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Auslander–Reiten sequences, locally free sheaves and Chebysheff polynomials

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Dedicated to Claus M. Ringel on the occasion of his 60th birthday

Abstract

Let R be the exterior algebra in $n+1$ variables over a field K. We study the Auslander–Reiten quiver of the category of linear R-modules, and of certain subcategories of the category of coherent sheaves over $\mathbf{P}^n$. If $n > 1$, we prove that up to shift, all but one of the connected components of these Auslander–Reiten quivers are translation subquivers of a $\mathbf{Z}A_\infty$-type quiver. We also study locally free sheaves over the projective n-space $\mathbf{P}^n$ for $n > 1$ and we show that each connected component contains at most one indecomposable locally free sheaf of rank less than n. Finally, using results from the theory of finite-dimensional algebras, we construct a family of indecomposable locally free sheaves of arbitrary large ranks, where the ranks can be computed using the Chebysheff polynomials of the second kind.

1. Introduction

It is well known that the derived category of coherent sheaves on $\mathbf{P}^n$ has Auslander–Reiten triangles, but that the category of coherent sheaves itself, does not have Auslander–Reiten sequences, unless $n = 1$. In this paper we show that the category $\mathcal{C}$ of coherent sheaves on $\mathbf{P}^n$ can be exhausted by an ascending chain of full subcategories $\mathcal{C}_i$, each having left Auslander–Reiten sequences, and we determine the structure of the Auslander–Reiten components. Moreover, we deal with the rank growth on Auslander–Reiten components in $\mathcal{C}_i$ and the existence of vector bundles of low ranks in each such component. We also construct a family of indecomposable locally free sheaves whose ranks can be computed using the Chebysheff polynomials of the second kind.

We recall first some definitions and preliminary results. Throughout this article, R will denote the exterior algebra in $n+1$ variables $x_0, x_1, \ldots, x_n$ over an algebraically closed field K. That is, $R = \bigwedge V$, where V is the $n+1$-dimensional K-vector space with the basis $\{x_0, x_1, \ldots, x_n\}$. It is easy to describe R in terms of generators and relations: $R = K\langle x_0, \ldots, x_n\rangle / \langle x_i x_j + x_j x_i, x_i^2\rangle_{i,j}$, where $K\langle x_0, \ldots, x_n\rangle$ is the free algebra in the variables $x_0, x_1, \ldots, x_n$. In this way, R is a graded K-algebra, that is, $R = R_0 \oplus R_1 \oplus \cdots$, where the K-dimension of R_i is finite and equals $\binom{n+1}{i}$, and where for each $i, j \geqslant 0$ we have $R_i R_j = R_{i+j}$. We also see that the initial subalgebra $R_0 \cong K$. The radical of R is the homogeneous ideal $J = R_1 \oplus R_2 \oplus \cdots$. By mod$R$ we denote the category of all finitely generated R modules, by GrR the category of all graded modules and degree zero morphisms, and by gr$R$ the subcategory of all finitely generated graded R modules and degree zero homomorphisms. For a graded R-module $M = M_i \oplus M_{i+1} \oplus \cdots$, as M_i is a R_0 summand of M/JM, we abuse the language, and say that M_i is the 'highest degree' part of M. If M is a graded R-module, we set its

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ith shift $M[i]$ to be the graded R-module such that for each j, $M[i]_j = M_{i+j}$. A graded R module M is a *linear* module, if it is generated in degree 0, and if it has a (minimal) free resolution:

$$\cdots \to \mathcal{P}_n \xrightarrow{f_n} \mathcal{P}_{n-1} \to \cdots \to \mathcal{P}_0 \xrightarrow{f_0} M \to 0$$

such that $\mathcal{P}_k$ is generated in degree k for each $k \geqslant 0$. Linear modules have also been called Koszul modules in the literature [GM96, GMRSZ98, Mar99, MZ03]. We denote by $\mathcal{K}_R$ the full subcategory of $\operatorname{gr} R$ consisting of the linear modules. The Yoneda ext-algebra of R (or its Koszul dual), is the cohomology algebra

