

***n*-DIMENSIONAL FIRST INTEGRAL AND SIMILARITY SOLUTIONS FOR TWO-PHASE FLOW**

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(Received 12 May, 1998; revised 1 October, 1999)

Abstract

This paper considers similarity solutions of the multi-dimensional transport equation for the unsteady flow of two viscous incompressible fluids. We show that in plane, cylindrical and spherical geometries, the flow equation can be reduced to a weakly-coupled system of two first-order nonlinear ordinary differential equations. This occurs when the two phase diffusivity $D(\theta)$ satisfies $(D/D')' = 1/\alpha$ and the fractional flow function $f(\theta)$ satisfies $df/d\theta = \kappa D^{n/2}$, where n is a geometry index (1, 2 or 3), α and κ are constants and primes denote differentiation with respect to the water content θ . Solutions are obtained for time dependent flux boundary conditions. Unlike single-phase flow, for two-phase flow with $n = 2$ or 3, a saturated zone around the injection point will only occur provided the two conditions $\int_0^1 D/(1-f) d\theta < \infty$ and $f'(1) \neq 0$ are satisfied. The latter condition is important due to the prevalence of functional forms of $f(\theta)$ in oil/water flow literature having the property $f'(1) = 0$.

1. Introduction

McWhorter and Sunada [9] presented a useful range of integral solutions to two-phase flow problems in both one and two dimensions based on the fractional flow function approach of McWhorter [8]. While these solutions are exact and apply for arbitrary capillary hydraulic properties, they do require iteration techniques to evaluate the saturation profiles. The one-dimensional solution of [9] is similar to that given by Sander *et al.* [19] which was based on the work of Parlange and Braddock [12] and Parlange [10]. The solution of [19] also applies for arbitrary capillary hydraulic

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properties and requires iteration to obtain the exact saturation profiles. However its advantage is that by incorporating the optimisation procedure of Parlange [11] an accurate first estimate of the saturation profile is obtained which renders iteration unnecessary.

There are now quite a few analytical closed-form exact solutions for two-phase flow. These are predominantly for one-dimensional flow problems and originate with Fokas and Yortsos [7] for $D = D_0(1 - \nu\theta)^{-2}$, (D_0 and ν constants), $df/d\theta = \kappa D$ and a constant flux boundary condition. Rogers *et al.* [16] then extended this to include the effect of gravity through a linear hydraulic conductivity function. Solutions were also obtained for two-phase absorption under Dirichlet boundary conditions by Sander *et al.* [18] using the same D and f as Fokas and Yortsos [7], and Sander *et al.* [17] with the same D but with $f(\theta) = (f_1 + f_2\theta + f_3\theta^2)/(1 - \nu\theta)$, where f_1 , f_2 and f_3 are constants.

In higher dimensions there seems to be only one exact explicit solution. This was found by Weeks *et al.* [24] in two dimensions, without gravity, for radially symmetric flow under a constant flux injection through a line source at $r = 0$. Their solution can be written as

$$\theta(r, t) = \begin{cases} (1 - \alpha r^2/(4D_0t))^{1/\alpha}, & r^2 < 4D_0t/\alpha, \\ 0, & r^2 \geq 4D_0t/\alpha, \end{cases} \tag{1.1}$$

for $D(\theta) = D_0\theta^\alpha$, $f(\theta) = \theta^{\alpha+1}$ and an injection flux $V = 2D_0/(\alpha + 1)$. Their solution was found by extending the first integral approach of Parlange *et al.* [15] from single-phase to two-phase flow. In this paper we seek to use the same approach to obtain first integral style similarity solutions, $(\theta(\phi); \phi = r/\sqrt{t})$, for three-dimensional radially symmetric two-phase flow under a time dependent point source flux boundary condition. For completeness we present the first integral solutions for all symmetric geometries in one, two and three dimensions through a geometry index parameter n . As such, the previous solutions of Sander *et al.* [19], $n = 1$, and Weeks *et al.* [24], $n = 2$, will be obtainable directly from the theory presented here.

In n -dimensions, the nonlinear transport equation for radially symmetric two-phase flow is given by [5]

$$\frac{\partial\theta}{\partial t} = \frac{1}{r^{n-1}} \frac{\partial}{\partial r} \left[r^{n-1} D(\theta) \frac{\partial\theta}{\partial r} - f(\theta)q_t \right], \tag{1.2}$$

where t is time, r is the radial coordinate and $n = 1, 2$ and 3 for plane, cylindrical and spherical geometries. The boundary and initial conditions considered are

$$t = 0, \quad r \geq 0, \quad \theta = 0, \tag{1.3a}$$

$$t > 0, \quad r = 0, \quad f q_t - r^{n-1} D(\theta) \partial\theta/\partial r = q_w(r = 0, t) = q_{w0}, \tag{1.3b}$$

$$t > 0, \quad r \rightarrow \infty, \quad \theta \rightarrow 0. \tag{1.3c}$$

In (1.2) and (1.3), $\theta(r, t)$ is the normalised volumetric water content with $0 \leq \theta \leq 1$ and is related to the actual water content Θ by $\theta = (\Theta - \Theta_r)/(\Theta_{sat} - \Theta_r)$, where Θ_{sat} and Θ_r are the saturated and residual water contents respectively. The flux of water (or wetting fluid) is denoted by q_w , non-wetting fluid q_{nw} and the total fluid flux $q_t = q_w + q_{nw}$. The volumetric flow rates Q_i ($i = w, nw, t$) are related to their fluxes q_i by $q_i = Q_i/G(\Theta_{sat} - \Theta_r)$. Both Q_t and the constant G depend upon n . In 1, 2 and 3 dimensions, G equals 1, $2\pi b$ (where b corresponds to the length of the line source), and 4π respectively. For similarity, the water injection takes place along a plane, line or point source subject to [12]

$$q_{wo} = Vt^{n/2-1}, \tag{1.4}$$

with V a constant.

