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POLYNOMIAL DECAY FOR SOLUTIONS OF HYPERBOLIC INTEGRODIFFERENTIAL EQUATIONS*

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Abstract. We consider a linear integrodifferential equation of second order in a Hilbert space and show that the solution tends to zero polynomially if the decay of the convolution kernel is polynomial. Both polynomials are of the same order.

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1. Introduction. In this paper, we investigate the following integrodifferential equation

$$\ddot{u}(t) = -Au(t) + \int_0^t g(t-s)Au(s)\,ds + f(t),$$
$$u(0) = x, \quad \dot{u}(0) = y \tag{1}$$

in a Hilbert space H. We generalize the result by Rivera and Gomez [**1**] on polynomial decay of the solutions. As in [**1**], $A : D(A) \to H$ is a self-adjoint operator. Our assumptions on g are as follows:

(g1) $g(t) \in C^3([0, +\infty))$, $g(t) > 0$ for all $t \geq 0$.
(g2) There exists $c_0 > 0$ such that $-c_0 g(t) \leq g'(t)$ for all $t \geq 0$.
(g3) There exist $c_1 > 0$ and $p > 1$ such that $g'(t) \leq -c_1 g^{1+\frac{1}{p}}(t)$ for all $t \geq 0$.
(g4) There exists $c_2 > 0$ such that $|g''(t)| \leq c_2 g(t)$ for all $t \geq 0$.
(g5) $G := \int_0^\infty g(\tau)d\tau < 1$.

It follows from (g3) and continuity of g in 0 that $g(t) \leq C(1+t)^{-p}$ for some $C > 0$. Unlike Rivera and Gomez, we do not need $p > 2$ and our assumption (g2) is also weak. In fact, in [**1**] one assumes $-c_0 g(t)^{1+\frac{1}{p}} \leq g'(t)$ which means that the behaviour of g in $+\infty$ is exactly the same as t^{-p}. This excludes kernels like $g(s) = (1+t)^{-p}\ln(1+t)$. In our case, the decay of g is anything between polynomial and exponential.

Throughout this paper, c and C are general positive constants independent of t; their values vary from expression to expression.

2. Main result. We introduce an energy functional and formulate the main result. Define

$$E(t, v) := \frac{1}{2}\left(\|v_t\|^2 + (1 - G(t))\left\|A^{1/2}v\right\|^2 + g \circ A^{1/2}v\right),$$

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where

$$(g \circ k)(t) := \int_0^t g(t-s)\|k(s)-k(t)\|^2\, ds$$

and

$$G(t) := \int_0^t g(s)\, ds.$$

THEOREM 2.1. *Let g satisfy (g1)–(g5) and A be a self-adjoint operator (such that* $D(A^r) \hookrightarrow D(A^s)$ *is compact for* $r > s$*). Let* $x \in D(A)$, $y \in D(A^{1/2})$ *and* $f \in C^1(\mathbb{R}_+, H)$ *such that*

$$\|f(t)\|^2 \le \frac{c_f}{(1+t)^p}$$

for a positive constant c_f. *Then there exists* $C_E > 0$ *such that the solution u of* (1) *satisfies*

$$E(t,u) \le C_E E(0,u) \frac{1}{(1+t)^p}. \tag{2}$$

First of all, according to [**2**], there exists a global solution $u \in C^2(\mathbb{R}_+, H) \cap C^1(\mathbb{R}_+, D(A^{1/2})) \cap C(\mathbb{R}_+, D(A))$ of (1) whenever $x \in D(A)$, $y \in D(A^{1/2})$. From now on, u is the solution of (1). Let us start proving Theorem 2.1. The following lemmas will be helpful.