$$R^! = \bigoplus_{n\geqslant 0} \operatorname{Ext}^n_R(R_0, R_0)$$

where the multiplication in $R^!$ is defined by the Yoneda product. It is well known that $R^!$ is isomorphic to the commutative polynomial algebra in $n+1$ variables $S = K[x_0, x_1, \ldots, x_n]$. It is also known that both the exterior algebra R, and the polynomial algebra S are Koszul algebras and we have mutually inverse Koszul dualities

$$\mathcal{K}_R \underset{\mathcal{F}}{\overset{\mathcal{E}}{\rightleftarrows}} \mathcal{K}_S$$

given by $\mathcal{E}(M) = \bigoplus_{n\geqslant 0} \operatorname{Ext}^n_R(M, R_0)$ with the obvious action of S on $\mathcal{E}(M)$, and with the duality $\mathcal{F}$ defined in a similar way; see also [BGS96].

In studying the linear modules over a finite-dimensional selfinjective Koszul algebra such as the exterior algebra, we will use Auslander–Reiten sequences. Recall that, given a subcategory $\mathcal{C}$ of $\operatorname{gr} R$, a nonsplit exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

in $\mathcal{C}$ is an Auslander–Reiten sequence, if A and C are indecomposable, and if every nonisomorphism $f\colon X \to C$ in $\mathcal{C}$ with X indecomposable can be lifted to B. Auslander–Reiten sequences (also called almost split sequences), exist in $\operatorname{gr} R$ and are unique up to isomorphism. We say that a subcategory $\mathcal{C}$ has *right* Auslander–Reiten sequences, if for each indecomposable nonprojective $C \in \mathcal{C}$, there exists an Auslander–Reiten sequence in $\mathcal{C}$ ending at C. We can similarly define the notion of having *left* Auslander–Reiten sequences in $\mathcal{C}$. A related notion, is that of an irreducible morphism in a subcategory $\mathcal{C}$ of $\operatorname{gr} R$. If M and N are indecomposable objects in $\mathcal{C}$, then a nonzero morphism $f\colon M \to N$, is irreducible, if for every factorization of f in $\mathcal{C}$, $f = gh$, then g is a splittable epimorphism, or h is a splittable monomorphism. A lot of the information dealing with Auslander–Reiten sequences in a subcategory $\mathcal{C}$, is contained in a directed graph called the Auslander–Reiten quiver of $\mathcal{C}$. For more facts about Auslander–Reiten sequences, irreducible morphisms, and Auslander–Reiten quivers, see [AR75, ARS95, Rin84]. We will always denote by $\operatorname{gr}_0 R$, the full subcategory of $\operatorname{gr} R$ consisting of all the modules generated in degree zero. Note that $\mathcal{K}_R$ is a full subcategory of $\operatorname{gr}_0 R$.

The paper is organized as follows: in § 2 we show that certain subcategories of the category of coherent sheaves over the projective n-space have left Auslander–Reiten sequences and we study their Auslander–Reiten quivers. This is done by first analyzing the Auslander–Reiten quiver of the category of linear R-modules, and then translating this knowledge over to $\operatorname{coh}(\mathbf{P}^n)$. We show that for $n > 1$, except for one component that is of type $\mathbf{Z}\Delta$, where Δ is a quiver of the generalized Kronecker algebra (that is Δ has two vertices and $n+1$ arrows from the first vertex to the second vertex), all the other stable connected components of the Auslander–Reiten quiver of $\mathcal{K}_R$ are full subquivers of a quiver of the form $\mathbf{Z}A_\infty$. We should remark here that Auslander–Reiten sequences were known to exist in certain interesting situations already; for instance, they exist in the category of coherent sheaves over a nonsingular projective curve (Auslander–Reiten in [AR87], also A. Schofield had an unpublished proof at around the same time). Also, Geigle and Lenzing have shown in [GL87], that

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for a weighted projective line $\mathbb{X}$, Auslander–Reiten sequences exist in the category $\mathrm{coh}(\mathbb{X})$. In § 3 we continue the study of the Auslander–Reiten quivers of $\mathcal{K}_R$ and we sharpen our results from the previous section. We also study the ranks of the locally free sheaves in $\mathrm{coh}(\mathbf{P}^n)$ (see also [BGG79]) by looking to corresponding modules over the exterior algebra, and, using the representation theory of the generalized Kronecker algebras, we construct a family of indecomposable locally free sheaves of arbitrary large ranks, and show that these ranks can be computed using the Chebysheff polynomials. Throughout this paper, R and S denote the exterior (respectively polynomial) algebra in $n+1$ variables. For general facts about coherent and locally free sheaves we refer to [Har87] and its exercises.