2. First integral

In this paper we will only be considering unidirectional displacement of a non-wetting phase by a wetting phase, therefore at $r = 0$, $q_{nw} = 0$ and $q_t = q_{wo}$. Defining the standard similarity variable

$$\phi = rt^{-1/2}, \tag{2.1}$$

then (1.2) reduces to

$$-\frac{\phi^n}{2} \frac{d\theta}{d\phi} = \frac{d}{d\phi} \left[\phi^{n-1} D \frac{d\theta}{d\phi} - f Q \right], \tag{2.2}$$

while (1.3a) and (1.3c) coalesce to $\theta = 0$, $\phi \rightarrow \infty$. The handling of boundary condition (1.3b) depends very much on n . Integrating (2.2) gives

$$-\frac{1}{2} \int_0^\infty \phi^n \frac{d\theta}{d\phi} d\theta = \left[f V - D \frac{d\theta}{d\phi} \right]_{\theta=\theta_s} = V, \tag{2.3}$$

where the last equality comes from combining (1.3b) and (1.4) and remembering $q_t = q_{wo}$. In the case of $n = 1$, at $\phi = 0$, $\theta = \theta_s$, where θ_s is constant with time and related to V through (2.3) (since in this case $\theta(\phi)$ is invertible) by [19]

$$\int_0^{\theta_s} \phi d\theta = 2V. \tag{2.4}$$

Here V is related to the well-known sorptivity S by $S = 2V$ and therefore (2.4) also imposes an upper limit to V defined by $\theta_s = 1$.

When we move to two- and three-dimensional flow regimes ($n = 2, 3$) a saturated zone will usually develop around the source [9]. In this zone $\theta = 1$, $d\theta/d\phi = 0$ and

$\theta(\phi)$ is not uniquely invertible. However if we define the edge of this saturated zone by $\phi = \phi_1$, then we can still invert the solution provided $\phi > \phi_1$ and then (2.3) and (2.4) are replaced by

$$\left[f V - \phi^{n-1} D \frac{d\theta}{d\phi} \right]_{\phi=\phi_1} = V, \quad n = 2, 3, \tag{2.5}$$

where ϕ_1 is obtained from

$$- \int_{\phi_1}^{\infty} \phi^n \frac{d\theta}{d\phi} d\phi = \int_0^1 \phi^n d\theta = 2V, \quad n = 2, 3.$$

To obtain a first integral solution we follow the method as outlined in Parlange *et al.* [13–15] and define the transformations

$$u = \phi D^{-1/2} \tag{2.6a}$$

and

$$dy = dD/D, \tag{2.6b}$$

where y is the independent variable. Equation (2.2) then becomes

$$\begin{aligned} \frac{u}{2} \left(\frac{u}{2} + u' \right)^2 - \left(\frac{u'}{2} + u'' \right) - \frac{VD^{-n/2}}{u^{n-1}} \left(\frac{u}{2} + u' \right)^2 \frac{df}{d\theta} \\ + \left(\frac{u}{2} + u' \right) \left(\frac{n}{2} + (n-1) \frac{u'}{u} \right) + \left(\frac{u}{2} + u' \right) h(\theta) = 0, \end{aligned} \tag{2.7}$$

where

$$h(\theta) = \frac{d}{d\theta} \left[\frac{D}{dD/d\theta} \right]. \tag{2.8}$$

If (2.7) is to lead to a first integral, then it is necessary that both the functions $h(\theta)$ and $VD^{-n/2}df/d\theta$ be independent of θ . This is easily achieved by setting $h(\theta) = 1/\alpha$ and $VD^{-n/2}df/d\theta = \beta$, where α and β are constants. It is hardly surprising that the condition $h(\theta) = 1/\alpha$ emerges from our analysis. It is well-known that this is the necessary condition for Lie group invariance of the nonlinear diffusion equation and Richards' equation [2, 21, 22] which is a property that known exact solutions possess. Integrating (2.8) shows that $D(\theta)$ must be of the form

$$D(\theta) = D_0(\theta + \mu)^\alpha \tag{2.9a}$$

or

$$D(\theta) = A e^{p\theta}, \tag{2.9b}$$

where μ , A and p are positive constants, and (2.9b) is the limiting case of $\alpha \rightarrow \infty$. If we now set $VD^{-n/2}df/d\theta = \beta$ and integrate using $f(0) = 0$ and $f(1) = 1$ then $f(\theta)$ is given by

$$f(\theta) = \frac{\int_0^\theta D^{n/2}d\sigma}{\int_0^1 D^{n/2}d\theta}, \tag{2.10}$$

with

$$\beta = \frac{V}{\int_0^1 D^{n/2}d\theta}, \tag{2.11}$$

and $D(\theta)$ given by either (2.9a) or (2.9b) only.

