SOME SERIES OF PARTIALLY BALANCED INCOMPLETE BLOCK DESIGNS

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1. Introduction. The use of incomplete block designs for estimating and judging the significance of the difference of treatment effects is now a standard statistical technique. A special kind of incomplete block design is the Partially Balanced Incomplete Block Design (PBIBD) introduced in **(3)**. A PBIBD is an incomplete block design that obeys the following conditions:

(1) there are b blocks of k distinct varieties each;

(2) there are v varieties, each replicated r times;

(3) given any variety, the remaining ones fall into sets of n_i varieties each $(i = 1, 2, \ldots, m)$ such that every variety of the ith set occurs λ_i times with the given variety, λ_i being independent of the given initial variety;

(4) given any two varieties which are ith associates (that is, occur together λ_i times), the number of varieties which are jth associates of the one and kth associates of the other is

$$p_{jk}^{\ i} = p_{kj}^{\ i}$$

and is independent of the original pair of varieties.

From the definition of a PBIBD, certain fundamental identities concerning the parameters of the design follow; for ease of reference, these are listed as

$$bk = rv, \tag{1.1}$$

$$v - 1 = \sum_{i=1}^{m} n_i, \tag{1.2}$$

$$r(k-1) = \sum_{i=1}^{m} \lambda_i n_i, \tag{1.3}$$

$$n_i - 1 = \sum_{k=1}^{m} p_{ik}^{\ i}, \tag{1.4}$$

$$n_j = \sum_{k=1}^{m} p_{jk}^{\ i}, \tag{1.5}$$

$$n_i p_{jk}^{\ i} = n_j p_{ik}^{\ j}. \tag{1.6}$$

In **(3)**, a module theorem was proved by which it is possible in certain cases to construct the entire PBIBD by adding elements of a module to a given initial block. The purpose of the present paper is to generalize the module theorem of **(3)** to the case $v \neq b$, and then to apply the methods of **(6)** to obtain some general series of PBIBD's.

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2. The general module theorem.

THEOREM 2.1. *Let M be an additive abelian group (module) consisting of n elements x_i and suppose that there exist m blocks $B_1, B_2, \ldots, B_m$ such that:*

(1) *every block contains k distinct elements;*

(2) *among the $mk(k-1)$ differences arising from the m blocks, just n_i of the non-zero elements of M are repeated λ_i times; we may denote these n_i elements by*

$$\alpha_1^{\ i}, \alpha_2^{\ i}, \ldots, \alpha_{n_i}^{\ i};$$

(3) *among the $n_i(n_i - 1)$ differences*

$$\alpha_u^{\ i} - \alpha_w^{\ i} \qquad (u, w = 1, 2, \ldots, n_i; u \neq w)$$

every element of the set

$$\alpha_1^{\ e}, \alpha_2^{\ e}, \ldots, \alpha_{n_e}^{\ e}$$

occurs $p_{ii}^{\ e}$ times, while among the $n_i n_j$ differences

$$\alpha_u^{\ i} - \alpha_w^{\ j} \quad (u = 1, 2, \ldots, n_i; w = 1, 2, \ldots, n_j),$$

every element of the set

$$\alpha_1^{\ e}, \alpha_2^{\ e}, \ldots, \alpha_{n_e}^{\ e}$$

occurs $p_{ij}^{\ e}$ times.
Then we can form blocks

$$B_{g\theta} \qquad (g = 1, 2, \ldots, m; \theta \in M)$$

of elements x', where $x' = x + \theta$ (x ranging over the elements of B_g). Since there are m initial blocks and θ assumes n values, we thus get mn blocks $B_{g\theta}$ ($B_{g0} = B_g$), and these blocks form a PBIBD with parameters

$$v = n, b = mn, r = mk, k; n_i, \lambda_i; P_i = (p_{jk}^{\ i}) \qquad (i = 1, 2, \ldots, m).$$

Proof. The proof of Theorem 2.1 is omitted, since it is an exact parallel of the proof given in **(3)** for the special case in which $m = 1, v = b$.

In the series which we shall derive from Theorem 2.1, it will be convenient to reserve the letter p to denote a prime; M will always be a Galois Field GF(p^c), and x will always denote a primitive element of M.