LEMMA 2.2. *Denote*

$$w(t) := u(t) - (g * u)(t). \tag{3}$$

Then there exist $K, k > 0$, *such that the following estimates hold for all* $t \in \mathbb{R}_+$ *(the values of k and K in different lines may be different).*

$$\begin{aligned}
\|w\|^2 &\le K(\|u\|^2 + g \circ u),\\
\|w_t\|^2 &\le K(\|u_t\|^2 + g(t)\|u\|^2 + g \circ u),\\
\left\|A^{1/2}w\right\|^2 &\le K\left(\left\|A^{1/2}u\right\|^2 + g \circ A^{1/2}u\right),\\
\|w_t\|^2 &\ge k\|u_t\|^2 - K(g(t)\|u\|^2 + g \circ u),\\
\left\|A^{1/2}w\right\|^2 &\ge k\left((1-G(t))\left\|A^{1/2}u\right\|^2\right) - Kg \circ A^{1/2}u.
\end{aligned}$$

Proof. To prove the first estimate we multiply (3) by w

$$\|w(t)\|^2 = (u(t), w(t)) - ((g * u)(t), w(t)). \tag{4}$$

For every $c > 0$ there exists $C > 0$ such that

$$(u, w) \le C\|u\|^2 + c\|w\|^2 \tag{5}$$

and

$$\begin{aligned}
\left|\int_0^t g(t-s)(u(s), w(t))\, ds\right| \le &\left|\int_0^t g(t-s)(u(s)-u(t), w(t))\, ds\right|\\
&+ \left|\int_0^t g(t-s)(u(t), w(t))\, ds\right|
\end{aligned}$$

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$$\le \int_0^t g(t-s)(C\|u(s)-u(t)\|^2 + c\|w(t)\|^2)\,ds + \int_0^t g(t-s)(C\|u(t)\|^2 + c\|w(t)\|^2)\,ds$$
$$\le C(g\circ u)(t) + cG(t)\|w(t)\|^2 + CG(t)\|u(t)\|^2 + cG(t)\|w(t)\|^2.$$

Inserting these estimates into (4) we obtain ($G(t) < 1$ by (g5))

$$(1-3c)\|w(t)\|^2 \le 2C\|u(t)\|^2 + C(g\circ u)(t).$$

Taking c small enough we have proved the first estimate with $K := 2C/(1-3c)$.

To show the second estimate, we multiply the derivative of (3) by w_t

$$\|w_t(t)\|^2 = (u_t(t), w_t(t)) - (\partial_t(g * u)(t), w_t(t)). \tag{6}$$

The first term on the right-hand side is estimated as in (5) and the second term can be rewritten as

$$g(0)(u(t), w_t(t)) + \int_0^t g'(t-s)(u(s)-u(t), w_t(t))\,ds + \int_0^t g'(t-s)(u(t), w_t(t))\,ds$$
$$= g(t)(u(t), w_t(t)) + \int_0^t g'(t-s)(u(s)-u(t), w_t(t))\,ds.$$

According to (g2), the integral term can be estimated by

$$c_0(C(g\circ u)(t) + cG(t)\|w(t)\|^2).$$

Inserting the estimates into (6) we obtain

$$(1-c-g(t)c-c_0c)\|w_t\|^2 \le C\|u_t(t)\|^2 + Cg(t)\|u(t)\|^2 + c_0Cg\circ u(t),$$

and taking c small enough we obtain the second estimate with $K := C(1+c_0)/(1-c-cg(0)-cc_0)$ (g is decreasing).

By the same technique we obtain the other three estimates. The third estimate follows by applying A to (3) and multiplying by w. To show the fourth and fifth estimates we differentiate (3), resp. apply A to (3), and then multiply by u_t, resp. u. In this proof we have applied assumptions (g1), (g2) and (g5). □

It is not important in Lemma 2.2 that u is the solution of (1). In fact, the estimates hold for all $u \in C(\mathbb{R}_+, D(A)) \cap C^1(\mathbb{R}_+, D(A^{1/2}))$ and the constants are independent of u.

