$ is identified with $\mathbb{C}t^{\lambda^\vee}w_0$. Recall also that the extended affine Weyl group $W \ltimes X^\vee$ acts on the lattice $\mathrm{Hom}\,(T \times \mathbb{G}_m, \mathbb{G}_m)$ in such a way that $\lambda^\vee \cdot \tilde{\omega}_0 = \lambda + \tilde{\omega}_0$ for all $\lambda^\vee \in X^\vee$ (see [PS, Proposition 4.9.5], for instance). Thus, for any dominant coweight $\lambda^\vee$ the restriction of $\mathcal{L}_G$ to the G-orbit $G \cdot e_{\lambda^\vee}$ is the line bundle $\mathcal{L}(\lambda)$ on $G/G_{\lambda^\vee}$. In particular the restriction of the line bundle $\mathcal{L}_i$ to $\mathrm{Gr}^{M_i}_{\lambda^\vee}$ is $O_{\mathbb{P}^1}(1)$. Thus $\mathbf{e}_1 = l \otimes \mathrm{id}$ and $\mathbf{e}_2 = \mathrm{id} \otimes l$, where l is the product by the first Chern class of $O_{\mathbb{P}^1}(1)$. The relation is obviously satisfied.

3.3. Assume that $\lambda^\vee$ is a quasi-minuscule dominant coweight. Observe that if G is of type $A_1 \times A_1$, A_2 or B_2, then the set of minuscule coweights is nonempty. Thus, from Section 2.3 we can assume that G is of type G_2. Let $\alpha_1^\vee$ be the long simple coroot, and let $\alpha_2^\vee$ be the short one. Then

$$\lambda^\vee = \alpha_1^\vee + 2\alpha_2^\vee, \qquad (\alpha_2, \alpha_1^\vee) = -3, \qquad (\alpha_1, \alpha_2^\vee) = -1.$$

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Set $\mathcal{L} = \mathcal{L}(\lambda)$ and $\bar{\mathcal{L}} = \mathcal{L} \cup \{e_0\}$. Then $\overline{\mathrm{Gr}}_{\lambda^\vee} \cap \mathrm{Gr}^{M_i}$ is the fixpoints set of Z_i on $\bar{\mathcal{L}}$, i.e.

$$\overline{\mathrm{Gr}}_{\lambda^\vee} \cap \mathrm{Gr}^{M_i} = \{e_0\} \cup \bigcup_{\mu^\vee \in W \cdot \lambda^\vee} \mathrm{Gr}^{M_i}_{\mu^\vee} \quad \text{where} \quad \mathrm{Gr}^{M_i}_{\mu^\vee} = {}^{Z_i}\mathcal{L}|_{M_i e_{\mu^\vee}}.$$

Assume that $\mu^\vee = w \cdot \lambda^\vee$ with $w \in W$.

(a) If $(\alpha_i, \mu^\vee) = 0$ then $\mathrm{Gr}^{M_i}_{\mu^\vee} = {}^{Z_i}\mathcal{L}|_{e_{\mu^\vee}}$. The torus T acts on the fiber $\mathcal{L}|_{e_{\mu^\vee}}$ by the character μ. Since $\mu^\vee \neq 0$ and $(\alpha_i, \mu^\vee) = 0$, necessarily $\mu(Z_i)$ is nontrivial. Thus, $\mathrm{Gr}^{M_i}_{\mu^\vee} = e_{\mu^\vee}$.

(b) If $(\alpha_i, \mu^\vee) = 2$ then $\mu^\vee = \alpha_i^\vee$ and $\mathrm{Gr}^{M_i}_{\mu^\vee} = \mathcal{L}|_{M_i e_{\alpha_i^\vee}}$. Moreover, since $\lambda^\vee$ is short and $\alpha_1^\vee$ is long we have $i = 2$.