3. Series with $4m(4\lambda \pm 1) + 1 = p^c$.

THEOREM 3.1. *If $v = 4m(4\lambda + 1) + 1 = p^c$, and if among the 2λ expressions*

$$x^{4ms} - 1 = x^{q_s} \qquad (s = 1, 2, \ldots, 2\lambda)$$

there are $\lambda + a$ even and $\lambda - a$ odd powers of x, then the design with parameters

$$v = 4m(4\lambda + 1) + 1, \quad b = mv, \quad r = m(4\lambda + 1), \quad k = 4\lambda + 1;$$
$$n_1 = n_2 = 2m(4\lambda + 1), \quad \lambda_1 = \lambda + a, \quad \lambda_2 = \lambda - a;$$
$$P_1 = \begin{pmatrix} \frac{1}{4}(v-5) & \frac{1}{4}(v-1) \\ \frac{1}{4}(v-1) & \frac{1}{4}(v-1) \end{pmatrix}, \quad P_2 = \begin{pmatrix} \frac{1}{4}(v-1) & \frac{1}{4}(v-1) \\ \frac{1}{4}(v-1) & \frac{1}{4}(v-5) \end{pmatrix}$$

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can be constructed from the initial blocks

$$(x^{2i}, x^{2i+4m}, \ldots, x^{2i+16\lambda m})$$

where i *ranges from* 0 *to* $m - 1$.

Proof. As in **(6**, Theorem 5.1**)**, we find, setting $d = 4\lambda - s + 1$, that

$$x^{q_d} = x^{q_s + 2m(d-s)}.$$

Divide the elements of GF(v) into two sets; set 1 will be composed of elements of the form x^{2i}, and set 2 will be composed of elements of the form x^{2i+1}. The differences

$$x^{2i+4mr+q_s}, x^{2i+2m(4\lambda-2s+1+2r)+q_s},$$

for s fixed, range over set 1 or set 2 according as q_s is even or odd. Hence, as s ranges, each element of set 1 occurs $\lambda_1 = \lambda + a$ times and each element of set 2 occurs $\lambda_2 = \lambda - a$ times. Also, the number of elements in set 1 = the number of elements in set 2 = $\frac{1}{2}(v - 1) = 2m(4\lambda + 1)$. Thus condition 2 of Theorem 2.1 is satisfied and $n_1 = n_2 = 2m(4\lambda + 1)$.

The differences between elements of set 1 have the form

$$x^{2t+2u} - x^{2t} = x^{2t}(x^{2u} - 1) \quad (t = 0, 1, \ldots, \tfrac{1}{2}(v - 3); u = 1, 2, \ldots, \tfrac{1}{2}(v-3)).$$

As t ranges, these differences cover set 1 or set 2 according as $x^{2u} - 1$ is an even or an odd power of x. Hence, if y of the expressions $x^{2u} - 1$ are odd powers of x, we have

$$p_{11}^2 = y, \quad p_{11}^1 = \tfrac{1}{2}(v - 3) - y.$$

The differences between elements of set 2 have the form

$$x^{2t+2u+1} - x^{2t+1} = x^{2t+1}(x^{2u} - 1).$$

These cover set 1 or set 2 according as $x^{2u} - 1$ is an odd or even power of x. Hence

$$p_{22}^1 = y, \quad p_{22}^2 = \tfrac{1}{2}(v - 3) - y,$$

and condition 3 of Theorem 2.1 is satisfied.

Using the identities 1.4 and 1.5, we obtain

$$p_{11}^1 + p_{12}^1 = n_1 - 1 = \tfrac{1}{2}(v - 3), \quad p_{12}^1 = y,$$

and

$$p_{21}^1 + p_{22}^1 = n_1 = \tfrac{1}{2}(v - 1), \quad p_{21}^1 = \tfrac{1}{2}(v - 1) - y.$$

Thus $y = \frac{1}{4}(v - 1) = p_{12}^1 = p_{21}^1$; therefore P_1 and P_2 are as stated in the theorem.

If $a = 0$, the resulting series is the completely balanced series given in **(6**, Theorem 5.1**)**.

Example. If $m = 1, \lambda = 3$, we take $x = 2$ and obtain a design for 53 varieties in blocks of 13 by adding the integers modulo 53 to the initial block

$$(1, 16, 44, 15, 28, 24, 13, 49, 42, 36, 46, 47, 10).$$

THEOREM 3.2. *If* $v = 4m(4\lambda - 1) + 1 = p^c$ *and if among the* $2\lambda - 1$ *expressions*

$$x^{4ms} - 1 = x^{q_s} \qquad (s = 1, 2, \ldots, 2\lambda - 1)$$

there are $\lambda - a$ *even and* $\lambda + a - 1$ *odd powers of* x, *then the design with parameters*

$$v = 4m(4\lambda - 1) + 1, \quad b = mv, \quad r = 4m\lambda, \quad k = 4\lambda;$$
$$n_1 = n_2 = 2m(4\lambda - 1), \quad \lambda_1 = \lambda - a + 1, \quad \lambda_2 = \lambda + a - 1;$$
$$P_1, P_2 \text{ (cf. 3.1)}$$

can be constructed from the initial blocks

$$(0, x^{2i}, x^{2i+4m}, \ldots, x^{2i+4m(4\lambda-2)}),$$

where i *ranges from* 0 *to* $m - 1$.