Define $\tilde{E}$ by

$$\tilde{E}(t) := \left(\|w_t\|^2 + \left\|A^{1/2}w\right\|^2 + g(0)(w, w_t) + \frac{g(0)}{2}\|w\|^2\right).$$

It follows from Lemma 2.2, Cauchy–Schwarz inequality and $\|u\| \le \|A^{1/2}u\|$ that

$$c\left(\|u_t\|^2 + \left\|A^{1/2}u\right\|^2\right) - C\left(g\circ A^{1/2}u + \|u\|^2 + g\circ u\right)$$
$$\le \tilde{E}(t) \le C\left(\|u_t\|^2 + \left\|A^{1/2}u\right\|^2 + g\circ A^{1/2}u\right) \tag{7}$$

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for some c, $C > 0$. Moreover, the derivatives of $E(t,u)$ and $\tilde{E}(t)$ satisfy the following estimates.

LEMMA 2.3. *It holds that*

$$\frac{d}{dt}E(t,u) = -\frac{1}{2}g(t)\left\|A^{1/2}u\right\|^2 + \frac{1}{2}g' \circ A^{1/2}u + (f, u_t) \tag{8}$$

and for every $\delta > 0$ *small enough there exists* $C_\delta > 0$ *such that*

$$\frac{d}{dt}\tilde{E}(t,u) \leq -\left(\frac{g(0)}{2} - \delta\right)\left(\|w_t\|^2 + \left\|A^{1/2}w\right\|^2\right) + C_\delta(g(t)\|u\|^2 + g \circ u) + \left(f, w_t + \frac{g(0)}{2}w\right). \tag{9}$$

Both parts of this lemma are proved in [**1**]. The equality (8) follows from multiplicating (1) by u_t and some computation; the inequality (9) can be proved in the same way as Lemma 3.2 in [**1**]. Assumptions (g1), (g2), (g4) and (g5) are applied.

LEMMA 2.4. *Let* $p > 1$ *and* $q \geq 0$. *Assume that* $g(t) \leq C_1(1+t)^{-p}$ *and* $\|k^2(t)\| \leq C_2(1+t)^{-q}$ *for some* C_1, $C_2 > 0$ *and all* $t \geq 0$. *If* $0 \leq q \leq 1$, *then for every* $1 > r > (1-q)/p$ *there exists* $K > 0$ *such that*

$$g \circ k \leq K\left(g^{1+\frac{1}{p}} \circ k\right)^{\frac{(1-r)p}{1+(1-r)p}} \quad \textit{for all } t \geq 0.$$

If $q > 1$, *then there exists* $K > 0$ *such that*

$$g \circ k \leq K\left(g^{1+\frac{1}{p}} \circ k\right)^{\frac{p}{1+p}} \quad \textit{for all } t \geq 0.$$

Proof. By Hölder inequality we have for $1 < a < +\infty$

$$\begin{aligned}(g \circ k)(t) &= \int_0^t g^{\frac{1+\frac{1}{p}}{a}}(t-s)\|k(s)-k(t)\|^{\frac{1}{a}} g(t-s)^{1-\frac{1+\frac{1}{p}}{a}}(t-s)\|k(s)-k(t)\|^{1-\frac{1}{a}}\,ds \\ &\leq \left(\int_0^t \left(g^{\frac{1+\frac{1}{p}}{a}}(t-s)\|k(s)-k(t)\|^{\frac{2}{a}}\right)^a ds\right)^{\frac{1}{a}} \\ &\quad\times \left(\int_0^t \left(g^{1-\frac{1+\frac{1}{p}}{a}}(t-s)\|k(s)-k(t)\|^{2-\frac{2}{a}}\right)^{\frac{a}{a-1}} ds\right)^{1-\frac{1}{a}} \\ &= \left(\int_0^t g^{1+\frac{1}{p}}(t-s)\|k(s)-k(t)\|^2\right)^{\frac{1}{a}} \left(\int_0^t g^{\frac{a-1-\frac{1}{p}}{a-1}}(t-s)\|k(s)-k(t)\|^2\,ds\right)^{1-\frac{1}{a}}.\end{aligned} \tag{10}$$

Here the first integral on the right-hand side is exactly $g^{1+\frac{1}{p}} \circ k$, so it remains to show that the second integral is bounded by a constant independent of t for an appropriate a.