The form of (2.7) suggests we take a new function ω ([13])

$$\omega = u/2 + u', \tag{2.12}$$

which combined with (2.9)–(2.11) allows (2.7) to be rewritten as the first-order differential equation

$$\left[u + \frac{2}{u} (n - 1 - \beta u^{2-n}) \right] \omega^2 + \left(1 + \frac{2}{\alpha} \right) \omega - (2\omega - u) \frac{d\omega}{du} = 0. \tag{2.13}$$

Therefore whenever D and f obey (2.9) and (2.10) a first integral exists. Equations (2.12) and (2.13) form a weakly-coupled system of first-order differential equations for the function $u(y)$. First consider the case of the exponential diffusivity as given by (2.9b). When $D \neq 0$ at $\theta = 0$, as in an exponential D , $\phi \rightarrow 0$ asymptotically as $\theta \rightarrow 0$. From (2.6a) $\phi \rightarrow \infty$ implies $u \rightarrow \infty$ and (2.13) can be approximated by $u\omega^2 - 2\omega d\omega/du \approx 0$ which has the dominant solution $\omega \approx \gamma \exp(u^2/4)$ with $\gamma \neq 0$. Clearly γ must depend not only on p in (2.9b) but also the surface flux V . Unfortunately there is no direct means of determining this dependence other than through iteration. However when $D(0) = 0$, iteration is unnecessary and γ is known directly, given by $\gamma = 0$ [12]. In this case it can be shown that as $u \rightarrow 0$, $\omega \rightarrow 0$ also.

In this paper we are only interested in solutions which do not require iteration and from now on we only consider $D(\theta)$ as in (2.9a) with $\mu = 0$ so that $D(0) = 0$. Thus from (2.9a), (2.10) and (2.11)

$$D(\theta) = D_0\theta^\alpha, \tag{2.14}$$

$$f(\theta) = \theta^{n\alpha/2+1}, \tag{2.15}$$

$$\beta = V(n\alpha/2 + 1)D_0^{-n/2}. \tag{2.16}$$

Since $D(0) = 0$, ϕ at $\theta = 0$ must be finite, $u \rightarrow \infty$ and ω^{-1} can be expanded as a series in powers of u in the form

$$\omega^{-1} = -\alpha u/2 + A_n + B_n u^{-1} + C_n u^{-2} + E_n u^{-3} + O(u^{-4}), \tag{2.17}$$

where A_n, B_n, C_n and E_n are constants which depend on the dimension n . The importance of (2.17) is that it provides the necessary boundary condition for the integration of (2.13) and therefore (2.12). If we substitute into (2.13) we obtain

$$\left(\frac{2}{\alpha} + 1\right) A_n - 2\beta u^{1-n} + 2\left(n + \left(\frac{1}{\alpha} + 1\right) B_n\right) u^{-1} + \left(\left(\frac{2}{\alpha} + 3\right) C_n + \frac{4}{\alpha} A_n\right) u^{-2} + \left[\left(\frac{2}{\alpha} + 4\right) E_n + \frac{8}{\alpha} \left(B_n + \frac{A_n^2}{\alpha}\right)\right] u^{-3} + O(u^{-4}) = 0. \tag{2.18}$$

For (2.18) to hold true, all coefficients of u must be zero. Thus, by setting n and grouping like terms, we find that the constants A_n, B_n, C_n and E_n take on the following values: for $n = 1$,

$$\begin{aligned} A_1 &= \frac{2\alpha\beta}{\alpha + 2}, & C_1 &= \frac{-8\alpha\beta}{(\alpha + 2)(3\alpha + 2)}, \\ B_1 &= \frac{-\alpha}{\alpha + 1}, & E_1 &= \frac{4\alpha[(\alpha + 2)^2 - 4\beta^2(\alpha + 1)]}{(\alpha + 1)(2\alpha + 1)(\alpha + 2)^2}, \end{aligned} \tag{2.19a}$$

for $n = 2$,

$$A_2 = 0, \quad B_2 = \frac{\alpha(\beta - 2)}{\alpha + 1}, \quad C_2 = 0, \quad E_2 = \frac{-4\alpha(\beta - 2)}{(\alpha + 1)(2\alpha + 1)} \tag{2.19b}$$

and for $n = 3$,

$$A_3 = 0, \quad B_3 = \frac{-3\alpha}{\alpha + 1}, \quad C_3 = \frac{2\alpha\beta}{3\alpha + 2}, \quad E_3 = \frac{12\alpha}{(\alpha + 1)(2\alpha + 1)}. \tag{2.19c}$$

For $n = 2$, the coefficients of all of the even powers of u (u^0, u^{-2} , etc.) are zero and for $\beta = 2, \omega = -2/\alpha u$ is the solution of (2.13). This leads to the fully analytic solution for (1.1) in cylindrical geometry given by $\theta(r, t) = (1 - \alpha r^2/4D_0t)^{1/\alpha}$ for $r^2 < 4D_0t/\alpha$ and $\theta(r, t) = 0$ for $r^2 \geq 4D_0t/\alpha$ [24]. No similar phenomena are found in either the plane or spherical cases.

