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ON CESÀRO AND ABEL SUMMABILITY FACTORS FOR INTEGRALS

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1. Introduction. Many results have been obtained about factors transforming integrals summable by ordinary and absolute Cesàro methods of non-negative orders into integrals summable by such methods (see **[4]**, **[2]**, **[6]**, **[3]**) and also into integrals summable by the ordinary and absolute Abel methods (see **[7]**). Since the Cesàro summability methods (C, α) and $|C, \alpha|$ for integrals are defined for $\alpha \geqq -1$, it is natural to try to extend the above mentioned results for $\alpha \geqq 0$ to the case $-1 \leqq \alpha < 0$. In this paper we restrict attention to the simplest case $\alpha = -1$, and classify the summability factors from $(C, -1)$ and $|C, -1|$ to $(C, -1)$, $|C, -1|$, (C, λ), $|C, \lambda|$, A and $|A|$, where $\lambda \geqq 0$ and A denotes Abel summability.

2. Notation and definitions. Let $M(a, b)$ and $L(a, b)$ denote respectively the Banach spaces of Lebesgue measurable essentially bounded functions on (a, b) and Lebesgue integrable functions on (a, b). Let $M = M(1, \infty)$, $L = L(1, \infty)$,

$$M_{\text{loc}} = \bigcap_{1 \leqq n < \infty} M(1, n) \quad \text{and} \quad L_{\text{loc}} = \bigcap_{1 \leqq n < \infty} L(1, n).$$

Denote by BV the Banach space of functions of bounded variation over $[1, \infty)$, by $BV(0, \infty)$ the space of functions of bounded variation over $(0, \infty)$, and by BV_{loc} the space of functions of bounded variation over $[1, n]$ for every finite $n > 1$. Suppose throughout that $\lambda \geqq 0$.

An integral $\int_1^\infty x(t)dt$ is said to be

(i) summable $(C, -1)$ [$|C, -1|$] if $x \in L_{\text{loc}}$ and

$$(1) \quad y(t) = \int_1^t x(u)du + tx(t)$$

is equivalent to a function tending to a finite limit as $t \to \infty$ [$\in BV$];

(ii) summable (C, λ) [$|C, \lambda|$] if $x \in L_{\text{loc}}$ and

$$\int_1^t \left(1 - \frac{u}{t}\right)^\lambda x(u)du$$

tends to a finite limit as $t \to \infty$ [$\in BV$];

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(iii) summable A [$|A|$] if $x \in L_{\text{loc}}$ and

$$\int_1^\infty e^{-su}x(u)du$$

is convergent for $s > 0$ and tends to a finite limit as $s \to 0+$ [$\in BV(0, \infty)$].

The symbol $(C, -1)$ will also be used to denote the linear space of functions x such that $\int_1^\infty x(t)dt$ is summable $(C, -1)$, and a similar use will be made of the symbols $|C, -1|$, (C, λ) etc. The set of summability factors from X to Y will be denoted by $(X; Y)$, i.e., $k \in (X; Y)$ if and only if $xk \in Y$ whenever $x \in X$.

A summary of the main results established is given at the end of the paper.

3. Preliminary results. Integrating (1) yields the identity

$$\frac{1}{t}\int_1^t y(u)du = \int_1^t x(u)du$$

from which it follows that $x \in (C, -1)$ if and only if $x \in (C, 0)$ and

$$\operatorname*{ess\,lim}_{t\to\infty} tx(t) = 0.$$

Likewise, it follows that $x \in |C, -1|$ if and only if $x \in |C, 0|$ and $tx(t) \in BV$. The inverse transformation to (1) is given by

$$x(t) = \frac{d}{dt}\left(\frac{1}{t}\int_1^t y(u)du\right) \quad \text{for a.a. } t \geqq 1.$$

If the function y defined by (1) is in BV_{loc}, then the inverse transformation has the following alternate form

$$x(t) = \frac{y(1)}{t^2} + \frac{1}{t^2}\int_1^t udy(u).$$

Moreover, given any function $y \in BV_{\text{loc}}$ the function x defined by

$$(2) \qquad x(t) = \frac{1}{t^2}\int_1^t udy(u)$$

satisfies (1) with $y(t) - y(1)$ in place of $y(t)$. Hence if $y \in BV_{\text{loc}}$ and $y(t)$ tends to a finite limit as $t \to \infty$, then (2) defines a function $x \in (C, -1)$; and if $y \in BV$, then $x \in |C, -1|$.

LEMMA 1. *Let*

$$f_1(t) = \frac{1}{t}\int_1^t k(u)du, \quad f_2(t) = t\int_t^\infty \frac{k(u)}{u^2}du \quad \textit{for } t \geqq 1,$$

where $k \in L_{\text{loc}}$. *Then*

(a) $f_1 \in M$ *if and only if* $f_2 \in M$;
(b) $f_1 \in BV$ *if and only if* $f_2 \in BV$.

Proof. (a) Suppose that $f_1 \in M$ and that $T \geqq t \geqq 1$. Then, on integration by parts,

$$\int_t^T \frac{k(u)}{u^2}du = \frac{1}{T^2}\int_1^T k(u)du - \frac{1}{t^2}\int_1^t k(u)du + 2\int_t^T \frac{f_1(u)}{u^2}du.$$

Letting $T \to \infty$, we deduce that the integral defining $f_2(t)$ converges and that

$$(3)\quad f_2(t) = -f_1(t) + 2t\int_t^\infty \frac{f_1(u)}{u^2}du,$$

and hence that $f_2 \in M$.

Conversely, if $f_2 \in M$ then the integral defining f_2 is convergent by hypothesis and integration by parts yields, for $t \geqq 1$,

$$(4)\quad f_1(t) = \frac{1}{t}\int_1^t u^2\frac{k(u)}{u^2}du = \frac{1}{t}\int_1^\infty \frac{k(u)}{u^2}du - f_2(t) + \frac{2}{t}\int_1^t f_2(u)du,$$

and so $f_1 \in M$.

(b) Suppose that $f_1 \in BV$. Then (3) holds, and to show that $f_2 \in BV$ we have to show that

$$\gamma(t) := t\int_t^\infty \frac{f_1(u)}{u^2}du \in BV.$$

Now

$$\gamma(t) = \int_0^1 f_1\left(\frac{t}{v}\right)dv$$

and hence, if $1 \leqq t_0 < t_1 < \ldots < t_n$, then

$$\sum_{i=1}^n |\gamma(t_i) - \gamma(t_{i-1})| \leqq \int_0^1 dv \sum_{i=1}^n \left|f_1\left(\frac{t_i}{v}\right) - f_1\left(\frac{t_{i-1}}{v}\right)\right|$$

$$\leqq \int_1^\infty |df_1(t)|,$$

so that $\gamma \in BV$.

