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A NOTE ON COMPLEMENTARY SUBSPACES IN c_0

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A well known result of A. Pelcyński [2] states that each subspace of c_0 which is isomorphic to c_0 and of infinite deficiency has a complementary subspace which is itself isomorphic to c_0. We are concerned here with the question of when there exists R, a subset of the integers, such that the complementary subspace X can actually be taken to be $c_0(R)$. That is, we are concerned with determining when the basis vectors for X can be chosen as a subset of the usual basis vectors for c_0. If $T: c_0 \to c_0$ is norm increasing and $||T|| < 2$, it is not hard to see, as we shall show, that Tc_0 admits a complement of the form $c_0(R)$. However, this bound cannot be improved; indeed, it is possible to construct norm increasing $T: c_0 \to c_0$ such that $||T|| = 2$ and yet Tc_0 admits no such complement. The construction of such a T is the main point of this note. This construction also enables us to dispose of a speculation of ours in [1].

Our notation is standard except for a few conventions. We denote the set of positive integers by ω. We denote by $c_0(\omega)$, or simply c_0, the Banach space under the sup norm of complex-valued sequences tending to 0. If $R \subset \omega$, we denote by $c_0(R)$ the subspace of c_0 consisting of sequences supported on R. We denote by E^R the operator of perpendicular projection on $c_0(R)$. That is, $E^R: c_0 \to c_0(R)$ is defined by

$$\begin{aligned}(E^R x)(n) &= x(n), \quad n \in R\\ &= 0, \qquad\ \ n \notin R.\end{aligned}$$

We denote by m the Banach space of bounded sequences and, considering m as the second dual of c_0, let $T^c: m \to m$ denote the extension of continuous $T: c_0 \to c_0$ as a second adjoint. We denote by δ_i the sequence which is 1 at i and 0 elsewhere. We make here two observations to which we will refer in the sequel.

(1) If $T: c_0 \to c_0$ is an isomorphism into (that is, except for a scalar factor, T is a norm increasing, bounded map), then necessary and sufficient that Tc_0 admit $c_0(R)$ as a complement is that $E^{\omega \backslash R}T: c_0 \to c_0(\omega\backslash R)$ be an isomorphism onto.

(2) If we extend the matrix $T: c_0 \to c_0$ to $T^c: m \to m$ by simply letting the matrix operate on m, then T^c is the extension by second adjoint of T and so, if $T^c x \in c_0$ for some $x \in m\backslash c_0$, then $T: c_0 \to c_0$ cannot be an isomorphism into.

The first observation is easily verified; the second is not trivial but is a standard result.

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The following easy proposition is essentially contained in [**1**] as Proposition 1.3, together with a reference to some relevant results of M. Cambern. We prove it here in order to show just how the complementary subspaces $c_0(R)$ arise. It is interesting that, despite the crudeness of the method, the bound $\rho < 2$ cannot be improved.

PROPOSITION 1. *Let $T: c_0 \to c_0$ be a norm increasing map such that $||T|| \leqq \rho < 2$. Then Tc_0 admits a complementary subspace of the form $c_0(R)$.*

Proof. To each i, we associate a single n_i such that $|T\delta_i(n_i)| \geqq 1$. If $||x|| = 1$ and $|x(i)| = 1$, then $|Tx(n_i)| \geqq 2 - \rho$. Indeed, if $||y|| = 1$ and $y(i) = 0$, then $|Ty(n_i)| \leqq \rho - 1$; for, otherwise, for some α of modulus 1 we would have $|T(\delta_i + \alpha y)(n_i)| > \rho$. Similarly, it is clear that, if $i \neq j$, then $n_i \neq n_j$. Hence, if we let $s = \{n_i\}$ we see that $E^sT: c_0 \to c_0(s)$ is within $\rho - 1$ of the isometry $T': c_0 \to c_0(s)$ defined by $(T'x)(n_i) = x(i)$. Hence, $E^sT: c_0 \to c_0(s)$ is an isomorphism onto. Therefore, by our observation (1), we see that $c_0(\omega\backslash s)$ is a complement to Tc_0. This completes our proof.

There are conditions other than norm conditions which would guarantee that Tc_0 admit a complement $c_0(R)$; for example, if Tc_0 is an isomorphic image of c_0 of finite deficiency n, it is not hard to see that there exists a set R of cardinality n such that $c_0(R)$ is a complement to Tc_0. We are interested here, however, in norm conditions.

We come now to the example which shows the bound $\rho < 2$ is best possible.