The definitions of $D(\theta)$ and y from (2.14) and (2.6b) allow (2.12) to be written as

$$\frac{du}{d \ln \theta} = \alpha \left(\omega - \frac{u}{2}\right), \tag{2.20}$$

which can be integrated to first order for u large ($\omega \rightarrow 0$) to give $u \approx c\theta^{-\alpha/2}$ (c being a constant) which combined with the definition of u gives $\phi = uD^{1/2} \approx c\theta^{-\alpha/2} D_0^{1/2} \theta^{\alpha/2}$ being finite at $\theta = 0$. Consequently a well-defined wetting front is moving into the soil, consistent with $D(0) = 0$. Equation (2.17) provides the starting condition at infinity (the position of the wetting front) so that (2.13) can be integrated backwards in the direction $u \rightarrow 0$. The initial value of u has to be chosen large enough so that

the higher order terms in (2.17) become negligible. The boundary condition given by either (2.3) or (2.5) depending on n , determines how far (2.13) is integrated backwards. Rewriting (2.5) as

$$V = -\phi^{n-1} D_w(\theta) \frac{d\theta}{d\phi}, \quad \phi = \phi_1, \tag{2.21}$$

then $D_w(\theta) = D(\theta)/(1 - f)$ is the usual single-phase (water) diffusivity.

Substituting (2.6a) for u in (2.20) and expanding the derivative term will give the following expression for the flux in terms of u and ω at the injection point $\phi = \phi_1$,

$$-\phi^{n-1} D_w(\theta) \left. \frac{d\theta}{d\phi} \right|_{\phi=\phi_1} = \left[\frac{-u^{n-1} D_0^{n/2} \theta^{n\alpha/2+1}}{\alpha\omega(1-f)} \right]_{\phi=\phi_1}, \tag{2.22}$$

or combining with (2.21)

$$V = \left[\frac{-u^{n-1} D_0^{n/2} \theta^{n\alpha/2+1}}{\alpha\omega(1-f)} \right]_{\phi=\phi_1}. \tag{2.23}$$

For $n = 1$, $\theta = \theta_s$ at $\phi = 0$ and with $f(\theta)$ from (2.15)

$$V = \frac{D_0^{1/2}}{\alpha\omega(0)(1 - \theta_s^{-(\alpha/2+1)})}. \tag{2.24}$$

As $V = S/2$, (2.24) provides a relationship between the sorptivity S and θ_s and is equivalent to (2.14) of Sander *et al.* [19].

For $n = 2$ or 3 , however, $\theta = 1$ at $\phi = \phi_1$ which means that the denominator of the right-hand side of (2.23) is zero due to $f(1) = 1$. Therefore, if we are to obtain a finite flux V from (2.23), then u must also be zero. The definitions of ϕ and u from (2.1) and (2.6a) show that $u = 0$ corresponds to $r = 0$. This means that no saturation zone develops around the source for the two- and three-dimensional problems. This occurs because of the functional form of D and f in (2.14) and (2.15) resulting in the sorptivity S , approximated by [11]

$$\left[2 \int_0^1 \frac{D}{1-f} d\theta \right]^{1/2},$$

being infinite. Consequently any flux of water injected at the source ($r = 0$) can be absorbed by the soil and no saturation zone forms. This is by no means the only restriction on the formation of a saturated zone. A saturated zone will also not occur for $n = 2$ and 3 if $f(\theta)$ has the property $f'(1) = 0$ even if S is finite. A proof and further discussion of saturated zone formation is given in a later section.

3. Asymptotic expansions for θ ($\phi \rightarrow 0$)

We can obtain expansions for $\theta(\phi)$ as $\phi \rightarrow 0$ or when θ is near θ_s for $n = 1$, and θ near 1 (saturation) for $n = 2$ and $n = 3$.

Case $n = 1$. The differential equation for $n = 1$ is

$$(2\omega - u) \frac{d\omega}{du} = \omega \left[(u - 2\beta)\omega + \frac{2}{\alpha} + 1 \right], \tag{3.1}$$

and we look for an expansion for $\omega(u \rightarrow 0)$ of the form

$$\omega = \omega_0(1 + a_1u + c_1u^2 + O(u^3)), \tag{3.2a}$$

where $\omega_0 = \omega(u = 0)$. Substituting (3.2a) into (3.1) and by matching terms of the order u^0 and u we find

$$a_1 = -\beta + (1/2 + 1/\alpha)\omega_0^{-1} \tag{3.2b}$$

and

$$c_1 = a_1(2 + 2/\alpha)(4\omega_0)^{-1} + 1/4 - a_1\beta - a_1^2/2. \tag{3.2c}$$