Proof. The differences involving the zero element are all distinct and cover the even powers of x once. Hence, from Theorem 3.1, $\lambda_1 = \lambda - a + 1$, $\lambda_2 = \lambda + a - 1$; also $n_1 = n_2 = \frac{1}{2}(v - 1) = 2m(4\lambda - 1)$. P_1 and P_2 are obtained as before, and have the same form.

If $a = 1$, a completely balanced series is obtained.

4. Series with $2am(2a\lambda \pm 1) + 1 = p^c$.

THEOREM 4.1. *If* $v = 2am(2a\lambda + 1) + 1 = p^c$, *and if among the exponents* q_s, *where*

$$x^{q_s} = x^{2ams} - 1 \qquad (s = 1, 2, \ldots, a\lambda),$$

the residue class of $(i - 1)$ *modulo* a *is represented* λ_i *times, then the design with parameters*

$$v = 2am(2a\lambda + 1) + 1, \; b = mv, \; r = m(2a\lambda + 1), \; k = 2a\lambda + 1;$$
$$n_i = 2m(2a\lambda + 1), \; \lambda_i; \; P_i,$$

can be constructed from the initial blocks

$$(x^{au}, x^{au+2am}, x^{au+4am}, \ldots, x^{au+4ma^2\lambda}),$$

where u *ranges from* 0 *to* $m - 1$. *The* p_{ij}^k *are the number of expressions*

$$x^{at+i-j} - 1 = x^z,$$

where $z \equiv k - j \pmod a$, *and* t *ranges from* 0 *to* $(v - a - 1)/a$.

Proof. The differences are

$$x^{au+2arm+q_s}, \; x^{au+am(2r+2a\lambda+1-2s)+q_s}.$$

Let set i be the set of powers which are congruent to $(i - 1)$ modulo a. There are a such sets; hence $n_i = (v - 1)/a = 2m(2a\lambda + 1)$. Also, since each element of set i occurs among the differences λ_i times, condition 2 of Theorem 2.1 is satisfied.

The number of times that every element of set k occurs among the differences in (set i − set j) is p_{ij}^k; but (set i − set j) consists of elements of the form

$$x^{at+aw+i-1} - x^{aw+j-1} = x^{aw+j-1}(x^{at+i-j} - 1) = x^{aw+j-1+z},$$

where

$$x^{at+i-j} - 1 = x^z.$$

These elements are in set k if and only if

$$j - 1 + z \equiv k - 1 \pmod a,$$

that is,

$$z \equiv k - j \pmod a.$$

The work of finding the p_{ij}^k can be simplified by noting that

$$x^{z_1} = x^{2am(2a\lambda+1)-at} - 1 = (x^{at} - 1)\, x^{am(2a\lambda+1)-at} = x^{z+am(2a\lambda+1)-at},$$

that is, $z \equiv z_1 \pmod a$. Hence we need the expressions

$$x^z = x^{at} - 1$$

only for $t = 0, 1, \ldots, (v-1)/2a$. The residue class of z corresponding to a given t is represented twice if $t < (v-1)/2a$ and is represented once if $t = (v-1)/2a$.

Example. Take $a = 3$, $m = \lambda = 1$; then $v = b = 43$, $r = k = 7$, $n_1 = n_2 = n_3 = 14$. From the equations in GF(43)

$$3^3 - 1 = 3^{17},\quad 3^6 - 1 = 3^{22},\quad 3^9 - 1 = 3^{34},\quad 3^{12} - 1 = 3,\quad 3^{15} - 1 = 3^{36},$$
$$3^{18} - 1 = 3^{23}, 3^{21} - 1 = 3^6,$$

we deduce that the q_s are 22, 1, and 23, that is, 1, 1, and 2 (mod 3). Hence $\lambda_1 = 0$, $\lambda_2 = 2$, $\lambda_3 = 1$.