2. The Auslander–Reiten quivers of $\mathcal{K}_R$ and of some subcategories of $\mathrm{coh}(\mathbf{P}^n)$

In this section, we use the structure of the Auslander–Reiten quiver of the category $\mathcal{K}_R$ of linear R-modules, to study some subcategories of the category $\mathrm{coh}(\mathbf{P}^n)$ of coherent sheaves over $\mathbf{P}^n$. We recall from [GMRSZ98], that if M is an indecomposable nonprojective linear module, then there exists an Auslander–Reiten sequence in the category $\mathcal{K}_R$ ending at M. This sequence is constructed by first looking at the usual Auslander–Reiten sequence ending at M in $\mathrm{gr}\, R$

$$0 \longrightarrow \tau M \longrightarrow F \longrightarrow M \longrightarrow 0$$

and then chopping off the part in negative degrees. Defining $\sigma M = (\tau M)_{\geqslant 0} = (\tau M)_0 \oplus (\tau M)_1 \oplus \cdots$ and $E = F_{\geqslant 0}$, we obtain a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \sigma M & \longrightarrow & E & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \tau M & \longrightarrow & F & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

It was proved in [GMRSZ98] that σM is an indecomposable linear module, and that the induced sequence

$$0 \longrightarrow \sigma M \longrightarrow E \longrightarrow M \longrightarrow 0$$

in $\mathcal{K}_R$ is an Auslander–Reiten sequence in $\mathcal{K}_R$. Thus, $\mathcal{K}_R$ has *right* Auslander–Reiten sequences. It is interesting to note that σM has Loewy length exactly two, hence it is also a R/J^2-module. We also recall that the sequence $0 \to \sigma M \to E \to M \to 0$ is also an Auslander–Reiten sequence when viewed in the subcategory $\mathrm{gr}_0\, R$. This will be used later in this section. Now using the Koszul duality, we observe that we can transport an Auslander–Reiten sequence from $\mathcal{K}_R$ into an Auslander–Reiten sequence in $\mathcal{K}_S$. Furthermore, by shifting it becomes evident that for all integers i, the subcategories $\mathcal{K}_S[i]$ of the category of finitely generated graded S-modules have *left* Auslander–Reiten sequences.

Let $Q\,\mathrm{gr}\, S$ be the quotient category of $\mathrm{gr}\, S$ modulo the finite length modules (or the category of *tails* of $\mathrm{gr}\, S$). Recall that $Q\,\mathrm{gr}\, S$ has the same objects as $\mathrm{gr}\, S$, and $\mathrm{Hom}_{Q\,\mathrm{gr}\, S}(M, N)$ is a direct limit of $\mathrm{Hom}_{\mathrm{gr}\, S}(M', N/N')$ where the limit is taken over all the pairs (M', N') such that M/M' and N' are of finite length. Let $\pi\colon\ \mathrm{gr}\, S \to Q\,\mathrm{gr}\, S$ denote the canonical functor. Then, if X and Y are two finitely generated graded S-modules, we see that we have

$$\mathrm{Hom}_{Q\,\mathrm{gr}\, S}(\pi(Y), \pi(X)) = \bigcup_{t \geqslant 0} \mathrm{Hom}_{\mathrm{gr}\, S}(Y_{\geqslant t}, X).$$

Now, recall the result of Avramov and Eisenbud ([AE92]) stating that for each finitely generated graded S-module X, there exists an integer k such that the shifted truncation $X_{\geqslant k}[k]$ is a linear S-module. A well-known theorem of Serre states that there exists an equivalence of categories $\Phi\colon Q\,\mathrm{gr}\,S \xrightarrow{\sim} \mathrm{coh}\,\mathbf{P^n}$. We have a diagram

$$\mathcal{K}_R \xrightarrow{\mathcal{E}} \mathcal{K}_S \xrightarrow{\pi} Q\,\mathrm{gr}\,S \xrightarrow{\Phi} \mathrm{coh}(\mathbf{P}^n)$$