Conversely, if $f_2 \in BV$ then (4) holds and so $f_1 \in BV$ since

$$\frac{1}{t}\int_1^t f_2(u)du = \int_0^1 f_2(tv)dv \in BV$$

provided we define $f_2(u) = 0$ for $0 \leqq u < 1$.

LEMMA 2. *Let*

$$g_1(t) = \frac{1}{t}\int_1^t u\,dk(u),\ g_2(t) = t\int_t^\infty \frac{dk(u)}{u} \text{ for } t \geqq 1,$$

where $k \in BV_{\text{loc}}$. *Then*

(a) $g_1 \in M$ *if and only if* $g_2 \in M$;

(b) $g_1 \in BV$ *if and only if* $g_2 \in BV$.

The proof is similar to that of Lemma 1. An immediate consequence of Lemma 2(a) is the following:

COROLLARY. *Let* $k \in BV_{\text{loc}}$. *Then*

$$\frac{1}{t}\int_1^t u|dk(u)| \in M$$

if and only if

$$t\int_t^\infty \frac{|dk(u)|}{u} \in M.$$

Parts (b) and (c) of the following lemma are due to Tatchell [**8**, Theorem 1], and Lorentz [**5**].

LEMMA 3. *Suppose that* $y \in BV_{\text{loc}}$ *and a function* w *on* $[1, \infty]$ *is defined by the transform*

$$w(t) = \int_1^\infty K(t, u)dy(u) \tag{5}$$

where, for every $t \geqq 1$, $K(t, u)$ *is bounded and continuous as a function of* u *on* $[1, \infty]$.

(a) *If* $w \in M$ *whenever* $y \in BV$, *then there is a constant* H *such that, for all* $u \geqq 1$ *and* a.a. $t \geqq 1$,

$$|K(t, u)| \leqq H.$$

(b) *If* $w \in L$ *whenever* $y \in BV$, *then there is a constant* H *such that, for all* $u \geqq 1$,

$$\int_1^\infty |K(t, u)|dt \leqq H.$$

(c) *If* $w(t)$ *tends to a finite limit as* $t \to \infty$ *whenever* $y(t)$ *tends to a finite limit as* $t \to \infty$, *then there are constants* c, t_0, H *such that*

$$\int_c^\infty |d_uK(t, u)| \leqq H \text{ for } t \geqq t_0, \tag{6}$$

$$|K(t, u)| \leqq H \text{ for } t \geqq t_0,\ u \geqq 1, \tag{7}$$

and, for every $u \geqq 1$,

$$K(t, u) \text{ tends to a finite limit as } t \to \infty. \tag{8}$$

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Proof of (a). Using Lemma 6 in **[8]**, with the Banach space B taken to be M, and the arguments in the proof of Lemma 1 in **[8]**, we observe that the hypotheses imply that (5) defines a bounded linear operator from BV to M. Hence there is a constant H such that

$$\operatorname*{ess\,sup}_{t \geqq 1} |w(t)| \leqq H(|y(1)| + \int_1^\infty |dy(u)|)$$

for every $y \in BV$. Taking the special case

$$y(u) = \begin{cases} 0 \text{ if } 1 \leqq u \leqq v, \\ 1 \text{ if } v < u, \end{cases}$$

we get that, for every $v \geqq 1$,

$$\operatorname*{ess\,sup}_{t \geqq 1} |K(t, v)| \leqq H,$$

and this is the required result.

The arguments in the proof of Theorem 3 in **[8]** can be used to establish:

LEMMA 4. *If, for every $t > 1$, $K(t, u)$ is continuous as a function of u on $[1, t]$, and the transform*

$$w(t) = \int_1^t K(t, u)dy(u)$$

defines a function $w \in BV$ whenever $y \in BV$, then there is a constant H such that, for all $u \geqq 1$,

$$|K(u, u)| + \int_u^\infty |d_t K(t, u)| \leqq H.$$

4. Summability factors from $|C, -1|$.

THEOREM 1. *In order that $k \in (|C, -1|; (C, \lambda))$ it is necessary and sufficient that*

$$(9) \quad k \in L_{\text{loc}} \quad \textit{and} \quad \frac{1}{t}\int_1^t k(u)du \in M.$$

Proof. Sufficiency. Suppose that (9) holds, that $x \in |C, -1|$, and that y satisfies (1). Then, using the inverse transformation to (1), we have that, for $t \geqq 1$,

$$\int_1^t x(u)k(u)du = y(1)\int_1^t \frac{k(u)}{u^2}du + \int_1^t \frac{k(u)}{u^2}du \int_1^u v dy(v).$$

By Lemma 1(a), (9) implies that

$$\int_1^\infty u^{-2}k(u)du$$

is convergent and so $xk \in (C, 0)$ if and only if the second term on the right-hand side of the above identity tends to a finite limit as $t \to \infty$. By Fubini's theorem,

$$\int_1^t \frac{k(u)}{u^2}du \int_1^u vdy(v) = \int_1^t vdy(v) \int_v^t \frac{k(u)}{u^2}du$$
$$= \int_1^t vdy(v)\left(\int_v^\infty \frac{k(u)}{u^2}du - \int_t^\infty \frac{k(u)}{u^2}du\right),$$

and, using (9) and Lemma 1(a),

$$\int_1^t \left|vdy(v)\int_v^\infty \frac{k(u)}{u^2}du\right| \leqq H \int_1^\infty |dy(v)|,$$

where

$$H = \sup_{v \geqq 1} \left|v\int_v^\infty \frac{k(u)}{u^2}du\right| < \infty.$$

Hence

$$\int_1^t vdy(v) \int_v^\infty u^{-2}k(u)du$$

tends to a finite limit as $t \to \infty$. Also

$$\left|\int_1^t vdy(v) \int_t^\infty \frac{k(u)}{u^2}du\right| \leqq \frac{H}{t}\left|\int_1^t vdy(v)\right|$$
$$= H\left|y(t) - \frac{y(1)}{t} - \frac{1}{t}\int_1^t y(v)dv\right| \to 0 \text{ as } t \to \infty,$$

since $y(t)$ tends to a finite limit as $t \to \infty$. Thus $xk \in (C, 0) \subset (C, \lambda)$.