To find the p_{ij}^k, take $i \equiv j \pmod 3$; p_{ii}^k is the number of expressions $x^{3t} - 1 = x^z$, where $z \equiv k - i \pmod 3$, and $t = 0, 1, \ldots, 7$. From the preceding tabulation of these expressions, it is seen that $z \equiv 0$ once for $t < 7$ and once for $t = 7$; $z \equiv 1$ three times; $z \equiv 2$ twice. Hence

$$p_{11}^1 = p_{22}^2 = p_{33}^3 = 3;\ p_{11}^2 = p_{22}^3 = p_{33}^1 = 6;\ p_{11}^3 = p_{22}^1 = p_{33}^2 = 4.$$

The remaining p's can be obtained from the relations 1.4, 1.5, and 1.6; thus

$$P_1 = \begin{pmatrix} 3 & 6 & 4 \\ 6 & 4 & 4 \\ 4 & 4 & 6 \end{pmatrix},\quad P_2 = \begin{pmatrix} 6 & 4 & 4 \\ 4 & 3 & 6 \\ 4 & 6 & 4 \end{pmatrix},\quad P_3 = \begin{pmatrix} 4 & 4 & 6 \\ 4 & 6 & 4 \\ 6 & 4 & 3 \end{pmatrix}.$$

In case $a = 1$, there is only one residue class modulo a; hence all the q_s fall in this single residue class and there is only one $\lambda_i = \lambda$. The resulting series is the completely balanced Series B of **(6)**.

THEOREM 4.2. *If $v = 2am(2a\lambda - 1) + 1 = p^c$, and if among the exponents q_s, where*

$$x^{q_s} = x^{2ams} - 1, \qquad (s = 1, 2, \ldots, a\lambda - 1),$$

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the residue class of $(i-1)$ *modulo* a *is represented* λ_i *times, then the design with parameters*

$$v = 2am(2a\lambda - 1) + 1,\ b = mv,\ r = 2am\lambda,\ k = 2a\lambda;$$
$$n_i = 2m(2a\lambda - 1),\ \lambda_i + \delta_{i1};\quad P_i$$

can be constructed from the initial blocks

$$(0, x^{au}, x^{au+2am}, \ldots, x^{au+2am(2a\lambda-2)}),$$

where u *ranges from* 0 *to* $m-1$.

Proof. Since the class of zero is represented once by all differences involving the zero element, the frequency of occurrence of the residue class of zero among the q_s is $\lambda_1 + 1$. The p_{ij}^k are found just as in Theorem 4.1.

5. Series with $am(a\lambda \pm 1) + 1 = p^c$.

THEOREM 5.1. *If* $v = am(a\lambda + 1) + 1 = p^c$, *where* a *is odd, and if among the exponents* q_s, *where*

$$x^{ams} - 1 = x^{q_s} \qquad (s = 1, 2, \ldots, a\lambda)$$

the residue class of $(i-1)$ *modulo* a *is represented* λ_i *times, then the design with parameters:*

$$v = am(a\lambda+1)+1,\ b = mv,\ r = m(a\lambda+1),\ k = a\lambda+1;\ n_i = m(a\lambda+1),\ \lambda_i;\ P_i$$

can be constructed from the initial blocks

$$(x^{au}, x^{au+am}, x^{au+2am}, \ldots, x^{au+ma^2\lambda}),$$

where u *ranges from* 0 *to* $m-1$.

Proof. The proof is similar to that of Theorem 4.1; the P_i can be found as before, except that here, since a is odd, there are no relations among the q_s to simplify the work.

THEOREM 5.2. *If* $v = am(a\lambda - 1) + 1 = p^c$, *where* a *is odd, and if among the exponents* q_s, *where*

$$x^{ams} - 1 = x^{q_s} \qquad (s = 1, 2, \ldots, a\lambda - 2)$$

the residue class of $(i-1)$ *modulo* a *is represented* λ_i *times, then the design with parameters*

$$v = am(a\lambda - 1) + 1,\ b = mv,\ r = am\lambda,\ k = a\lambda;\ n_i = m(a\lambda - 1),\ \lambda_i + 2\delta_{i1};\ P_i$$

can be constructed from the initial blocks

$$(0, x^{au}, x^{au+am}, \ldots, x^{au+am(a\lambda-2)})$$

where u *ranges from* 0 *to* $m-1$.

Proof. The proof is similar to that of Theorem 4.2.

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6. Series with $ma + 1 = p^c$. To every element y^f of $GF(p^c)$, let there correspond n varieties

$$y_1^f, y_2^f, \ldots, y_n^f.$$

Then designs with three associate classes can be formed by taking as first associates of any variety

$$y_u^{f_1}$$

all varieties

$$y_u^{f_2} \qquad (f_1 \neq f_2);$$

the second associates are all varieties

$$y_w^{f_1} \qquad (u \neq w);$$

the third associates are all varieties

$$y_w^{f_2}.$$

Thus the first associates of a variety are all varieties giving rise to pure differences; the second associates are all varieties giving rise to zero mixed differences; the third associates are all other varieties giving rise to non-zero mixed differences.

