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A NOTE ON QUADRATIC FORMS AND THE u-INVARIANT

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The u-invariant of a field F, $u = u(F)$, is defined to be the maximum of the dimensions of anisotropic quadratic forms over F. If F is a non-formally real field with a finite number q of square classes then it is known that $u \leqq q$. The purpose of this note is to give some necessary and sufficient conditions for equality in terms of the structure of the Witt ring of F.

In what follows, F will be a field of characteristic different from two and $\dot{F}$ denotes the multiplicative group of F. The subgroup of nonzero squares in F is denoted $\dot{F}^2$ and G denotes the square class group $\dot{F}/\dot{F}^2$. If $a \in F$ we let $[a]$ denote the image of a in G. The order of G will be written $q = q(F)$. Note that if $q < \infty$ then $q = 2^n$ for some $n \geqq 0$. If F is not formally real then the *level* (or stufe) of F is the smallest positive integer $s = s(F)$ such that -1 is a sum of s squares in F. If ϕ is a quadratic form over F we write $\phi \cong \langle a_1, \ldots, a_n \rangle$ to mean ϕ is isometric to an orthogonal sum $\langle a_1 \rangle \perp \ldots \perp \langle a_n \rangle$ where $\langle a_i \rangle$ denotes the one dimensional space F with form $(x, y) \mapsto a_i xy$. The *Witt ring* of anisotropic forms over F is denoted by $W(F)$ (for a definition, see [**5**, pp. 14–15]) and $I(F)$ denotes the ideal of $W(F)$ consisting of all even dimensional forms. For any $n \geqq 1$, the ideal $I^n(F) = I(F) \ldots I(F)$, n times, is generated by the 2^n-dimensional forms

$$\bigotimes_{i=1}^{n} \langle 1, a_i \rangle, a_i \in \dot{F} \qquad \text{(Pfister forms)}.$$

The mapping $[a] \mapsto \langle a \rangle$ of G into $W(F)$ is injective and induces a surjective ring homomorphism from the integral group ring $\mathbf{Z}[G]$ onto $W(F)$ which will be denoted by Ψ. Finally, if the level s of F is finite then by a theorem of Pfister, $W(F)$ is a $\mathbf{Z}/2s\mathbf{Z}$-algebra [**5**, 8.1, p. 45].

As mentioned, the u-invariant of F is defined to be the maximum of the dimensions of anisotropic forms over F (for a more general definition, see [**4**]). If no such maximum exists, $u(F)$ is taken to be ∞; for example, when F is formally real. Thus $u(F)$ is the least positive integer (or ∞) such every $u + 1$ dimensional quadatic form over F is isotropic. If $u < 2^n$ then 2^n-dimensional forms $\otimes_{i=1}^{n} \langle 1, a_i \rangle$, $a_i \in \dot{F}$, must be isotropic and hence, by a result of Witt, equal to 0 in $W(F)$ [**5**, pp. 22–23]. Thus $I^n(F) = 0$, so whenever u is finite, $I(F)$ is a nilpotent ideal.

Kneser has shown that if F is a non-formally real field with $q = q(F) < \infty$

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then $u \leqq q$ (For a proof, see Math. Review 15–500, [**5**, 8.4, p. 47], or [**4**, Proposition A1]). Thus if $q = 2^n$ then $I^{n+1}(F) = 0$.

THEOREM. *Let F be a non-formally real field with $q = 2^n$. Then the following statements are equivalent:*

(1) $u = q$.

(2) *Either $s = 1$ and $W(F)$ is an $\mathbf{F}_2$-vector space of dimension q or $s = 2$ and $W(F)$ is a free $\mathbf{Z}/4\mathbf{Z}$-module of rank $q/2$.*

(3) *Either $s = 1$ and $W(F) \cong \mathbf{F}_2[G]$ or $s = 2$ and $W(F) \cong (\mathbf{Z}/4\mathbf{Z})[H]$, where H is any subgroup of index 2 in G with $[-1] \notin H$.*

(4) *$I^n(F) \neq 0$, i.e. $n + 1$ is the index of nilpotency of $I(F)$.*

Proof. The equivalence of (1) and (2) follows from [**7**, Proposition 5.10, Theorem 5.13, Proposition 5.15].

(2) $\Rightarrow$ (3) Let $\Psi : \mathbf{Z}[G] \rightarrow W(F)$ be the natural surjection.

If $s = 1$ then Ψ induces a surjective mapping $\Psi^* : \mathbf{F}_2[G] \rightarrow W(F)$. Since $\dim_{\mathbf{F}_2}\mathbf{F}_2[G] = q = \dim_{\mathbf{F}_2} W(F)$, Ψ^* is an isomorphism.

If $s = 2$ then Ψ induces a surjection $\Psi^* : \mathbf{Z}/4\mathbf{Z}[G] \rightarrow W(F)$. Let H be any subgroup of index 2 in G with $[-1] \notin H$. Then $G = H \times \{[1], [-1]\}$ so if $\Psi^{**} : \mathbf{Z}/4\mathbf{Z}[H] \rightarrow W(F)$ is the restriction of Ψ to $\mathbf{Z}/4\mathbf{Z}[H]$ then Ψ^{**} is also surjective. Since H has $q/2$ elements $\mathbf{Z}/4\mathbf{Z}[H]$ and $W(F)$ are both finite sets with the same number of elements. Hence Ψ^{**} is an isomorphism.