An estimate of ω_0 for arbitrary α is not possible, however by following Parlange *et al.* [13] we can find estimates to ω_0 for α large. The exponent α is usually large for soils, for example $\alpha \geq 4$, and we note from (2.17) that for large α , $u\omega$ behaves like α^{-1} . Substituting for $u\omega$ into the right-hand side of (3.1) and then neglecting $O(\alpha^{-1})$ terms, (3.1) simplifies to

$$\frac{d\omega}{du} = \frac{\omega - 2\beta\omega^2}{2\omega - u}.$$

Integrating (3.1) along with the boundary condition $u\omega \rightarrow -2/\alpha$ as $u \rightarrow \infty$, an estimate for ω_0 is obtained by solving the nonlinear equation

$$\frac{1 - 2\beta\omega_0}{2\beta^2} \ln(1 - 2\beta\omega_0) - \frac{2}{\alpha} + \left(\frac{4\beta}{\alpha} + \frac{1}{\beta} \right) \omega_0 = 0. \tag{3.3}$$

Finally combining (2.6a), (2.14), (2.20) and (3.2a) and keeping only the dominant first two terms shows that as $\phi \rightarrow 0$, $\theta(\phi)$ behaves like $\theta = \theta_s + (\alpha\omega_0 D_0^{1/2} \theta_s^{\alpha/2})^{-1} \phi$, with ω_0 estimated from (3.3).

Case $n = 2$. In two dimensions, the first-order approximation for $\omega(u)$ is given by $\omega = \omega_0 u^{1-\beta}$ [24] and we look for the second-order term in the form

$$\omega = \omega_0 u^{1-\beta} + c_2 u^d, \tag{3.4}$$

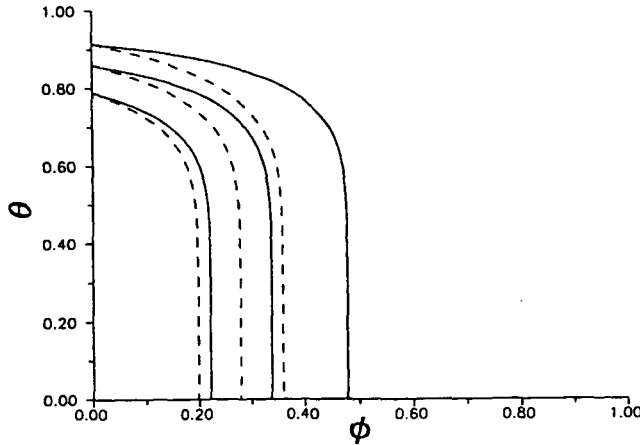


FIGURE 1. Profiles of $\theta(\phi)$ when $n = 1$ for one- and two-phase flow with $\alpha = 8$ and θ_s equivalent to $\beta = 0.5, 1$ and 2 .

where the constants c_2 and d can be found by substituting (3.4) into (2.13) and grouping like terms. For consistency we find $d = 1$ and the two-term expansion for $\omega(u)$ is $\omega = \omega_0 u^{1-\beta} + (1/2)(2 - \beta + 2/\alpha)u/\beta$. For $n = 2$, it is also possible to get a first-order estimate of ω_0 for the cases where α is large. As for $n = 1$, large α means that we can take $u\omega \approx -2/\alpha$, then substituting into (2.22) and neglecting terms of $O(\alpha^{-1})$ we have

$$\frac{d\omega}{du} = \frac{2(1 - \beta)\omega/u + 1}{2 - u/\omega}, \tag{3.5}$$

which can be integrated with the boundary condition $u\omega \rightarrow 2/\alpha$ as $u \rightarrow \infty$, resulting in the first-order approximation

$$\omega_0 = -\frac{1}{\beta} \left(\frac{2\beta}{\alpha} \right)^{\beta/2}.$$

The next term in the series can also be found by using the solution of (3.5) to estimate the $O(\alpha^{-1})$ terms initially neglected. For $1 < \beta < \alpha^2$ the two-term expansion for ω_0 is

$$\omega_0 = -\frac{1}{\beta(1 - 1/\alpha)} \left(\frac{2\beta(1 - 1/\alpha)}{\alpha} \right)^{\beta(1-1/\alpha)/2}.$$

It should also be noted that for $\beta = 1$, the differential equation for $\omega(u)$ reduces to (13) of [13]. Hence, a three-term expansion for ω_0 in this case ($\beta = 1, \alpha$ large) is given by (23) of [13], that is,

$$\omega_0^2 = 2\alpha^{-1} + 3\alpha^{-2} + (6\alpha^3)^{-1}.$$

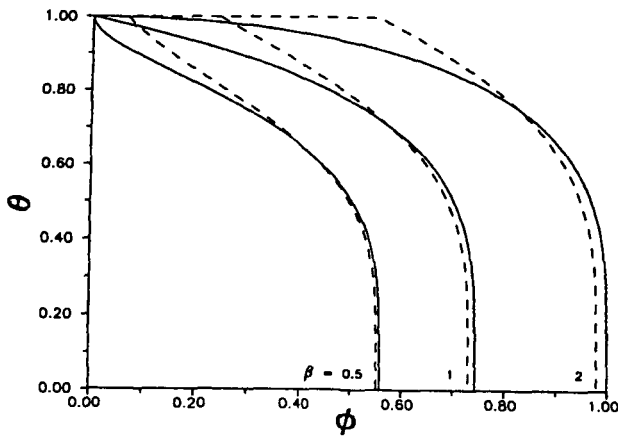


FIGURE 2. Profiles of $\theta(\phi)$ for $n = 2$, $\alpha = 4$ and β as marked.