THEOREM 6.1. *If $ma + 1 = p^c$, then the design with parameters*

$$v = n(ma + 1),\ b = \tfrac{1}{2}mv(n-1),\ r = am(n-1),\ k = 2a;$$
$$n_1 = ma,\ n_2 = n-1,\ n_3 = ma(n-1),\ \lambda_1 = (a-1)(n-1),\ \lambda_2 = ma,$$
$$\lambda_3 = a - 1;$$

$$P_1 = \begin{pmatrix} ma-1 & 0 & 0 \\ 0 & 0 & n-1 \\ 0 & n-1 & (ma-1)(n-1) \end{pmatrix},$$

$$P_2 = \begin{pmatrix} 0 & 0 & ma \\ 0 & n-2 & 0 \\ ma & 0 & ma(n-2) \end{pmatrix},$$

$$P_3 = \begin{pmatrix} 0 & 1 & ma-1 \\ 1 & 0 & n-2 \\ ma-1 & n-2 & (ma-1)(n-2) \end{pmatrix}$$

can be constructed from the initial blocks

$$(x_u^i, x_u^{i+m}, \ldots, x_u^{i+(a-1)m}, x_w^i, x_w^{i+m}, \ldots, x_w^{i+(a-1)m}),$$

where i ranges from 0 to $m-1$, and (u, w) are the $\tfrac{1}{2}n(n-1)$ pairs of integers selected from 1 to n.

Proof. The pure differences of the type (u, u), arising from all blocks with a fixed u and w, are each repeated $a - 1$ times **(6,** Theorem 2.1**)**. Since a fixed u occurs with $n - 1$ values of w, $\lambda_1 = (n-1)(a-1)$. The zero mixed differences of the type (u, w) occur a times for i fixed, and ma times in all; hence

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$\lambda_2 = ma$. The non-zero mixed differences of the type (u, w) occur $\lambda_3 = a - 1$ times (**6**, Theorem 2.1). Since there are $\frac{1}{2}n(n-1)$ pairs (u, w),

$$b = \tfrac{1}{2}m(ma+1)\,n(n-1).$$

The number of first associates of a variety $y_u^{f_1}$ is the number of varieties of the form

$$y_u^{f_2};$$

thus $n_1 = ma$. Also, n_2 is the number of varieties of the form

$$y_w^{f_1},$$

that is, $n_2 = n - 1$; $n_3 = ma(n-1) =$ the number of varieties of the form

$$y_w^{f_2}.$$

Two first associates have the form

$$y_u^{f_1}, y_u^{f_2};$$

hence $p_{11}{}^1$ is the number of varieties of the form

$$y_u^{f_3} \qquad (f_1 \neq f_3, f_2 \neq f_3),$$

that is, $ma - 1$. Also, $p_{12}{}^1 = p_{13}{}^1 = p_{22}{}^1 = 0$. We obtain $p_{23}{}^1$ as the number of varieties of the form

$$y_w^{f_1} \qquad (f_1 \text{ fixed});$$

hence $p_{23}{}^1 = n - 1$. The number of varieties of the form

$$y_w^{f_3} \qquad (u \neq w)$$

gives us $p_{33}{}^1 = (ma-1)(n-1)$. This completes P_1 and the matrices P_2 and P_3 can be found in a similar way.

THEOREM 6.2. *If $ma + 1 = p^c$, then the design with parameters*

$v = n(ma+1)$, $b = m(n-1)v$, $r = m(a+1)(n-1)$, $k = a+1$;
$n_1 = am$, $n_2 = n-1$, $n_3 = ma(n-1)$, $\lambda_1 = (a-1)(n-1)$, $\lambda_2 = 0$, $\lambda_3 = 2$;
P_1, P_2, P_3 (cf. 6.1)

can be constructed from the initial blocks

$$(x_u^{i}, x_u^{i+m}, \ldots, x_u^{i+(a-1)m}, 0_w),$$

where i ranges from 0 to $m-1$ and (u, w) are the $n(n-1)$ permutations of the integers from 1 to n, taken two at a time.

Proof. Each pure difference of the type (u, u) occurs $a - 1$ times for w fixed (**6**, Theorem 2.1) and $(n-1)(a-1)$ times in all. The non-zero mixed differences of the type (u, w) occur twice, since u and w can be interchanged; the zero mixed differences do not occur at all. The n_i and P_i are found as in Theorem 6.1 and have the same values.

COROLLARY 6.21. *If we permit $u = w$, we obtain a series with parameters*

$v = n(am + 1)$, $b = mnv$, $r = mn(a + 1)$, $k = a + 1$; n_i (cf. 6.2), $\lambda_1 = n(a - 1) + 2, \lambda_2 = 0, \lambda_3 = 2$; P_i (cf. 6.2).