Necessity. Suppose $k \in (|C, -1|; (C, \lambda))$. For any $t > 1$,

$$\chi_{[1,t]} \in |C, -1|$$

and hence

$$k\chi_{[1,t]} \in (C, \lambda),$$

so that $k \in L_{\text{loc}}$. Given any $y \in BV$, if we define x by (2) then $x \in |C, -1|$ and, for $t \geqq 1$,

$$w(t) := \int_1^t \left(1 - \frac{v}{t}\right)^\lambda k(v)x(v)dv$$
$$= \int_1^t \left(1 - \frac{v}{t}\right)^\lambda \frac{k(v)}{v^2}dv \int_1^v udy(u)$$
$$= \int_1^t udy(u) \int_u^t \left(1 - \frac{v}{t}\right)^\lambda \frac{k(v)}{v^2}dv$$

by Fubini's theorem. Thus $w \in M$ whenever $y \in BV$, and so, by Lemma 3(a), there is a constant H such that, for all $u \geqq 1$ and a.a. $t > u$,

$$\left| u \int_u^t \left(1 - \frac{v}{t}\right)^{\lambda} \frac{k(v)}{v^2} dv \right| \leqq H.$$

Since the left-hand side is a continuous function of t, the inequality must in fact hold whenever $t \geqq u \geqq 1$. Consequently, for $1 \leqq u \leqq U \leqq t$,

$$\left| u \int_u^U \left(1 - \frac{v}{t}\right)^{\lambda} \frac{k(v)}{v^2} dv \right| \leqq 2H.$$

Letting $t \to \infty$ we obtain, by dominated convergence, that

$$\left| u \int_u^U \frac{k(v)}{v^2} dv \right| \leqq 2H$$

and hence, by Cauchy's criterion, that

$$\int_u^{\infty} v^{-2} k(v) dv$$

is convergent. Now let $U \to \infty$ to obtain that, for all $u \geqq 1$

$$\left| u \int_u^{\infty} \frac{k(v)}{v^2} dv \right| \leqq 2H$$

and hence, by Lemma 1(a), that (9) holds.

THEOREM 2. *In order that* $k \in (|C, -1|; (C, -1))$ *it is necessary and sufficient that* $k \in L_{\text{loc}}$ *and*

(10) $k \in M(c, \infty)$ *for some* $c \geqq 1$.

Proof. Sufficiency. Suppose that $k \in L_{\text{loc}}$ and that $x \in |C, -1|$. Then

$$\lim_{t \to \infty} t x(t) = 0$$

and, by Theorem 1, $xk \in (C, 0)$. Thus if (10) also holds, then

$$\operatorname*{ess\,lim}_{t \to \infty} t k(t) x(t) = 0$$

and so $xk \in (C, -1)$.

Necessity. Suppose that $k \in (|C, -1|; (C, -1))$. That $k \in L_{\text{loc}}$ follows from Theorem 1 since $(C, -1) \subset (C, 0)$. Assume (10) to be false. Then there is a strictly increasing sequence of positive integers $\{n_i\}$ and a sequence of open intervals $\{I_i\}$ such that $m(I_i) < 1$, $I_i \subset (n_i, n_{i+1})$ and

$$\operatorname*{ess\,sup}_{t \in I_i} |k(t)| > 2^i \quad \text{for } i = 1, 2, \ldots.$$

Define a function x by setting

$$x(t) = \begin{cases} \dfrac{1}{t2^i} \text{ for } t \in I_i, \ i = 1, 2, \ldots, \\ 0 \text{ for all other } t \geqq 1. \end{cases}$$

Then $tx(t) \in BV$ and

$$\int_1^\infty |x(t)|dt = \sum_{i=1}^\infty \frac{1}{2^i} \int_{I_i} \frac{dt}{t} < \sum_{i=1}^\infty \frac{1}{2^i} = 1,$$

so that $x \in |C, -1|$. On the other hand

$$\operatorname*{ess\,sup}_{t \in I_i} |tx(t)k(t)| \geqq 1 \quad \text{for } i = 1, 2, \ldots,$$

and so $xk \notin (C, -1)$. This contradicts the hypothesis that

$$k \in (|C, -1|; (C, -1)),$$

and thus (10) is necessary.

THEOREM 3. *In order that* $k \in (|C, -1|; A)$ *it is necessary and sufficient that* (9) *hold.*

Proof. Sufficiency. This follows from Theorem 1 since A is regular.

Necessity. Suppose that $k \in (|C, -1|; A)$ and that $s > 0$. Then, for all $x \in |C, -1|$,

$$\int_1^\infty e^{-sv}x(v)k(v)dv$$

is convergent so that $e^{-sv}k(v) \in (|C, -1|; (C, 0))$. Hence, by Theorem 1,

$$e^{-sv}k(v) \in L_{\text{loc}}$$

so that $k \in L_{\text{loc}}$. In addition, by Lemma 1(a), there is a constant H_s such that

$$\left| V \int_V^\infty e^{-sv}\frac{k(v)}{v^2}dv \right| \leqq H_s \quad \text{for all } V \geqq 1. \tag{11}$$

Suppose now that $y \in BV$. If x is defined by (2), then $x \in |C, -1|$ and, for $V \geqq 1$,

$$\begin{aligned} \int_1^V e^{-sv}x(v)k(v)dv &= \int_1^V e^{-sv}\frac{k(v)}{v^2}dv \int_1^v udy(u) \\ &= \int_1^V udy(u)\left(\int_u^\infty - \int_V^\infty \right) e^{-sv}\frac{k(v)}{v^2}dv \end{aligned}$$

by Fubini's theorem. Further, by (11),

$$\lim_{V\to\infty} \int_1^V udy(u) \int_V^\infty e^{-sv}\frac{k(v)}{v^2}dv = 0$$

since, on integration by parts,

$$\int_1^V udy(u) = o(V) \quad \text{as } V \to \infty.$$

Hence, if we let $V \to \infty$ in the above identity and observe that the left-hand side tends to a finite limit since $xk \in A$, we obtain

$$\int_1^\infty e^{-sv}x(v)k(v)dv = \int_1^\infty udy(u) \int_u^\infty e^{-sv}\frac{k(v)}{v^2}dv,$$

the left-hand side being a continuous bounded function of s in $(0, \infty)$ whenever $y \in BV$. It follows, by Lemma 3(a), that there is a constant H such that for all $u \geqq 1$ and a.a. $s > 0$

$$\left| u \int_u^\infty e^{-sv}\frac{k(v)}{v^2}dv \right| \leqq H.$$

By continuity the inequality holds for all $s > 0$, and hence if $1 \leqq u \leqq U$ then

$$\left| u \int_u^U e^{-sv}\frac{k(v)}{v^2}dv \right| \leqq 2H$$

and so, letting $s \to 0+$,

$$\left| u \int_u^U \frac{k(v)}{v^2}dv \right| \leqq 2H.$$

Hence, by Cauchy's criterion,

$$\int_u^\infty v^{-2}k(v)du$$

is convergent, and letting $U \to \infty$ we obtain that, for all $u \geqq 1$,

$$\left| u \int_u^\infty \frac{k(v)}{v^2}dv \right| \leqq 2H$$

so that, by Lemma 1(a), (9) holds.