The expansions for $\theta(\phi \rightarrow 0)$ as $\phi \rightarrow 0$ is given by combining (2.6a), (2.14), (2.20) and (3.4) as [24]

$$\theta \approx 1 + \frac{D_0^{-\beta/2}}{\alpha\omega_0\beta} \phi^\beta. \quad (3.6)$$

By differentiating (3.6) it is possible to see that in two dimensions the value of β determines the behaviour of $d\theta/d\phi$ as $\phi \rightarrow 0$. There are three distinct types of behaviour of $d\theta/d\phi$, (i) $d\theta/d\phi \rightarrow -\infty$ for $0 < \beta < 1$, (ii) $d\theta/d\phi \rightarrow (D_0^{1/2}\alpha\omega_0)^{-1}$ for $\beta = 1$ and (iii) $d\theta/d\phi \rightarrow 0$ for $\beta > 1$. The case $0 < \beta < 1$ is particularly interesting as it implies a rapid decline in the water content away from the injection point and that overall an S-shaped water profile will result. This type of behaviour has often been noticed in two-phase flow experiments (see Bond and Collis-George [3], for example). The definition of β shows that S-shaped profiles will predominantly occur under small injection rates V , which is also in agreement with the findings of McWhorter and Sunada [9].

Case $n = 3$. In three dimensions, the differential equation for $d\omega/du$ is

$$\frac{d\omega}{du} = \frac{\omega}{-2\omega - u} \left[\left(u + \frac{2}{u} \left(2 - \frac{\beta}{u} \right) \right) \omega + \frac{2}{\alpha} + 1 \right]. \quad (3.7)$$

For u small the dominant term in (3.7) is given through $(2 - \beta/u)\omega/u$. Neglecting other terms and integrating gives a first-order approximation for ω as

$$\omega = \omega_0 u^2 e^{\beta/u}. \quad (3.8)$$

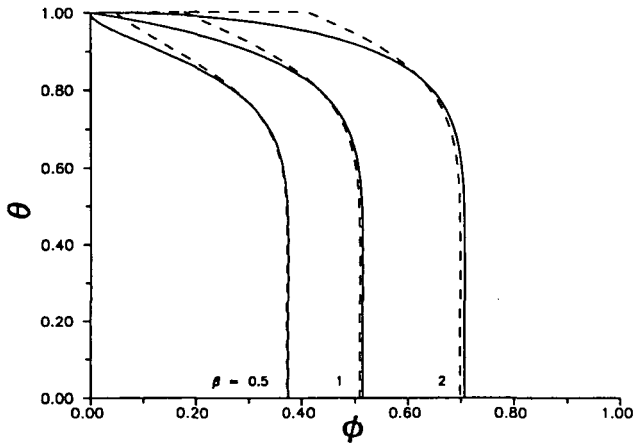


FIGURE 3. Profiles of $\theta(\phi)$ for $n = 2$, $\alpha = 8$ and β as marked.

Then from (2.6a), (2.14), (2.20) and (3.8) the first-order approximation to θ for $\phi \rightarrow 0$ is

$$\theta = 1 + (\alpha\omega_0\beta)^{-1} e^{-\beta/(D_0^{1/2}\phi)}. \quad (3.9)$$

Differentiating (3.9) shows that $d\theta/d\phi$ at $\phi = 0$ for $n = 3$ is always zero. As yet estimates to ω_0 for $n = 3$ have not been found.

4. Discussion

In general, (2.12) and (2.13) cannot be integrated analytically. Using (2.17) as the starting condition (2.13) is integrated backwards to $u = 0$ using the fourth-order Runge-Kutta method to obtain $\omega(u)$. Having $\omega(u)$, (2.12) can be integrated using Simpson's rule for $u(y)$ using the boundary condition $u = 0$, $y = \alpha \ln(\theta_s)$ for $n = 1$, or $u = 0$, $y = 0$ for $n = 2$ and 3. Equation (2.6) then finally gives $\phi(\theta)$ or $\theta(\phi)$ parametrically. When $\beta > 1$ for $n = 2$ and 3, (3.4) and (3.8) show that $\omega \rightarrow -\infty$ as $u \rightarrow 0$, then for these cases the integration is taken back only as far as $u = u^*$, $u^* > 0$. When $\omega(u)$ is known numerically, (3.4) when $n = 2$ or (3.8) when $n = 3$ can be rearranged to estimate ω_0 numerically. As $u \rightarrow 0$ these expansions become more accurate, thus u^* is defined by the value of u giving ω_0 accurate to four significant figures. Equation (2.12) is still integrated as before but with the expansions used for $\omega(u)$ when $0 \leq u \leq u^*$ and the numerical solution for $u > u^*$. All figures presented in this section are accurate to four significant figures.