Proof. Each pure difference of the type (u, u) arises $a + 1$ times from the blocks in which $u = w$.

COROLLARY 6.22. *If $n = a + 1$ in Theorem 6.2 and the block $(0_1, 0_2, \ldots, 0_{a+1})$ is included twice in the set of initial blocks, we obtain a group-divisible design with parameters*

$v = (a + 1)(am + 1)$, $b = \{m(a + 1) + 2\}(am + 1)$, $r = ma(a + 1)+2$, $k = a + 1$;
$g = a + 1$, $h = am + 1$; $\lambda_1 = a(a - 1)$, $\lambda_2 = 2$,

where g is the number of groups in the GD *design, h is the number of treatments in a group.*

Proof. To combine the second and third classes of a PBIBD, the following conditions must hold:

(1) $\lambda_2 = \lambda_3$,
(2) $p_{13}^2 + p_{12}^2 = p_{13}^3 + p_{12}^3$,
(3) $p_{22}^2 + p_{33}^2 + p_{23}^2 + p_{32}^2 = p_{22}^3 + p_{33}^3 + p_{23}^3 + p_{32}^3$.

These conditions are satisfied; so classes 2 and 3 can be combined to give a PBIBD with the stated parameters. For this design,

$$n_1 = ma, n_2 = (ma + 1)(n - 1);$$
$$P_1 = \begin{pmatrix} ma - 1 & 0 \\ 0 & (ma + 1)a \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & ma \\ ma & (ma + 1)(a - 1) \end{pmatrix}.$$

COROLLARY 6.23. *If $n = a + 1$ in Corollary 6.21 and the block $(0_1, 0_2, \ldots, 0_{a+1})$ is included twice in the set of initial blocks, we obtain a* GD *design with parameters*

$v = (am + 1)(a + 1)$, $b = \{m(a + 1)^2 + 2\}(ma + 1)$, $r = (a + 1)^2m+2$, $k = a + 1$;
$g = a + 1$, $h = am + 1$; $\lambda_1 = a^2 + 1$, $\lambda_2 = 2$.

Proof. Similar to that of Corollary 6.22.

COROLLARY 6.24. *If in Theorem 6.2 the initial blocks are replaced by*

$$(0_u, x_u^{i}, x_u^{i+m}, \ldots, x_u^{i+(a-1)m}, 0_w) \qquad (i = 0, 1, \ldots, m - 1)$$

where (u, w) are permutations of the integers from 1 to n, taken two at a time, then the resulting design is a GD *design with parameters*

$v = n(am + 1)$, $b = m(n - 1)v$, $r = (a + 2)(n - 1)m$, $k = a + 2$;
$g = n$, $h = am + 1$; $\lambda_1 = (a + 1)(n - 1)$, $\lambda_2 = 2$.

Proof. Here each zero mixed difference of the type (u, w) also occurs twice; hence classes 2 and 3 can be combined as in Corollary 6.22.

Example. Take $a = 2$, $m = 1$, in Corollary 6.23; the GD design has parameters $v = 9$, $b = 33$, $r = 11$, $k = 3$; $g = h = 3$; $\lambda_1 = 5$, $\lambda_2 = 2$. The initial blocks are $(1, 2, 0)$, $(1, 2, 3)$, $(1, 2, 6)$, $(4, 5, 0)$, $(4, 5, 3)$, $(4, 5, 6)$, $(7, 8, 0)$, $(7, 8, 3)$, $(7, 8, 6)$, $(0, 3, 6)$, $(0, 3, 6)$, where we set $y_1 = y$, $y_2 = y + 3$, $y_3 = y + 6$. The three groups of three varieties are 0, 1, 2; 3, 4, 5; and 6, 7, 8. Thus, for instance, 3 occurs five times with 4 and 5 and twice with all the other varieties.