For a linear R-module M, denote by $\mathbf{G}^{(i)}(M)$ the composition $\Phi\pi(\mathcal{E}(M))[i]$. We set $\mathbf{G}(M) = \mathbf{G}^{(0)}(M) = \Phi\pi\mathcal{E}(M)$. Then, we know that to each graded R-module M corresponds a coherent sheaf $\mathbf{G}(M)$, and if we also assume that a linear module $M \neq R$ is indecomposable, then so is the sheaf $\mathbf{G}(M)$, as its endomorphism ring is local. Then, we can use the preceding discussion to show that, to each indecomposable coherent sheaf $\mathcal{M}$, corresponds an indecomposable *linear* nonprojective R-module M such that $\mathbf{G}^{(i)}(M) = \mathcal{M}$ for some integer i. It is also well known that we can express the quotient category $Q\,\mathrm{gr}\,S$ as a union $\bigcup_{i\in\mathbf{Z}} \pi(\mathcal{K}_S[i])$, and that for each $i \in \mathbf{Z}$, we have $\pi(\mathcal{K}_S[i]) \subseteq \pi(\mathcal{K}_S[i-1])$. For a graded S-module X, we denote by $\widetilde{X} = \Phi\pi(X)$. By applying Serre's theorem we obtain

$$\mathrm{coh}(\mathbf{P}^n) = \bigcup_{i\in\mathbf{Z}} \widetilde{\mathcal{K}_S[i]}.$$

We will show that each of the subcategories $\widetilde{\mathcal{K}_S[i]}$ have left Auslander–Reiten sequences and we will describe the shape of their Auslander–Reiten quivers. Before we do this, we want to describe the Auslander–Reiten quiver of $\mathcal{K}_R$. Recall the following from [GMRSZ98]: let $\mathcal{L}_R$ be the full subcategory of $\mathrm{gr}_0\,R$ consisting of all of the modules X having a linear presentation, that is a free presentation of the form

$$P_1 \longrightarrow P_0 \longrightarrow X \longrightarrow 0$$

where P_1 is generated in degree 1, and P_0 is generated in degree zero. It was shown that the functor $R/J^2 \otimes_R\, ?\colon \mathcal{L}_R \longrightarrow \mathrm{gr}_0\,R/J^2$ is an equivalence of categories. Applying this to our situation, we obtain a nonsplit exact sequence

$$0 \longrightarrow \sigma M \xrightarrow{\tilde{f}} E/J^2E \xrightarrow{\tilde{g}} M/J^2M \longrightarrow 0$$

whose end terms are indecomposable. We have the following result [GMRSZ98].

Theorem 2.1. *Let $M \neq R$ be an indecomposable linear module, and let*

$$0 \longrightarrow \sigma M \xrightarrow{f} E \xrightarrow{g} M \longrightarrow 0$$

be the Auslander–Reiten sequence ending at M in $\mathcal{K}_R$. Let $E = E_1 \oplus \cdots \oplus E_k$ be a decomposition of E into indecomposable summands, and let $f = [f_1 \quad f_2 \quad \ldots \quad f_k]$ and $g = [g_1 \quad g_2 \quad \ldots \quad g_k]^{\mathrm{T}}$. Then:

(i) *the induced sequence*

$$0 \longrightarrow \sigma M \xrightarrow{\tilde{f}} E/J^2E \xrightarrow{\tilde{g}} M/J^2M \longrightarrow 0$$

is the Auslander–Reiten sequence ending at M/J^2M in $\mathrm{gr}_0\,R/J^2$;

(ii) *the number of indecomposable summands of E, equals the number of indecomposable summands of E/J^2E.*

As an immediate application of this result we see that if M is a linear module of Loewy length 2, then the Auslander–Reiten sequence in $\mathcal{K}_R$ ending at M, coincides with that in $\mathrm{gr}_0\,R/J^2$. In particular, if R is the exterior algebra in two variables only, and if M is an indecomposable nonprojective linear module, then the Auslander–Reiten sequences ending at M in $\mathcal{K}_R$, in $\mathrm{gr}_0\,R/J^2$ and in $\mathrm{gr}\,R$ coincide. Let us assume now that $M \neq R$ is indecomposable in $\mathcal{K}_R$. Then M/J^2M is again an indecomposable R/J^2 module, so we have to analyze the Auslander–Reiten quiver of R/J^2. By [Rei75], the algebra R/J^2 is stably equivalent to the (hereditary) generalized Kronecker algebra with $n+1$ arrows, so we obtain that M/J^2M either belongs to the preprojective or preinjective components