Figures 1 to 6 show the $\theta(\phi)$ profiles as a function of α and V for all three dimensions. We have also included in these figures the one-phase moisture profiles

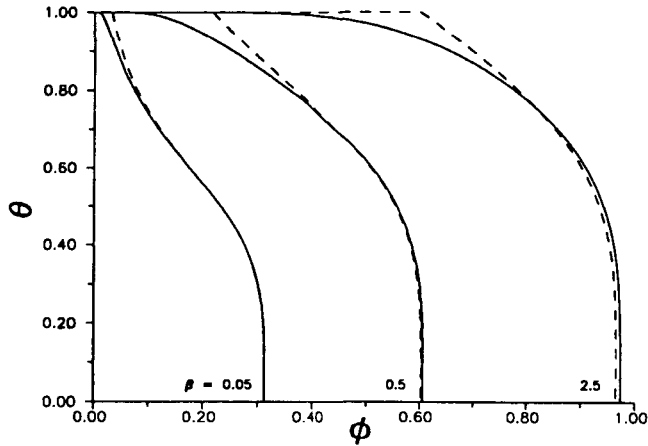


FIGURE 4. Profiles of $\theta(\phi)$ for $n = 3, \alpha = 4$ and β as marked.

(dashed lines) for the purposes of comparison. The single-phase n -dimensional flow equations are given in Parlange and Braddock [12] and can be obtained directly from (2.13) by setting $\beta = 0$. However in both two- and three-dimensional one-phase flow conditions a saturated zone will develop around $r = 0$ in which case (2.13) (with $\beta = 0$) is integrated only as far back as $u = u_1, \omega = \omega_1$, where u_1, ω_1 satisfy [12]

$$\omega_1^{-1} u^{n-1} = -\alpha V D_0^{-n/2}. \tag{4.1}$$

Note that (4.1) is obtainable directly from (2.23) by simply taking $f = 0$. If $n = 1$, then (2.24) is replaced by

$$\theta_s^{1+\alpha/2} = -\alpha V \omega(0) D_0^{-1/2}. \tag{4.2}$$

In Figure 1 ($n = 1$) both the one- and two-phase profiles (solid line) are plotted for the same surface water content θ_s , hence the fluxes V will be different and given by (4.2) for single-phase and (2.24) for two-phase flow. In Figures 2 to 6 both the single and two-phase flow profiles (solid lines) have the same injection flux V determined from (2.16) as $V = \beta D_0^{n/2} (1 + n\alpha/2)^{-1}$ once β, α and n are given. Without loss of generality D_0 is taken as 1 in all the figures.

Figures 2 to 6 show that the effect of increasing α or increasing geometry dimension through n is to reduce the influence of the non-wetting phase on the saturation profile $\theta(\phi)$. Secondly the effect of the non-wetting phase is predominantly to alter the shape and position of the saturation profile around $\theta = 1$ and it has very little influence over the position or shape of the tail of the wetting front itself. The reason for this is clearly seen when we look at the functional form of $f(\theta)$, that is, $f(\theta) = \theta^{1+n\alpha/2}$. As either

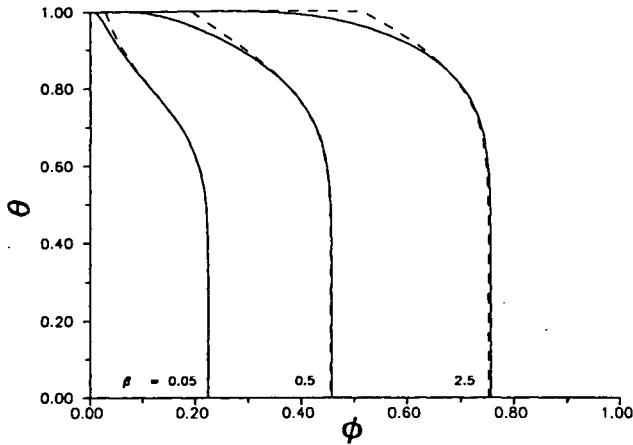


FIGURE 5. Profiles of $\theta(\phi)$ for $n = 3$, $\alpha = 8$ and β as marked.

α or n increases and in particular for α large, $f(\theta)$ will only be significantly different from 0 for θ near 1. Thus $f(\theta)$ starts to behave like a delta function and therefore the non-wetting phase has little or no influence on the saturation profile.

The other feature we notice in Figures 2 to 6 is the S-shaped profiles which occur when $n = 2$ or 3. For $n = 2$ these profiles will occur for $\beta < 1$ (Equation (3.6)) or for a flux $V \leq (1 + \alpha)^{-1}$. These profiles therefore only arise for small injection fluxes, and the larger the diffusivity exponent α , the smaller the flux V will need to be. It is also clear that S-shaped profiles will develop when $n = 3$ for a small injection flux. However with $n = 3$, while the value of β necessary for S-shaped profiles is much smaller than its value when $n = 2$, there is no limiting value of β below which these profiles will always occur. This can be seen in Figures 4 and 5 for $\beta = 0.05$, where for $\alpha = 4$ the profile is markedly S-shaped but at $\alpha = 8$ it is barely so and in fact disappears once β increases to 0.1.

The S-shaped profiles also appear for single-phase flow in two and three dimensions and are therefore not due to the presence of the second fluid. If anything, the effect of the second fluid has been to diminish the magnitude of the positive curvature in $\theta(\phi)$ for small ϕ when $n = 2$ or 3. We see this most strikingly in Figure 6 for $n = 3$, $\alpha = 2$ and $\beta = 0.5$, where the single-phase profile is S-shaped and the two-phase profile is not.