7. Series with $2m(2\lambda \pm 1) + 1 = p^c$.

THEOREM 7.1. *If $2m(2\lambda + 1) + 1 = p^c$, then the design with parameters*

$$v = \{2m(2\lambda + 1) + 1\}n, b = \tfrac{1}{2}(n - 1)mv, r = m(2\lambda + 1)(n - 1),$$
$$k = 4\lambda + 2;$$
$$n_1 = 2m(2\lambda + 1), n_2 = n - 1, n_3 = n_1(n - 1), \lambda_1 = (n - 1)\lambda,$$
$$\lambda_2 = m(2\lambda + 1), \lambda_3 = \lambda;$$

$$P_1 = \begin{pmatrix} n_1 - 1 & 0 & 0 \\ 0 & 0 & n - 1 \\ 0 & n - 1 & (n_1 - 1)(n - 1) \end{pmatrix},$$

$$P_2 = \begin{pmatrix} 0 & 0 & n_1 \\ 0 & n - 2 & 0 \\ n_1 & 0 & n_1(n - 2) \end{pmatrix},$$

$$P_3 = \begin{pmatrix} 0 & 1 & n_1 - 1 \\ 1 & 0 & n - 2 \\ n_1 - 1 & n - 2 & (n_1 - 1)(n - 2) \end{pmatrix}$$

can be constructed from the initial blocks

$$(x_u{}^i, x_u{}^{i+2m}, \ldots, x_u{}^{i+4m\lambda}, x_w{}^i, x_w{}^{i+2m}, \ldots, x_w{}^{i+4m\lambda}),$$

where i ranges from 0 to $m - 1$ and (u, w) are the $\tfrac{1}{2}n(n - 1)$ pairs of integers selected from 1 to n.

Proof. By (**6**, Theorem 3.1), the pure differences of the type (u, u) occur λ times for a fixed w and $\lambda_1 = (n - 1)\lambda$ times in all. Similarly, the non-zero mixed differences of the type (u, w) occur $\lambda_3 = \lambda$ times while the zero mixed differences of the type (u, w) occur $(2\lambda + 1)$ times for a fixed i, that is, $\lambda_2 = m(2\lambda + 1)$ times in all. The method employed in Theorem 6.1 gives the values for the n_i and shows that the P_i have the same form as in that theorem, with ma replaced by $2m(2\lambda + 1)$.

Example. $m = 2$, $n = 3$, $\lambda = 1$, gives a design with $v = 39$, $b = 78$,

$$r = 12, k = 6; n_1 = 12, n_2 = 2, n_3 = 24, \lambda_1 = 2, \lambda_2 = 6, \lambda_3 = 1;$$

$$P_1 = \begin{pmatrix} 11 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 22 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & 0 & 12 \\ 0 & 1 & 0 \\ 12 & 0 & 12 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 0 & 1 & 11 \\ 1 & 0 & 1 \\ 11 & 1 & 11 \end{pmatrix}.$$

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The initial blocks are

$$(2_u{}^i, 2_u{}^{i+4}, 2_u{}^{i+8}, 2_w{}^i, 2_w{}^{i+4}, 2_w{}^{i+8}),$$

that is, reduced modulo 13,

$$(i+1)(1_u, 3_u, 9_u, 1_w, 3_w, 9_w), \qquad i = 0 \text{ or } 1.$$

The pairs (u, w) are the pairs $(1, 2)$, $(1, 3)$, and $(2, 3)$. Setting $y_1 = y$; for $y = 0, 1, \ldots, 12$; $y_2 = y + 13$; $y_3 = y + 26$; we find that the initial blocks are

$$(1, 3, 9, 14, 16, 22), \quad (1, 3, 9, 27, 29, 35), \quad (2, 6, 5, 15, 19, 18),$$
$$(2, 6, 5, 28, 32, 31), \quad (14, 16, 22, 27, 29, 35), \quad (15, 19, 18, 28, 32, 31).$$

The other blocks are generated by addition modulo 13; thus the first block generates $(2, 4, 10, 15, 17, 23)$, $(3, 5, 11, 16, 18, 24), \ldots,$ $(0, 2, 8, 13, 15, 21)$. Given any variety, say 1, its first associates are $0, 2, 3, \ldots, 12$; its second associates are 14 and 27; the remaining varieties are third associates.

THEOREM 7.2. *If* $2m(2\lambda + 1) + 1 = p^c$, *then the design with parameters*

$$v = \{2m(2\lambda + 1) + 1\}n, b = (n-1)mv, r = m(2\lambda + 2)(n-1),$$
$$k = 2\lambda + 2; n_1 \text{ (cf. 7.1)}, \lambda_1 = (n-1)\lambda, \lambda_2 = 0, \lambda_3 = 1; P_i \text{ (cf. 7.1)}$$

can be constructed from the initial blocks

$$(x_u{}^i, x_u{}^{i+2m}, \ldots, x_u{}^{i+4m\lambda}, 0_w),$$

where i *ranges from* 0 *to* $m - 1$ *and* (u, w) *are the* $n(n-1)$ *permutations of the integers from* 1 *to* n, *taken two at a time.*

Proof. Proceed as in Theorem 6.2.