THEOREM 4. *In order that* $k \in (|C, -1|; |C, -1|)$ *it is necessary and sufficient that*

$$(12) \quad k \in M \cap BV_{\text{loc}} \quad \textit{and} \quad \frac{1}{t}\int_1^t u|dk(u)| \in M.$$

Proof. Sufficiency. Suppose that (12) holds and that $x \in |C, -1|$. Then $x \in L$ and $k \in M$ imply that $xk \in L$. In order to show that $xk \in |C, -1|$

it remains to prove that $tx(t)k(t) \in BV$. Using the inverse transformation to (1) we have, for $t \geqq 1$

$$tx(t)k(t) = y(1)\frac{k(t)}{t} + \frac{k(t)}{t}\int_1^t udy(u)$$

where $y \in BV$. Since

$$\int_1^\infty \left|d\left(\frac{k(t)}{t}\right)\right| \leqq \int_1^\infty \frac{|k(t)|}{t^2}dt + \int_1^\infty \frac{|dk(t)|}{t} < \infty$$

by (12) and the corollary to Lemma 2, it remains to show that $\gamma \in BV$ where

$$\gamma(t) = \frac{k(t)}{t}\int_1^t udy(u).$$

Let $1 = t_0 < t_1 < \ldots < t_n$. Then, for $i = 1, 2, \ldots, n$

$$\gamma(t_i) - \gamma(t_{i-1}) = \alpha_i + \beta_i$$

where

$$\alpha_i = \left(\frac{k(t_i)}{t_i} - \frac{k(t_{i-1})}{t_{i-1}}\right)\int_1^{t_{i-1}} udy(u)$$

and

$$\beta_i = \frac{k(t_i)}{t_i}\int_{t_{i-1}}^{t_i} udy(u).$$

Now

$$\sum_{i=1}^n |\beta_i| \leqq K\sum_{i=1}^n \frac{1}{t_i}\int_{t_{i-1}}^{t_i} u|dy(u)|$$

$$\leqq K\int_1^{t_n} |dy(u)| \leqq K\int_1^\infty |dy(u)|$$

where

$$K = \sup_{t \geqq 1} |k(t)|.$$

Further

$$\sum_{i=1}^n |\alpha_i| \leqq \sum_{i=1}^n \left|\frac{k(t_i)}{t_i} - \frac{k(t_{i-1})}{t_{i-1}}\right| \int_1^{t_{i-1}} u|dy(u)|$$

$$= \int_1^{t_{n-1}} u|dy(u)| \sum_{i=i_u}^n \left|\frac{k(t_i)}{t_i} - \frac{k(t_{i-1})}{t_{i-1}}\right|$$

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where i_u is the index such that $t_{i_u-2} < u \leqq t_{i_u-1}$. But for $1 \leqq u \leqq t_{n-1}$,

$$u \sum_{i=i_u}^{n} \left| \frac{k(t_i)}{t_i} - \frac{k(t_{i-1})}{t_{i-1}} \right| \leqq u \int_u^\infty \left| d\left(\frac{k(t)}{t} \right) \right|$$

$$\leqq u \int_u^\infty \frac{|k(t)|}{t^2} dt + u \int_u^\infty \frac{|dk(t)|}{t},$$

which is bounded for $u \geqq 1$ in view (12) and the corollary to Lemma 2. Hence there is a constant c such that

$$\sum_{i=1}^{n} |\gamma(t_i) - \gamma(t_{i-1})| \leqq \sum_{i=1}^{n} |\alpha_i| + \sum_{i=1}^{n} |\beta_i| \leqq c$$

for all choices of $t_0, t_1, \dots, t_n$, and so $\gamma \in BV$.

Necessity. Suppose that $k \in (|C, -1|; |C, -1|)$. For any $t > 1$,

$$\chi_{[1,t]} \in |C, -1|$$

so that

$$k\chi_{[1,t]} \in |C, -1|.$$

Hence $uk(u) \in BV_{\text{loc}}$ and so $k \in BV_{\text{loc}}$. Given any $y \in BV$, define x by (2) so that $x \in |C, -1|$ and, for $t \geqq 1$,

$$w(t) := tx(t)k(t) = \frac{k(t)}{t} \int_1^t u dy(u).$$

Then $w \in BV$ whenever $y \in BV$ and consequently, by Lemma 4, there is a constant H such that, for all $u \geqq 1$,

$$|k(u)| + u \int_u^\infty \left| d\left(\frac{k(t)}{t} \right) \right| \leqq H,$$

and so

$$u \int_u^\infty \frac{|dk(t)|}{t} \leqq u \int_u^\infty \frac{|k(t)|}{t^2} dt + u \int_u^\infty \left| d\left(\frac{k(t)}{t} \right) \right| \leqq 2H.$$

It follows, by the corollary to Lemma 2, that (12) holds.

THEOREM 5. *In order that $k \in (|C, -1|; |C, 0|)$ it is necessary and sufficient that*

$$(13) \quad k \in L_{\text{loc}} \quad \text{and} \quad \frac{1}{t} \int_1^t |k(u)| du \in M.$$

Proof. Sufficiency. Suppose that (13) holds and that $x \in |C, -1|$. By Lemma 1(a) there is a constant H such that, for all $t \geqq 1$,

$$t\int_t^\infty \frac{|k(u)|}{u^2}du \leqq H.$$

Using the inverse transformation to (1), we have, for $u \geqq 1$,

$$x(u)k(u) = y(1)\frac{k(u)}{u^2} + \frac{k(u)}{u^2}\int_1^u tdy(t),$$

where $y \in BV$. Hence

$$\int_1^\infty |x(u)k(u)|du \leqq H|y(1)| + \int_1^\infty t|dy(t)|\int_t^\infty \frac{|k(u)|}{u^2}du$$
$$\leqq H|y(1)| + H\int_1^\infty |dy(t)| < \infty.$$

Thus $xk \in |C, 0| = L$.

Necessity. Suppose that $k \in (|C, -1|; |C, 0|)$. Since $|C, 0| \subset (C, 0)$ it follows, by Theorem 1, that $k \in L_{\text{loc}}$. Given any $y \in BV$, define x by (2) so that $x \in |C, -1|$ and for $t \geqq 1$,

$$w(t) := \int_1^t x(v)k(v)dv$$
$$= \int_1^t \frac{k(v)}{v^2}dv\int_1^v udy(u)$$
$$= \int_1^t udy(u)\int_u^t \frac{k(v)}{v^2}dv.$$

Then $w \in BV$ whenever $y \in BV$ and so, by Lemma 4, there is a constant H such that, for all $u \geqq 1$,

$$u\int_u^\infty \frac{|k(v)|}{v^2}dv \leqq H.$$

By Lemma 1(a), this implies (15).