PROPOSITION 2. *Let $T: c_0 \to c_0$ be defined by*

$$(Tx)(2n-1) = x(n) - \sum_{j=n+1}^{n!} x(j)/(n! - n),$$

$$(Tx)(2n) = x(n) + \sum_{j=n+1}^{n!} x(j)/(n! - n).$$

Then $T: c_0 \to c_0$ is norm increasing, $||T|| = 2$, but Tc_0 admits no complementary subspace of the form $c_0(R)$.

Proof. It is clear that T is norm increasing and that $||T|| = 2$. By observation (1), we see that necessary and sufficient that Tc_0 admit a complementary subspace $c_0(R)$ is that $E^sT: c_0 \to c_0(s)$ be an isomorphism, where $s = \omega\backslash R$. We now show that no such s can exist. Suppose, for the sake of contradiction, that there is such an s. Then there must exist $\eta > 0$ such that

$$\inf_{||x||=1} ||E^sTx|| > \eta.$$

Then for all $n > 2/\eta$, either $2n$ or $2n - 1$ would be in s; for, otherwise, $||E^sT\delta_n|| < \eta$. Now if we assume that $2n \in s$, we can see that $2n - 1 \notin s$, and vice versa.

For, suppose that there were $x \in c_0$ such that $\delta_{2n} = E^sTx$. Then, for each $m > n$, we would have either $Tx(2m - 1) = 0$ or $Tx(2m) = 0$. But if $x \in c_0$, then for $\epsilon > 0$, we have

$$\left| \sum_{j=M+1}^{M!} x(j)/(M! - M) \right| < \epsilon,$$

for large enough M and so, since either $Tx(2M) = 0$ or $Tx(2M - 1) = 0$, we see that $|x(M)| < \epsilon$. We now have

$$\left| \sum_{j=M}^{(M-1)!} x(j)/(M - 1)! - (M - 1) \right| < \epsilon,$$

and so we see that $|x(M - 1)| < \epsilon$, and so on, back to $|x(n + 1)| < \epsilon$. Since ϵ is arbitrary, we see that $x(n + k) = 0$ and so $Tx(2n) = Tx(2n - 1)$. Therefore, either $2n \in s$ or $2n - 1 \in s$, but not both. So we are left with our hypothetical s consisting of exactly one of $2n$ or $2n - 1$, for all large enough n. But now, having such an s, we see that there is a bounded sequence $y = (\pm 1, \pm 1, \pm 1, \ldots)$, with appropriate choice of sign in each coordinate depending on s, such that $(E^sT)^c y \in c_0$.

Indeed, we can define y as follows: For $j \leqq 6$, we let $y(j) = 1$. For $j > 6$, we let

$$y(j) = y(n) \qquad (n - 1)! < j \leqq n! \quad \text{if } 2n - 1 \in s$$

$$y(j) = -y(n) \qquad (n - 1)! < j \leqq n! \quad \text{if } 2n \in s.$$

(The rapidly increasing indices in the summation allow each average to be approximately $y(n)$ or $-y(n)$, as desired, despite the fact that the first block of terms in the sum is already determined.)

By our observation (2) concerning second adjoints, we see that $E^sT\colon c_0 \to c_0(s)$ is not an isomorphism, contradicting our hypothesis on s and, therefore, showing that no suitable s exists. (Of course, for $\epsilon > 0$ we could simply taper the sequence y sufficiently slowly to 0 to construct directly a $\tilde{y} \in c_0$ such that $||\tilde{y}|| = 1$ and $||E^sT\tilde{y}|| < \epsilon$ and, thus also, show that $E^sT\colon c_0 \to c_0(s)$ is not an isomorphism.)

Hence, we have shown that there cannot exist s such that $c_0(\omega \backslash s)$ is a complement to Tc_0, and our proof is complete.

In [**1**], we speculated on the possibility that if $T\colon c_0 \to c_0$ is an isomorphism into, then there must exist s such that $(E^sT)^c\colon m \to m(s)$ is an isomorphism onto. But our example is just such an isomorphism for which no such s exists. Our observation (2) on second adjoints and the fact that $E^sT\colon c_0 \to c_0$ is assumed to be an isomorphism into, indicate that, if $[(E^sT)^c x](n) = 0$ for large enough n, then $x \in c_0$. Therefore, in the one step where we explicitly use c_0, showing that only one of $2n$ or $2n - 1$ is in s, we may, from the assumption that $x \in m$, deduce $x \in c_0$ and conclude the same contradiction as in Proposition 2.

References

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2. A. Pełcyński, *Projections in certain Banach spaces*, Studia Math. *19* (1960), 209–228.

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