Saturation zones. We saw from (2.23) that our two- and three-dimensional first integral solutions do not develop a saturated zone around the injection point. This is due in part to the diffusivity required for a first integral to exist, that is, (2.9) is not zero at $\theta = 1$. While the traditional single-phase diffusivity D_w is a monotonically increasing function of θ , the two-phase diffusivity $D(\theta)$ is zero at both $\theta = 0$ and

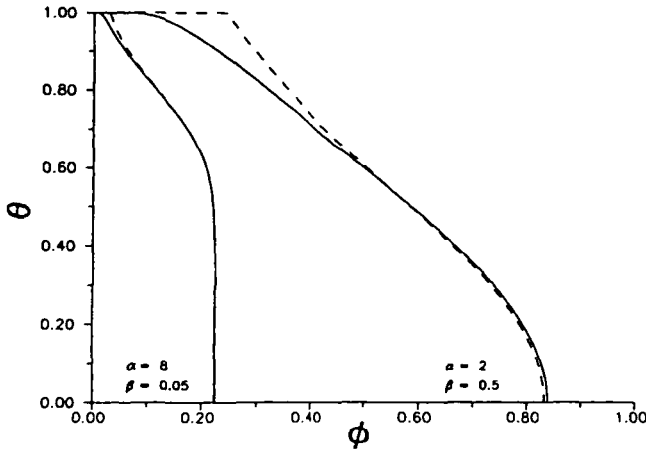


FIGURE 6. Profiles of $\theta(\phi)$ for $n = 3$, with α and β as marked.

$\theta = 1$. With D and D_w related by $D_w = D/(1 - f)$, then from (2.14) and (2.15) not only does $D_w(1) \rightarrow \infty$ but also

$$\int_0^1 D_w d\theta \rightarrow \infty. \tag{4.3}$$

For D_w increasing rapidly with θ we know from [11] that the integral in (4.3) is approximately $S^2/2$. Consequently if the soil's sorptivity is infinite it will not be possible for a saturated zone to develop. It is quite straightforward to derive this condition for arbitrary $D(\theta)$ with $D(1) = 0$ rather than just the particular form of $D(\theta)$ necessary for a first integral. Integrating (2.2) we have

$$\psi(\theta) = \int_0^\theta \frac{D d\sigma}{V(1 - f) - (1/2) \int_\sigma^1 \phi^n d\rho}, \tag{4.4}$$

where

$$\psi(\theta) = \begin{cases} \phi, & \text{for } n = 1, \\ \ln(\phi_f/\phi), & \text{for } n = 2, \\ \phi^{-1} - \phi_f^{-1}, & \text{for } n = 3, \end{cases} \tag{4.5}$$

with ϕ_f being the position of the wetting front. Without loss of generality we assume $D(0) = 0$ so that ϕ_f is finite. Since $\phi(\theta)$ is a monotonically decreasing function of θ we know that the integrand in (4.4) is always positive, hence

$$V(1 - f) \geq \frac{1}{2} \int_\theta^1 \phi^n d\sigma, \tag{4.6}$$

and therefore at $\theta = 1$ (4.4) has the lower bound

$$\int_0^1 \frac{D d\theta}{V(1-f) - (1/2) \int_0^1 \phi^n d\rho} \geq \frac{1}{V} \int_0^1 D_w d\theta. \quad (4.7)$$

If the single-phase diffusivity D_w satisfies (4.3), then from (4.5) and (4.6), $\psi \rightarrow \infty$ and the position of the saturated zone ϕ_1 ($\theta = 1$, $\phi = \phi_1$) must be at zero.

In single-phase flow, having the integral in (4.7) being finite is sufficient for a saturated zone to develop, however for two-phase flow a further condition is required. We see this by first Taylor expanding the integrand of (4.4) around $\theta = 1$ as

$$\begin{aligned} & \frac{D}{V(1-f) - (1/2) \int_0^1 \phi^n d\rho} \\ & \approx \frac{D'(1) + (\theta - 1)D''(1)/2}{V[\phi_1^2/(2V) - f'(1)] + (\theta - 1)[(n/2)\phi_1^{n-1}(d\phi/d\theta)|_{\theta=1} - f''(1)]/2}. \end{aligned} \quad (4.8)$$

Since (4.8) must be positive and $D'(1) \leq 0$, then $\phi_1^2/(2V) \leq f'(1)$, so that if $f'(1) = 0$, ϕ_1 must also be zero and again no saturated zone forms. It is worthwhile noting that it is very common in the oil/water and petroleum reservoir modelling literature to use functional forms of $f(\theta)$ which have the property $f'(1) = 0$ [1, 4, 6, 20, 23].

In conclusion we have extended the n -dimensional single-phase first integral similarity solutions of Parlange and Braddock [12] to two-phase flow when $D(\theta) = D_0\theta^\alpha$ and $f(\theta) = \theta^{1+n\alpha/2}$. We have also shown that S-shaped saturation profiles develop at low injection fluxes V in both two and three dimensions and that for a saturated zone to develop around the injection point $r = 0$, the hydraulic functions $D(\theta)$ and $f(\theta)$ must satisfy the two conditions of $\int_0^1 D/(1-f) d\theta < \infty$ and $f'(1) \neq 0$.

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