THEOREM 6. *In order that $k \in (|C, -1|; |C, \lambda|)$ where $\lambda > 0$ it is necessary and sufficient that $k \in L_{\text{loc}}$ and that there be a constant H such that, for all $t \geqq 1$,*

$$(14)\quad t\int_t^\infty \frac{du}{u^{\lambda+1}}\left|\int_t^u (u-v)^{\lambda-1}\frac{k(v)}{v}dv\right| \leqq H.$$

Proof. Sufficiency. It is shown in [3] that $a \in |C, \lambda|$ if and only if $a \in L_{\text{loc}}$ and

$$(15)\quad \int_1^\infty \frac{du}{u^{\lambda+1}}\left|\int_1^u (u-v)^{\lambda-1}va(v)dv\right| < \infty.$$

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Suppose that $x \in |C, -1|$ and that (14) holds. As before we have, for $u \geqq 1$,

$$(16) \quad x(u)k(u) = y(1)\frac{k(u)}{u^2} + \frac{k(u)}{u^2}\int_1^u tdy(t)$$

where $y \in BV$. It follows from (14) with $t = 1$ that (15) holds with $a(v) = v^{-2}k(v)$ so that $u^{-2}k(u) \in |C, \lambda|$. To show that the other term on the right-hand side of (16) is also in $|C, \lambda|$ we observe that, in view of (14),

$$\int_1^\infty \frac{du}{u^{\lambda+1}}\left|\int_1^u (u-v)^{\lambda-1}\frac{k(v)}{v}dv\int_1^v tdy(t)\right|$$
$$= \int_1^\infty \frac{du}{u^{\lambda+1}}\left|\int_1^u tdy(t)\int_t^u (u-v)^{\lambda-1}\frac{k(v)}{v}dv\right|$$
$$\leqq \int_1^\infty t|dy(t)|\int_t^\infty \frac{du}{u^{\lambda+1}}\left|\int_t^u (u-v)^{\lambda-1}\frac{k(v)}{v}dv\right|$$
$$\leqq H\int_1^\infty |dy(t)| < \infty.$$

Thus (15) holds with $a(v) = x(v)k(v)$, and so $xk \in |C, \lambda|$.

Necessity. Suppose that $k \in (|C, -1|; |C; \lambda|)$. Then

$$k \in (|C, -1|; (C, \lambda))$$

and so, by Theorem 1, $k \in L_{\text{loc}}$. Given any $y \in BV$, if we define x by (2) then $x \in |C, -1|$ and, for a.a. $t \geqq 1$,

$$w(t) := \frac{1}{t^{\lambda+1}}\int_1^t (t-v)^{\lambda-1}vx(v)k(v)dv$$
$$= \frac{1}{t^{\lambda+1}}\int_1^t (t-v)^{\lambda-1}\frac{k(v)}{v}dv\int_1^v udy(u)$$
$$= \frac{1}{t^{\lambda+1}}\int_1^t udy(u)\int_u^t (t-v)^{\lambda-1}\frac{k(v)}{v}dv$$
$$= \int_1^\infty K(t, u)dy(u)$$

where

$$K(t, u) = \frac{u}{t^{\lambda+1}}\int_u^t (t-v)^{\lambda-1}\frac{k(v)}{v}dv$$

whenever $1 \leqq u \leqq t$ and the integral exists in the Lebesgue sense, and $K(t, u) = 0$ otherwise. (Note that the integral may fail to exist on a set of measure zero when $0 < \lambda < 1$.) Then $w \in L$ whenever $y \in BV$ and so, by Lemma 3(b), there is a constant H such that, for all $u \geqq 1$,

$$H \geqq \int_1^\infty |K(t, u)|dt = u \int_u^\infty \frac{dt}{t^{\lambda+1}} \left| \int_u^t (t - v)^{\lambda-1} \frac{k(v)}{v} dv \right|,$$

i.e., (14) holds.

THEOREM 7. *If* $\lambda \geqq 1$ *then*

$$(|C, -1|; |C, \lambda|) = (|C, -1|; (C, 0)).$$

Proof. For any $\lambda \geqq 0$,

$$(|C, -1|; |C, \lambda|) \subset (|C, -1|; (C, \lambda)) = (|C, -1|; (C, 0))$$

by Theorem 1. To complete the proof we have to show that

$$(|C, -1|; (C, 0)) \subset (|C, -1|; |C, 1|),$$

since $|C, 1| \subset |C, \lambda|$ for $\lambda \geqq 1$. By Theorems 1 and 6 this means that we have to show that (9) or, by Lemma 1(a), that $k \in L_{\text{loc}}$ and

$$(17) \quad \left| t \int_t^\infty \frac{k(u)}{u^2} du \right| \leqq H \quad \text{for } t \geqq 1$$

implies that

$$(18) \quad t \int_t^\infty \frac{du}{u^2} \left| \int_t^u \frac{k(v)}{v} dv \right| \leqq H_1 \quad \text{for } t \geqq 1,$$

H and H_1 being constants. Suppose that (17) holds. Then, on integrating by parts,

$$\int_t^u \frac{k(v)}{v} dv = t \int_t^\infty \frac{k(w)}{w^2} dw - u \int_u^\infty \frac{k(w)}{w^2} dw + \int_t^u dv \int_v^\infty \frac{k(w)}{w^2} dw$$

and hence, by (17), we have, for $t \geqq 1$,

$$t \int_t^\infty \frac{du}{u^2} \left| \int_t^u \frac{k(v)}{v} dv \right| \leqq 2Ht \int_t^\infty \frac{du}{u^2} + Ht \int_t^\infty \frac{du}{u^2} \int_t^u \frac{dv}{v} \leqq 2H + H = 3H,$$

i.e., (18) holds.

THEOREM 8. $(|C, -1|; |A|) = (|C, -1|; (C, 0))$.

Proof. Suppose that $k \in (|C, -1|; (C, 0))$ and that $x \in |C, -1|$. Then, by Theorem 7, $xk \in (C, 0) \cap |C, 1|$, and so, by Theorem 3 in [7], $xk \in |A|$. Hence

$$(|C, -1|; (C, 0)) \subset (|C, -1|; |A|) \subset (|C, -1|; A).$$

Further, by Theorems 1 and 3,

$$(|C, -1|; A) = (|C, -1|; (C, 0)),$$

and the required identity follows.

5. Summability factors from $(C, -1)$.

THEOREM 9. *In order that* $k \in ((C, -1); (C, \lambda))$ *it is necessary and sufficient that*

$$(19)\quad k \in M_{\text{loc}} \quad \textit{and} \quad \frac{1}{t}\int_1^t k(u)du \in BV.$$