Denote

$$r := \frac{a-1-\frac{1}{p}}{a-1}.$$

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Then $r \in (-\infty, 1)$. Since $\|k(s)-k(t)\|^2 \le 2(\|k(s)\|^2 + \|k(t)\|^2)$, we can split the last integral in (10) into sum of two terms.

$$2\int_0^t g^r(t-s)\|k(s)\|^2\,ds + 2\int_0^t g^r a - 1(t-s)\|k(t)\|^2\,ds. \tag{11}$$

Let $0 < q \le 1$. Then the first term in (11) is estimated as follows.

$$\begin{aligned}
&\int_0^t g^r(t-s)\|k(s)\|^2\,ds \\
&\quad\le C_1^r C_2 \int_0^t (1+t-s)^{-pr}(1+s)^{-q}\,ds \\
&\quad\le C_1^r C_2 \left(\int_0^t ((1+s)^{-q})^{\frac{1+\varepsilon}{q}}\,ds\right)^{\frac{q}{1+\varepsilon}} \left(\int_0^t ((1+t-s)^{-pr})^{\frac{1+\varepsilon}{1+\varepsilon-q}}\,ds\right)^{1-\frac{q}{1+\varepsilon}} \le C,
\end{aligned}$$

provided

$$pr\frac{1+\varepsilon}{1+\varepsilon-q} > 1, \text{ i.e., } r > \frac{1-q}{p}$$

since $\varepsilon > 0$ is arbitrary. For the second term in (11), it holds

$$\int_0^t g^r(t-s)\|k(t)\|^2\,ds \le C_1C_2(1+t)^{-q}\int_0^t (1+t-s)^{-pr}\,ds \le C(1+t)^{-q-pr+1}.$$

This is bounded if

$$1-q-pr \le 0, \text{ i.e., } r \ge \frac{1-q}{p}.$$

Hence, if $1 > r > (1-q)/p$ we have

$$g \circ k \le K\big(g^{1+\frac{1}{p}} \circ k\big)^{\frac{1}{a}} = K\big(g^{1+\frac{1}{p}} \circ k\big)^{\frac{(1-r)p}{1+(1-r)p}}.$$

If $q = 0$, then the first term in (11) can be estimated in the same way as the second term. If $q > 1$, then the second integral in (10) is estimated by

$$2g(0)\left(\int_0^t \|k(s)\|^2\,ds + t\|k(t)\|^2\right) \le 2g(0)\tilde{C}(1+t)^{1-q} \le K,$$

provided $a \ge 1 + \frac{1}{p}$. The assertion for $q > 1$ follows. □

LEMMA 2.5. *Let $p > 1$ and $k > 0$ such that $\|f(t)\|^2 \le k(1+t)^{-p-1}$ and $g \le k(1+t)^{-p}$. Let $1 \ge q \ge 0$ such that $\|A^{1/2}u(t)\|^2 \le k(1+t)^{-q}$. Then $\|A^{1/2}u(t)\| \le K(1+t)^{-\tilde{q}}$ for some $K > 0$ and $\tilde{q} = q + \varepsilon$, where $\varepsilon > 0$ is small enough, depending on p but independent of q.*

Proof. By the previous Lemma we have

$$g \circ A^{1/2}u(t) \le C\big(g^{1+\frac{1}{p}} \circ A^{1/2}u(t)\big)^{\frac{(1-r)p}{1+(1-r)p}} \tag{12}$$

for all $1 > r > (1-q)/p$. Take $L(t) := \nu E(t,u) + \tilde{E}(t)$ for $\nu > 0$ large enough. The following estimate follows from Lemma 2.3 by applying Cauchy–Schwarz inequality

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to the terms containing f, assumption (g3) to the term containing g' and Lemma 2.2 to the terms containing w.

$$\frac{d}{dt}L(t) \le \nu\left(-\frac{1}{2}g(t)\left\|A^{1/2}u(t)\right\|^2 - \frac{1}{2}g^{1+\frac{1}{p}} \circ A^{1/2}u(t)\right) - C\big((1-G(t))\left\|A^{1/2}u(t)\right\|^2. + (1-\delta)\|u_t\|^2\big) + g(t)\|u(t)\|^2 + C_\delta\big(g \circ u + g \circ A^{1/2}u\big) + \nu C_\delta \|f\|^2.$$