(c) If $(\alpha_i, \mu^\vee) = 1$ then $\mathrm{Gr}^{M_i}_{\mu^\vee} = M_i e_{\mu^\vee}$ because

$$\big(\mu(Z_i) = 1 \text{ and } \mu^\vee \in R^\vee\big) \Rightarrow \mu^\vee \in \mathbb{Z}\alpha_i^\vee \Rightarrow (\alpha_i, \mu^\vee) \neq 1.$$

In Case (b) we get $(i = 2)$

$$S^{M_i}_{\alpha_i^\vee} \cap \overline{\mathrm{Gr}}_{\lambda^\vee} = \mathcal{L}|_{U_i e_{\alpha_i^\vee}}, \qquad S^{M_i}_{-\alpha_i^\vee} \cap \overline{\mathrm{Gr}}_{\lambda^\vee} = e_{-\alpha_i^\vee},$$

$$S^{M_i}_{0} \cap \overline{\mathrm{Gr}}_{\lambda^\vee} = \bar{\mathcal{L}}^\times|_{e_{-\alpha_i^\vee}}, \quad \text{and} \quad \overline{\mathrm{Gr}}^{M_i}_{\alpha_i^\vee} = \bar{\mathcal{L}}|_{M_i e_{\alpha_i^\vee}},$$

where the upperscript $\times$ means than the zero section has been removed. In Case (c) we get

$$S^{M_i}_{\mu^\vee} \cap \overline{\mathrm{Gr}}_{\lambda^\vee} = U_i e_{\mu^\vee}, \quad S^{M_i}_{\mu^\vee - \alpha_i^\vee} \cap \overline{\mathrm{Gr}}_{\lambda^\vee} = e_{\mu^\vee - \alpha_i^\vee}, \quad \text{and} \quad \overline{\mathrm{Gr}}^{M_i}_{\mu^\vee} = U_i e_{\mu^\vee} \cup e_{\mu^\vee - \alpha_i^\vee}.$$

Thus, for any $\mu^\vee \in X^\vee$, Claim (2.2.a) and Lemma 2.2 imply that

(d) if $\mathbf{e}_1(H^*_c(S_{\mu^\vee}, \mathcal{IC}_{\lambda^\vee})) \neq \{0\}$ then $(\alpha_1, \mu^\vee) = -1$, or $\mu^\vee = 0$, or $\mu^\vee = -\alpha_1^\vee$,

(e) if $\mathbf{f}_2(H^*_c(S_{\mu^\vee}, \mathcal{IC}_{\lambda^\vee})) \neq \{0\}$ then $(\alpha_2, \mu^\vee) = 1$, or $\mu^\vee = 0$, or $\mu^\vee = \alpha_2^\vee$.

Observe that in Case (d) the identity (2.4.c) and Lemma 2.2 imply indeed that $\mu^\vee \neq 0, -\alpha_1^\vee$, because

$$H^*_c(S_{\alpha_1^\vee}, \mathcal{IC}_{\lambda^\vee}) = H^*_c(S_{-\alpha_1^\vee}, \mathcal{IC}_{\lambda^\vee}) = \{0\}.$$

Thus, if $\mathbf{f}_2\mathbf{e}_1(H^*_c(S_{\mu^\vee}, \mathcal{IC}_{\lambda^\vee})) \neq \{0\}$ then $(\alpha_1, \mu^\vee) = -1$ and $(\alpha_2, \mu^\vee + \alpha_1^\vee) = 1$. We get $(\alpha_2, \mu^\vee) = 4$. This is not possible since $\mu^\vee \in \Omega(\lambda^\vee)$ and $\lambda^\vee$ is quasi-minuscule. Similarly, if $\mathbf{e}_1\mathbf{f}_2(H^*_c(S_{\mu^\vee}, \mathcal{IC}_{\lambda^\vee})) \neq 0$ then $(\alpha_1, \mu^\vee) = -2$. This is not possible either. Thus, the relation $[\mathbf{e}_1, \mathbf{f}_2] = 0$ is obviously satisfied. The relation $[\mathbf{e}_2, \mathbf{f}_1] = 0$ is proved in the same way.

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