Proof. Sufficiency. Suppose that (19) holds and that $x \in (C, -1)$. Let

$$\alpha(t) = \frac{d}{dt}\left(\frac{1}{t}\int_1^t k(u)du\right) \quad \text{and} \quad \beta(t) = \int_1^t \alpha(u)du,$$

so that, for a.a. $t \geqq 1$,

$$(20)\quad k(t) = t\alpha(t) + \beta(t).$$

By (19), $\alpha \in L \cap M_{\text{loc}}$ and $\beta \in BV$. It follows, since

$$\operatorname*{ess\,lim}_{t\to\infty} tx(t) = 0,$$

that

$$tx(t)\alpha(t) \in (C, 0)$$

and, since $x \in (C, 0)$, that

$$x\beta \in (C, 0).$$

Hence, by (20),

$$xk \in (C, 0) \subset (C, \lambda).$$

Necessity. Suppose that $k \in ((C, -1); (C, \lambda))$. For $T > 1$, if $x \in L(1, T)$ and $x(t) = 0$ for $t > T$, then $x \in (C, -1)$ so that $xk \in (C, \lambda)$ and in particular $xk \in L(1, T)$. Hence, by a theorem of Lebesgue, it is necessary that $k \in M_{\text{loc}}$. Now suppose that $y \in BV_{\text{loc}}$ and that $y(t)$ tends to a finite limit as $t \to \infty$. If x is defined by (2), then $x \in (C, -1)$ and, as in the proof of the necessity part of Theorem 1, for $t \geqq 1$,

$$w(t) := \int_1^t \left(1 - \frac{v}{t}\right)^\lambda k(v)x(v)dv = \int_1^\infty K(t, u)dy(u)$$

where

$$K(t, u) = \begin{cases} u\displaystyle\int_u^t \left(1 - \frac{v}{t}\right)^\lambda \frac{k(v)}{v^2}dv & \text{if } 1 \leqq u \leqq t, \\ 0 \text{ if } u > t. \end{cases}$$

Then $w(t)$ tends to a finite limit as $t \to \infty$ whenever $y \in BV_{\text{loc}}$ and $y(t)$ tends to a finite limit as $t \to \infty$. Hence, by Lemma 3(c), K satisfies (6), (7) and (8). As in the proof of Theorem 1, (7) implies that

$$\int_1^\infty v^{-2}k(v)dv$$

is convergent, so that, for $u \geqq 1$,

$$\lim_{t\to\infty} K(t, u) = u\int_u^\infty \frac{k(v)}{v^2}dv =: \gamma(u).$$

Now, by (6), whenever $c = u_0 < u_1 < \ldots < u_n$ and $t \geqq t_0$ we have

$$\sum_{i=1}^n |K(t, u_i) - K(t, u_{i-1})| \leqq H,$$

and hence

$$\sum_{i=1}^n |\gamma(u_i) - \gamma(u_{i-1})| \leqq H.$$

It follows that $\gamma \in BV$ and so, by Lemma 1(b), (19) holds.

THEOREM 10. *In order that* $k \in ((C, -1); (C, -1))$ *it is necessary and sufficient that* (19) *hold and that* $k \in M$.

Proof. Sufficiency. Suppose $x \in (C, -1)$. Then, by Theorem 9, (19) implies that $xk \in (C, 0)$, and $k \in M$ implies that

$$\operatorname*{ess\,lim}_{t\to\infty} tx(t)k(t) = 0.$$

Thus (19) and $k \in M$ imply that $xk \in (C, -1)$.

Necessity. Suppose $k \in ((C, -1); (C, -1))$. Since

$$|C, -1| \subset (C, -1) \subset (C, 0),$$

it follows from Theorem 9 that $k \in M_{\text{loc}}$ and from Theorem 2 that $k \in M(c, \infty)$ for some $c \geqq 1$. Hence $k \in M$.

THEOREM 11. *In order that* $k \in ((C, -1); A)$ *it is necessary and sufficient that* (19) *hold.*

Proof. Sufficiency. This follows from Theorem 9 since $(C, 0) \subset A$.

Necessity. Suppose that $k \in ((C, -1); A)$ and that $s > 0$. Then

$$e^{-sv}k(v) \in ((C, -1); (C, 0))$$

so that, by Theorem 9, $k \in M_{\text{loc}}$. Suppose now that $y \in BV_{\text{loc}}$ and that $y(t)$ tends to a finite limit as $t \to \infty$. If x is defined by (2), then $x \in (C, -1)$ and, as in the proof of the necessity part of Theorem 3,

$$\int_1^\infty e^{-sv}x(v)k(v)dv = \int_1^\infty udy(u)\int_u^\infty e^{-sv}\frac{k(v)}{v^2}dv.$$

Since the left-hand side tends to a finite limit as $s \to 0+$ whenever $y \in BV_{\text{loc}}$ and $y(t)$ tends to a finite limit as $t \to \infty$, it follows, by Lemma 3(c), that (6), (7) and (8) hold with

$$K(t, u) = u\int_u^\infty e^{-v/t}\frac{k(v)}{v^2}dv.$$

It follows from (7), as in the proof of Theorem 3, that

$$\lim_{t\to\infty} K(t, u) = u\int_u^\infty \frac{k(v)}{v^2}dv$$

and then from (6), as in the proof of Theorem 9, that

$$u\int_u^\infty v^{-2}k(v)dv \in BV.$$

Thus, by Lemma 1(b), (19) must hold.

THEOREM 12. *In order that* $k \in ((C, -1); |C, 0|)$ *it is necessary and sufficient that*

$$(21)\quad k \in M_{\text{loc}} \quad \textit{and} \quad \frac{k(t)}{t} \in L.$$

Proof. Sufficiency. Suppose that (21) holds and that $x \in (C, -1)$. Then there is a number $c \geqq 1$ such that $tx(t) \in M(c, \infty)$, and hence

$$\int_1^\infty |x(t)k(t)|dt = \int_1^c |x(t)k(t)|dt + \int_c^\infty \left|tx(t)\frac{k(t)}{t}\right|dt < \infty.$$

Thus $xk \in (C, 0)$.

Necessity. Suppose $k \in ((C, -1); |C, 0|)$. Then $k \in ((C, -1); (C, 0))$ so that $k \in M_{\text{loc}}$ by Theorem 9. Let $\{\alpha_n\}$ be any sequence of positive numbers decreasing to 0, and define

$$x(t) = \frac{(-1)^n\alpha_n}{t} \quad \text{for } n \leqq t < n + 1, n = 1, 2, \dots$$

Then $tx(t) \to 0$ as $t \to \infty$ and, for $2 \leqq N \leqq T < N + 1$,

$$\int_1^T x(t)dt = \sum_{n=1}^{N-1}(-1)^n\alpha_n\log\left(1 + \frac{1}{n}\right) + (-1)^N\alpha_N\log\left(\frac{T}{N}\right)$$

so that $x \in (C, 0)$ by Leibniz's test. Hence $x \in (C, -1)$ and so $xk \in |C, 0|$, i.e.,

$$\sum_{n=1}^{\infty} \alpha_n \int_n^{n+1} \frac{|k(t)|}{t} dt < \infty.$$

Since $\{\alpha_n\}$ could be any sequence decreasing to 0, we must have

$$\sum_{n=1}^{\infty} \int_n^{n+1} \frac{|k(t)|}{t} dt = \int_1^{\infty} \frac{|k(t)|}{t} dt < \infty$$

(otherwise

$$\alpha_n = 1/\left(1 + \int_n^{n+1} \frac{|k(t)|}{t} dt\right)$$

would give an example of a function $x \in (C, -1)$ for which $xk \notin |C, 0|$). Thus (21) must hold.

The proof of the next theorem is an adaptation of the proof for the series analogue given in [**1**].