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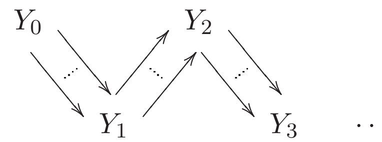
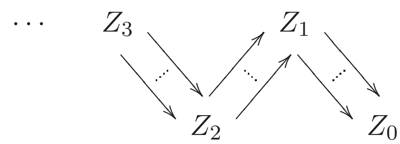
of $\mathrm{gr}_0\, R/J^2$, or it lies in a $\mathbf{Z}A_\infty$ component (see [Rin84]). The 'preprojective' component has the form



where $Y_0 = R/J^2$, and the 'preinjective' component has the form



where $Z_1 = \mathrm{soc}^2 R$ is the unique indecomposable injective R/J^2-module, and $Z_0 = K$ is the unique graded R/J^2-simple module. We show first that one of these possibilities cannot occur.

Lemma 2.2. *Assume that $n > 1$. If $M \neq R$ is indecomposable in $\mathcal{K}_R$, then M/J^2M cannot lie in the preprojective component of $\mathrm{gr}_0\, R/J^2$.*

Proof. If it does, it follows from the previous result that σM is also in the preprojective component. This component has only two σ-orbits, one of them being the σ-orbit of R/J^2. If $\sigma(M)$ is in the orbit of R/J^2, then by induction, R/J^2 is the σ of a linear module, therefore itself must be a linear module. We know, however, that if $n > 1$, R/J^2 is not in $\mathcal{K}_R$, and we obtain a contradiction. Assume that σM is in the orbit of $Y_1 = \tau^{-1}_{R/J^2}(K)$ where K is the trivial graded R-module. This implies by induction that Y_1 is linear. On the other hand, it is easy to verify that over R, $Y_1 = \tau^{-1}_{R/J^2}(K)$ cannot be a linear module since this would imply that σY_1 is simple and this is impossible. □

We have enough to describe the shape of the Auslander–Reiten quivers of $\mathcal{K}_R$ and of the subcategories $\mathcal{K}_S[i]$ of $\mathrm{gr}\, S$. Recall that if Γ is the Auslander–Reiten quiver of some additive category, then the *cone* of an object M is the full subquiver of Γ consisting of all the predecessors of M in Γ. Using our previous results and the preceding remarks, it is not hard to see that, if $M \neq R$ is an indecomposable module in $\mathcal{K}_R$, then the cones of M in $\mathcal{K}_R$ and of M/J^2M in $\mathrm{gr}\, R/J^2$ are isomorphic via an isomorphism that can be easily shown to take monomorphisms into monomorphisms and epimorphisms into epimorphisms. Moreover, as the R-modules $I = \mathrm{soc}^2 R$ and K are linear, the entire preinjective component of $\mathrm{gr}_0\, R/J^2$ consists of linear R-modules. Putting together these facts gives us the following result.

Proposition 2.3.

(i) *Let R denote the exterior algebra in $n+1$ variables, where $n > 1$. The Auslander–Reiten quiver of $\mathcal{K}_R$ has a connected component that coincides with the preinjective component of $\mathrm{gr}_0\, R/J^2$, a component consisting only of the module R, and all the remaining connected components are full subquivers of a quiver of type $\mathbf{Z}A_\infty$.*

(ii) *Let S denote the polynomial algebra in $n+1$ variables where $n > 1$. For each integer i, the Auslander–Reiten quiver of $\mathcal{K}_S[i]$ has one preprojective component, and all the other components are full subquivers of a quiver of type $\mathbf{Z}A_\infty$.*

We will see in the following section that we can actually say more about the shape of the components of $\mathcal{K}_R$ and of $\mathcal{K}_S[i]$ that are full subquivers of a $\mathbf{Z}A_\infty$-type quiver. Namely, we will show that in the exterior algebra case, these components are of the type $\mathbf{N}^-A_\infty$, and that in the symmetric algebra case they are of the type $\mathbf{N}A_\infty$.