Here $0 < \delta < 1$ and $C, C_\delta > 0$. By $\|u(t)\| \le c\|A^{1/2}u(t)\|$ and (12) we obtain for ν large enough

$$\frac{d}{dt}L(t) \le -C\big((1-G(t))\left\|A^{1/2}u(t)\right\|^2 + \|u_t\|^2\big) - C\left(g \circ A^{1/2}u(t)\right)^{\frac{1+(1-r)p}{(1-r)p}} + \nu C_\delta \|f\|^2$$
$$\le -C\big((1-G(t))\left\|A^{1/2}u(t)\right\|^2 + \|u_t\|^2 + g \circ A^{1/2}u(t)\big)^{\frac{1+(1-r)p}{(1-r)p}} + \nu C_\delta \|f\|^2.$$

Since $\tilde{E}(t) \le cE(t,u)$, we obtain

$$\frac{d}{dt}L(t) \le -C(L(t))^{\frac{1+(1-r)p}{(1-r)p}} + \nu C_\delta \|f\|^2.$$

Hence,

$$L(t) \le CL(0)(1+t)^{(1-r)p} \quad \text{and also} \quad \left\|A^{1/2}u(t)\right\|^2 \le CL(0)(1+t)^{(1-r)p}. \tag{13}$$

Let $0 < \tilde{\varepsilon} < 1 - 1/p$. Set $r := (1-q)/p + \tilde{\varepsilon}$ and $\tilde{q} := (1-r)p$. Then $1 > r > (1-q)/p$ and $\tilde{q} = q + (p - 1 - \tilde{\varepsilon}p) > q$. We have proved the assertion with $\tilde{q} = q + \varepsilon$, where $\varepsilon = p - 1 - \tilde{\varepsilon}p > 0$ is independent of q. □

LEMMA 2.6. *There exists $C > 0$ such that $\|A^{1/2}u(t)\| \le C$ for all $t \ge 0$.*

Proof. According to Theorem 5.1 in **[2]**, the solution v of the homogeneous equation

$$\ddot{u}(t) = -Au(t) - \int_0^t g(t-s)Au(s)\,ds,$$
$$u(0) = x, \quad \dot{u}(0) = y,$$

satisfies $v, \dot{v} \in L^2(\mathbb{R}_+, X)$. Integrating (1) we obtain

$$\dot{u}(t) = -\int_0^t \tilde{G}(t-s)Au(s)\,ds + F(t) + y,$$

where $\tilde{G}(\cdot)$ is the primitive function of g with $G(0) = 1$ and $F(t) := \int_0^t f(s)\,ds$. It follows that the solution of the inhomogeneous equation is given by

$$u(t) := v(t) + \int_0^t (F(t-s) + y)v(s)\,ds.$$

Hence,

$$\dot{u}(t) = \dot{v}(t) + (F(0) + y)v(t) + \int_0^t f(t-s)v(s).$$

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Since v, $\dot{v} \in L^2$ and $f \in L^1$, we obtain that $\dot{u} \in L^2$. Now it follows from (8) that

$$E(t, u) \leq E(0, u) + \int_0^t \|f(s)\| \cdot \|\dot{u}(s)\| \, ds \leq E(0, u) + \|f\|_2 \|\dot{u}\|_2.$$

Hence, $\|A^{1/2}u\|^2$ is bounded. □

We will finish the proof of Theorem 2.1. Since $\|A^{1/2}u(t)\|$ is bounded, i.e., assumptions of Lemma 2.5 hold, we obtain

$$\left\|A^{1/2}u(t)\right\| \leq c(1+t)^{-q}$$

for some $q > 1$ by applying Lemma 2.5 finitely many times. Then by Lemma 2.4 we obtain (12) with $r = 0$ and the proof of Lemma 2.5 yields (see (13))

$$L(t) \leq CL(0)(1+t)^p.$$

Estimating $\tilde{E}$ according to (7), we obtain $\nu E(t, u) + \tilde{E}(t) \geq (\nu - C)E(t, u)$. Hence, (2) holds.

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