Using proofs similar to those used in the Corollaries to Theorem 6.2, we obtain

COROLLARY 7.21. *If we permit* $u = w$ *in Theorem* 7.2, *we obtain a series with*

$$v = \{2m(2\lambda + 1) + 1\}n, \; b = mnv, \; r = mn(2\lambda + 2), \; k = 2\lambda + 2;$$
$$n_1 = 2m(2\lambda+1), n_2 = n-1, n_3 = n_1(n-1), \; \lambda_1 = n\lambda+1, \lambda_2 = 0, \lambda_3 = 1.$$

COROLLARY 7.22. *If in Theorem* 7.2 *we set* $n = 2\lambda + 2$ *and include the block*

$$(0_1, 0_2, \ldots, 0_{2\lambda+2})$$

among the initial blocks, then we obtain a GD *design with parameters*

$$v = (2\lambda + 2)\{2m(2\lambda + 1) + 1\},$$
$$b = \{m(2\lambda + 2)(2\lambda + 1) + 1\}\{2m(2\lambda + 1) + 1\},$$
$$r = (2\lambda + 2)(2\lambda + 1)m + 1, \; k = 2\lambda + 2;$$
$$g = 2\lambda + 2, \; h = 2m(2\lambda + 1) + 1; \; \lambda_1 = \lambda(2\lambda + 1), \lambda_2 = 1.$$

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COROLLARY 7.23. *If in Corollary* 7.21 *we set* $n = 2\lambda + 2$ *and include the initial block*

$$(0_1, 0_2, \ldots, 0_{2\lambda+2}),$$

then we obtain a GD *design with parameters*

$$v = (2\lambda + 2)\{2m(2\lambda + 1) + 1\},$$
$$b = \{m(2\lambda + 2)^2 + 1\}\{2m(2\lambda + 1) + 1\},$$
$$r = (2\lambda + 2)^2 m + 1, k = 2\lambda + 2;$$
$$g = 2\lambda + 2, h = 2m(2\lambda + 1) + 1;\ \lambda_1 = (2\lambda + 2)\lambda + 1, \lambda_2 = 1.$$

THEOREM 7.3. *If* $2m(2\lambda - 1) + 1 = p^c$, *then the design with parameters*

$$v = \{2m(2\lambda - 1) + 1\}n, b = \tfrac{1}{2}mv(n - 1), r = 2m(n - 1)\lambda, k = 4\lambda;$$
$$n_1 = 2m(2\lambda - 1), n_2 = n - 1, n_3 = n_1(n - 1),$$
$$\lambda_1 = (n - 1)\lambda, \lambda_2 = 2m\lambda, \lambda_3 = \lambda;$$

$$P_1 = \begin{pmatrix} n_1 - 1 & 0 & 0 \\ 0 & 0 & n - 1 \\ 0 & n - 1 & (n_1 - 1)(n - 1) \end{pmatrix},$$

$$P_2 = \begin{pmatrix} 0 & 0 & n_1 \\ 0 & n - 2 & 0 \\ n_1 & 0 & n_1(n - 2) \end{pmatrix},$$

$$P_3 = \begin{pmatrix} 0 & 1 & n_1 - 1 \\ 1 & 0 & n - 2 \\ n_1 - 1 & n - 2 & (n_1 - 1)(n - 2) \end{pmatrix}$$

can be constructed from the initial blocks

$$(0_u, x_u{}^i, x_u{}^{i+2m}, \ldots, x_u{}^{i+4(\lambda-1)m}, 0_w, x_w{}^i, x_w{}^{i+2m}, \ldots, x_w{}^{i+4(\lambda-1)m}),$$

where i *ranges from* 0 *to* $m - 1$ *and* (u, w) *are the* $\tfrac{1}{2}n(n - 1)$ *pairs of integers selected from* 1 *to* n.

Proof. Proceed as in Theorem 7.1.

THEOREM 7.4. *If* $2m(2\lambda - 1) + 1 = p^c$, *then the design with parameters*

$$v = \{2m(2\lambda-1)+1\}n, b = mv(n-1), r = (2\lambda+1)m(n-1), k = 2\lambda+1;$$
$$n_i \text{ (cf. 7.3)}, \lambda_1 = (n - 1)\lambda, \lambda_2 = 2, \lambda_3 = 1;\ P_i \text{ (cf. 7.3)}$$

can be constructed from the initial blocks

$$(0_u, x_u{}^i, x_u{}^{i+2m}, \ldots, x_u{}^{i+4m\lambda}, 0_w),$$

where i *ranges from* 0 *to* $m - 1$ *and* (u, w) *are the* $n(n - 1)$ *permutations of the integers from* 1 *to* n, *taken two at a time.*

Proof. Proceed as in Theorem 7.2.

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