THEOREM 13. *In order that* $k \in ((C, -1); |A|)$ *it is necessary and sufficient that* (21) *hold.*

Proof. Sufficiency. This follows from Theorem 12 since $|C, 0| \subset |A|$.

Necessity. Suppose $k \in ((C, -1); |A|)$ and define $\alpha(t)$, $\beta(t)$ as in (20). Since $|A| \subset A$, we have, by Theorem 11, that $\alpha \in L \cap M_{\text{loc}}$ and $\beta \in BV$. Suppose that $s > 0$ and that $x \in (C, -1)$. For $t \geqq 1$, let

(22) $$x_1(t) = \int_1^t x(u)du, \quad y(t) = x_1(t) + tx(t).$$

Then

$$\int_1^{\infty} e^{-st}x(t)k(t)dt = \int_1^{\infty} e^{-st}x(t)\beta(t)dt + \int_1^{\infty} e^{-st}tx(t)\alpha(t)dt.$$

The first integral is of bounded variation over $(0, \infty)$ by hypothesis, and so is the final integral since

$$\int_1^{\infty} |tx(t)\alpha(t)|dt < \infty.$$

Hence

$$I(s) := \int_1^{\infty} e^{-st}x(t)\beta(t)dt \in BV(0, \infty).$$

On integration by parts,

$$I(s) = \int_1^{\infty} x_1(t)e^{-st}(\alpha(t) - s\beta(t))dt$$

and

$$\int_1^{\infty} e^{-st}x_1(t)\alpha(t)dt \in BV(0, \infty)$$

since

$$\int_1^\infty |x_1(t)\alpha(t)|dt < \infty.$$

Thus

$$I_1(s) = s\int_1^\infty e^{-st}x_1(t)\beta(t)dt \in BV(0, \infty).$$

Further

$$I_1(s) = s\int_1^\infty e^{-st}\frac{\beta(t)}{t}dt \int_1^t y(u)du$$

$$= s\int_1^\infty y(u)du \int_u^\infty e^{-st}\frac{\beta(t)}{t}dt,$$

the interchange in order of integration being justified since

$$\int_1^\infty |y(u)|du \int_u^\infty e^{-st}\frac{|\beta(t)|}{t}dt$$

$$\leqq \sup_{u \geqq 1}\left(\frac{1}{u}\int_1^u |y(v)|dv\right) \cdot \int_1^\infty e^{-st}|\beta(t)|dt < \infty.$$

Next

$$I_1(s) = I_2(s) + I_3(s)$$

where

$$I_2(s) = s\int_1^\infty y(u)du \int_u^\infty \frac{e^{-st}}{t}(\beta(t) - \beta(u))dt,$$

$$I_3(s) = s\int_1^\infty y(u)\beta(u)du \int_u^\infty \frac{e^{-st}}{t}dt.$$

We prove first that $I_2 \in BV(0, \infty)$. Observe that

$$\int_0^\infty \left|\frac{\partial}{\partial s}(se^{-st})\right|ds = \int_0^{1/t}\frac{\partial}{\partial s}(se^{-st})ds - \int_{1/t}^\infty \frac{\partial}{\partial s}(se^{-st})ds = \frac{2}{et},$$

so that

$$\int_0^\infty |I_2'(s)|ds \leqq \frac{2}{e}\int_1^\infty |y(u)|du \int_u^\infty |\beta(t) - \beta(u)|\frac{dt}{t^2}$$

$$\leqq \frac{2}{e}\int_1^\infty |y(u)|du \int_u^\infty \frac{dt}{t^2}\int_u^\infty |d\beta(v)|$$

$$= \frac{2}{e}\int_1^\infty |d\beta(v)|\frac{1}{v}\int_1^v |y(u)|du < \infty.$$

Thus $I_2 \in BV(0, \infty)$ and so $I_3 \in BV(0, \infty)$ whenever $x \in (C, -1)$. For $u \geqq 1$, let

$$w_s(u) = \frac{\partial}{\partial s}\left(\int_u^\infty \frac{se^{-st}}{t}dt\right) = \int_u^\infty \frac{e^{-st}}{t}dt - e^{-su},$$

so that

$$I_3'(s) = \int_1^\infty y(u)\beta(u)w_s(u)du$$

and hence

$$\int_0^\infty ds\left|\int_1^\infty y(u)\beta(u)w_s(u)du\right| < \infty$$

whenever $y \in B$ where B denotes the Banach space of bounded measurable functions z on $[1, \infty)$ such that $z(t)$ tends to a finite limit as $t \to \infty$, the norm being given by

$$\|z\| = \sup_{t \geqq 1} |z(t)|.$$

This is so, since if $y \in B$ and x is defined by

$$x(t) = \frac{d}{dt}\left(\frac{1}{t}\int_1^t y(u)du\right) \text{ for a.a. } t > 1,$$

then (22) holds for a.a. $t > 1$ and $x \in (C, -1)$. For $n \geqq 1$ and each fixed $s > 0$, define a continuous linear functional $f_n : B \to \mathbf{C}$ by

$$f_n(y) = \int_1^n y(u)\beta(u)w_s(u)du.$$

Since, for every $y \in B$,

$$I_3'(s) = \lim_{n\to\infty} f_n(y) = \int_1^\infty y(u)\beta(u)w_s(u)du,$$

this defines a continuous linear functional on B for each fixed $s > 0$. Hence, by Lemma 1 in [**8**], since $I_3' \in L(0, \infty)$ there is a constant H such that, for every $y \in B$,

$$(23)\quad \int_0^\infty ds\left|\int_1^\infty y(u)\beta(u)w_s(u)du\right| \leqq H \sup_{u \geqq 1} |y(u)|.$$

Now take

$$y(u) = \begin{cases} \dfrac{|a(u)|}{a(u)} \operatorname{sgn} \beta(u) \text{ for } 1 \leqq u \leqq m, \\ 0 \text{ for } u > m, \end{cases}$$

where

$$a(u) = \int_0^\infty s^i w_s(u)ds = \frac{-\Gamma(2+i)}{2u^{1+i}} \quad \text{for } u \geqq 1.$$

Then

$$\begin{aligned}\frac{|\Gamma(2+i)|}{2}\int_1^m \frac{|\beta(u)|}{u}du &= \int_1^m a(u)y(u)\beta(u)du \\ &= \int_1^m y(u)\beta(u)du \int_0^\infty s^i w_s(u)ds \\ &= \int_0^\infty s^i ds \int_1^m y(u)\beta(u)w_s(u)du \\ &\leqq \int_0^\infty ds \left|\int_1^m y(u)\beta(u)w_s(u)du\right| \\ &\leqq H \sup_{m \geqq u \geqq 1} |y(u)| \leqq H\end{aligned}$$

by (23), the inversion in order of integration being justified since

$$\int_1^m |\beta(u)|du \int_0^\infty |w_s(u)|ds \leqq 2\int_1^m \frac{|\beta(u)|}{u}du < \infty.$$

It follows that

$$\frac{|\Gamma(2+i)|}{2}\int_1^\infty \frac{|\beta(u)|}{u}du \leqq H$$

and hence, by (20), that (21) holds.