(3) $\Rightarrow$ (4) is immediate.

(4) $\Rightarrow$ (1). If $u < q = 2^n$ then as remarked earlier, $I^n(F) = 0$.

Remarks. (1) In [**7**, § 5], C. Cordes investigated fields satisfying the conditions of the theorem and called them $\bar{C}$-fields. In that paper he gave several other equivalent conditions. In particular, he has shown that F is a $\bar{C}$-field if and only if for any anisotropic form ϕ over F, Card $D(\phi) = \dim \phi$, where $D(\phi) = \{[a] \in G | a \text{ is represented by } \phi\}$.

(2) Let A be a complete discrete valuation ring with field of fractions F and residue field k of characteristic not 2. Then an easy application of Hensel's lemma shows that $q(k) = 2^n$ if and only if $q(F) = 2^{n+1}$. Moreover, a theorem of Springer [**5**, 7.1, p. 43] gives an isomorphism $W(F) \cong W(k) \oplus W(k)$ of abelian groups. From this it is easy to see that k satisfies the conditions of the theorem with $u(k) = q(k) = 2^n$ if and only if F does with $u(F) = q(F) = 2^{n+1}$.

Examples. (0) If F is algebraically closed then $u(F) = q(F) = 1$.

(1) Any finite field (of char $\neq 2$) satisfies the conditions of the theorem with $u = q = 2$.

(2) If F is a local field with finite residue field of characteristic not 2 then $u(F) = q(F) = 4$.

(3) If $F = Q_2$, the field of 2-adic numbers, then $u(F) = 4$, $q(F) = 8$.

(4) If k is a field with $u(k) = q(k) = 2^n$ and $F = k((t_1)) \ldots ((t_r))$, the field of iterated power series over k then $u(F) = q(F) = 2^{n+r}$.

The paper concludes with a related result regarding the values of quadratic forms over F.

PROPOSITION. *For a field F the following statements are equivalent:*

(1) *For* $a \notin -\dot{F}^2$, $D(\langle 1, a\rangle) = \{[1], [a]\}$.

(2) *If* $\phi \cong \langle a_1, \ldots, a_n\rangle$ *is anisotropic then* $D(\phi) = \{[a_1], \ldots, [a_n]\}$.

(3) *The kernel of the mapping* $\Psi : \mathbf{Z}[G] \to W(F)$ *is generated by* $[1] + [-1]$.

(4) *Either F is formally real, pythagorean, and* $W(F) \cong \mathbf{Z}[H]$, *where H is a subgroup of index two in G with* $[-1] \notin H$ *or* $s(F) = 1$ *and* $W(F) \cong \mathbf{F}_2[G]$.

Proof. An easy induction gives the equivalence of (1) and (2). If $-1 \notin \dot{F}^2$ then by [**6**, Theorem 1], (1), (3), and the formally real case of (4) are equivalent. Thus it suffices to assume $-1 \in \dot{F}^2$, i.e., $s(F) = 1$, and show the equivalence of (1), (3), and the non-formally real case of (4).

(1) $\Rightarrow$ (3). As is well-known, the kernel of Ψ is generated by $[1] + [-1]$ and all elements of the form

$$g(a, x, y) = ([1] + [a])([1] - [x^2 + ay^2])$$

with $x, y \in F$ and $a, x^2 + ay^2 \in \dot{F}$ (see, for example, [**5**, 6.1, p. 41]). If $a \notin -\dot{F}^2$ then by (1), $[x^2 + ay^2] = [a]$ or $[1]$, so in either case $g(a, x, y) = 0$. Hence any non zero generator is either $[1] + [-1]$ or has the form $([1] + [-1])([1] - [b])$, with $b \in \dot{F}$, proving (3).

(3) $\Rightarrow$ (4). Since $-1 \in \dot{F}^2$, $[1] + [-1] = 2$ in $\mathbf{Z}[G]$ so $W(F) \cong \mathbf{Z}[G]/2\mathbf{Z}[G] \cong \mathbf{F}_2[G]$.

(4) $\Rightarrow$ (1). If $[b] \in D(\langle 1, a\rangle)$ then $\langle 1, a\rangle \cong \langle b, ab\rangle$ so $\langle 1\rangle + \langle a\rangle = \langle b\rangle + \langle ab\rangle$ in $W(F)$. Since $\{\langle x\rangle\}_{x\in\dot{F}}$ is a basis for $W(F)$ over $\mathbf{F}_2$ and $\langle a\rangle \neq \langle -1\rangle = \langle 1\rangle$ it follows that $\langle b\rangle = \langle 1\rangle$ or $\langle b\rangle = \langle a\rangle$, i.e. $[b] = [1]$ or $[b] = [a]$, proving (1).

Remark. Formally real fields satisfying the conditions of the proposition have been studied in [**1**; **2**; **3**; **6**]. Elman and Lam have called such fields superpythagorean.

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