THEOREM 14. $((C, -1); |A|) = ((C, -1); |C, \lambda|)$.

Proof. Suppose $k \in ((C, -1); |C, \lambda|)$ and that $x \in (C, -1)$. Then

$$k \in ((C, -1); (C, \lambda)) = ((C, -1); (C, 0)),$$

by Theorem 1. Thus $xk \in (C, 0) \cap |C, \lambda|$ and so, by Theorem 3 in [7], $xk \in |A|$, so that

$$k \in ((C, -1); |A|).$$

Hence

$$((C, -1); |C, \lambda|) \subset ((C, -1); |A|).$$

Further, by Theorems 12 and 13,

$$((C, -1); |A|) = ((C, -1); |C, 0|) \subset ((C, -1); |C, \lambda|),$$

and the required identity follows.

THEOREM 15. *In order that* $k \in ((C, -1); |C, -1|)$ *it is necessary and sufficient that* $k(t) = 0$ *for all* $t \geqq 1$.

Proof. Sufficiency. This is immediate provided we adopt the convention that $x(t)k(t) = 0$ whenever $k(t) = 0$ even when $x(t)$ is infinite or undefined.

Necessity. Suppose $k(c) \neq 0$ for some $c \geqq 1$. Define $x(t) = 0$ for $t \geqq 1$, $t \neq c$ and $x(c) = \infty$. Then $x \in (C, -1)$ but $xk \notin |C, -1|$ since

$$tx(t)k(t) \notin BV.$$

In Theorem 15 the necessity for k to be identically zero is a trivial consequence of the definition of $|C, -1|$. The following theorem shows that the condition on k cannot be significantly relaxed by enlarging the space $|C, -1|$ in a natural way.

THEOREM 16. *In order that*

$$\int_1^t x(u)k(u)du + tx(t)k(t)$$

be equivalent to a function in BV *whenever* $x \in (C, -1)$ *it is necessary and sufficient that*

$$(24) \quad k(t) = 0 \quad \textit{for a.a. } t \geqq 1.$$

Proof. Sufficiency. This is immediate.

Necessity. Assume (24) to be false. Then there is a strictly increasing sequence $\{c_n\}$, a sequence of positive numbers $\{\epsilon_n\}$, and a sequence of measurable sets $\{E_n\}$, such that $c_1 \geqq 1$, and, for $n = 1, 2, \ldots,$

$$|k(t)| \geqq \epsilon_n \quad \text{for } t \in E_n \subset (c_n, c_{n+1}) \text{ and } 0 < m(E_n) \leqq 1.$$

Let r_n be an even positive integer such that

$$\epsilon_n r_n \geqq n,$$

and define

$$E_{n,i} = (c_{n,i-1}, c_{n,i}) \cap E_n$$

where

$$c_n = c_{n,0} < c_{n,1} < \ldots < c_{n,r_n} = c_{n+1},$$

the numbers $c_{n,i}$ being chosen so that

$$\int_{E_{n,i}} \frac{dt}{tk(t)}$$

is constant for $i = 1, 2, \ldots, r_n$.

Now define

$$x(t) = \begin{cases} \dfrac{(-1)^i \epsilon_n}{ntk(t)} \text{ for } t \in E_{n,i},\ i = 1, 2, \ldots, r_n,\ n = 1, 2, \ldots, \\ 0 \text{ for other } t \geqq 1. \end{cases}$$

Then

$$\int_{c_n}^{c_{n+1}} x(t)dt = 0$$

and, for $c_n < T < c_{n+1}$,

$$\left| \int_{c_n}^{T} x(t)dt \right| \leqq \frac{m(E_n)}{n} \leqq \frac{1}{n},$$

so that

$$\int_1^T x(t)dt \to 0 \text{ as } T \to \infty.$$

Also,

$$|tx(t)| \leqq 1/n \quad \text{for } c_n \leqq t < c_{n+1},$$

and so

$$tx(t) \to 0 \quad \text{as } t \to \infty.$$

Hence $x \in (C, -1)$. On the other hand if $\gamma(t) = tx(t)\,k(t)$ for a.a. $t \geqq 1$, then the variation

$$V_{c_n}^{c_{n+1}}(\gamma(t)) \geqq \frac{2\epsilon_n r_n}{n} \geqq 2,$$

and so $\gamma \notin BV$. Thus (24) is necessary.

Remark. Theorem 12 is a special case of Theorem 14 and is not needed to prove Theorem 14. The proof of Theorem 12 has been included since it is much simpler than that of Theorem 14 which uses the necessity part of Theorem 13 in an essential way.

6. Summary. Collecting the above results we obtain:

$$\text{I.} \quad (|C, -1|; (C, \lambda)) = (|C, -1|; A)$$
$$= (|C, -1|; |C, \lambda + 1|)$$
$$= (|C, -1|; |A|) =: S,$$

and

$$k \in S \Leftrightarrow k \in L_{\text{loc}}$$

and

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$$\frac{1}{t}\int_1^t k(u)du \in M.$$

II. $((C, -1); (C, \lambda)) = ((C, -1); A) =: T,$

and

$$k \in T \Leftrightarrow k \in M_{\text{loc}}$$

and

$$\frac{1}{t}\int_1^t k(u)du \in BV.$$

III. $((C, -1); |C, \lambda|) = ((C, -1); |A|) =: U,$

and

$$k \in U \Leftrightarrow k \in M_{\text{loc}}$$

and

$$\frac{k(t)}{t} \in L.$$

IV. $k \in (|C, -1|; (C, -1)) \Leftrightarrow k \in L_{\text{loc}} \cap M(c, \infty)$ for some $c \geqq 1$.

V. $k \in (|C, -1|; |C, -1|) \Leftrightarrow k \in BV_{\text{loc}} \cap M$

and

$$\frac{1}{t}\int_1^t u|dk(u)| \in M.$$

VI. $k \in (|C, -1|; |C, 0|) \Leftrightarrow k \in \mathrm{L}_{\text{loc}}$

and

$$\frac{1}{t}\int_1^t |k(u)|du \in M$$

VII. $k \in (|C, -1|; |C, \lambda|)$ for $0 < \lambda < 1 \Leftrightarrow k \in L_{\text{loc}}$

and

$$t\int_t^\infty \frac{du}{u^{\lambda+1}}\left|\int_t^u (u-v)^{\lambda-1}\frac{k(v)}{v}dv\right| \in M.$$

VIII. $k \in ((C, -1); (C, -1)) \Leftrightarrow k \in M$

and

$$\frac{1}{t}\int_1^t k(u)du \in BV.$$

IX. $k \in ((C, -1); |C, -1|) \Leftrightarrow k(t) = 0$ for all $t \geqq 1$.

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