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If $n = 1$, the exterior algebra R is of tame representation type, and the stable Auslander–Reiten quiver of the category of linear R-modules consists of a preinjective component, and of tubes of rank one. If $n > 1$, as the preinjective component of $\mathcal{K}_R$ consists of linear modules of Loewy length at most two, it turns out that each linear module $M \neq R$ of Loewy length three or higher must lie in a 'regular' component. The following result yields examples of modules in the regular components of $\mathcal{K}_R$ lying at the mouths of these components (see also [GMRSZ98]).

Proposition 2.4. *Let $M \neq R$ be an indecomposable linear R-module and assume that M has no simple submodule concentrated in degree 1. Let*

$$0 \longrightarrow \sigma M \xrightarrow{f} E \xrightarrow{g} M \longrightarrow 0$$

be the Auslander–Reiten sequence in $\mathcal{K}_R$ ending at M. Then, the middle term E is indecomposable.

Proof. Assume that E decomposes into $k > 1$ indecomposable summands. Then $f = [f_1 \quad f_2 \quad \dots \quad f_k]$ and $g = [g_1 \quad g_2 \quad \dots \quad g_k]^{\mathrm{T}}$. It is clear that each composition $g_i f_i$ is nonzero. On the other hand, it was proved in [MZ03], that τM, and therefore σM too, are cogenerated in degree 1, that is their socles are concentrated entirely in degree 1. Hence, we obtain a contradiction to the fact that M has no cogenerators in degree 1. □

Theorem 2.5. *For each integer i, the subcategory $\widetilde{\mathcal{K}_S[i]}$ of the category $\mathrm{coh}(\mathbf{P}^n)$ has left Auslander–Reiten sequences.*

Proof. It is enough to show that there are left Auslander–Reiten sequences in $\pi(\mathcal{K}_S[i])$. Let A be an indecomposable module in $\mathcal{K}_S$ and let $0 \to A \to B \to C \to 0$ be an Auslander–Reiten sequence in $\mathcal{K}_S[i]$. It is easy to see that the induced exact sequence

$$0 \to \pi(A) \to \pi(B) \to \pi(C) \to 0$$

does not split in the quotient category, and that its end terms are indecomposable. Let N be an indecomposable module in $\mathcal{K}_S[i]$, and let $h\colon \pi(A) \to \pi(N)$ be a non-isomorphism. Consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi(A) & \xrightarrow{\hat{f}} & \pi(B) & \xrightarrow{\hat{g}} & \pi(C) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \hat{h}} & & & & & & \\ & & \pi(N) & & & & & & \end{array}$$

We may assume that these morphisms are induced from the following diagram in $\mathcal{K}_S[i-t]$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J^t A & \xrightarrow{f} & J^t B & \xrightarrow{g} & J^t C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle h} & & & & & & \\ & & J^t N & & & & & & \end{array}$$

for some $t > 0$. Passing back over to R using the Koszul duality we have the following diagram in $\mathcal{K}_R[-t]$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^t\mathcal{E}(C[i]) & \longrightarrow & \Omega^t\mathcal{E}(B[i]) & \longrightarrow & \Omega^t\mathcal{E}(A[i]) & \longrightarrow & 0 \\ & & & & & & \uparrow{\scriptstyle \mathcal{E}(h)} & & \\ & & & & & & \Omega^t\mathcal{E}(N[i]) & & \end{array}$$

It is easy to see that $\mathcal{E}(h)$ does not factor through a projective module in $\mathcal{K}_R[t]$, so we can use the equivalence of categories $\Omega^{-1}\colon \underline{\mathrm{gr}}R \to \underline{\mathrm{gr}}R$. As the sequence

$$0 \longrightarrow \mathcal{E}(C[i]) \longrightarrow \mathcal{E}(B[i]) \longrightarrow \mathcal{E}(A[i]) \longrightarrow 0$$

is an Auslander–Reiten sequence in $\mathcal{K}_R$, and the module $\mathcal{E}(N[i])$ is linear, it follows that the morphism $\Omega^{-t}\mathcal{E}(h)$ can be lifted, so $\hat{h}$ can be extended and the proof is complete. □

We use now the above theorem, and Proposition 2.3, to describe the shapes of the Auslander–Reiten quivers of the categories $\widetilde{\mathcal{K}_S[i]}$.

Theorem 2.6. *For each integer i, the Auslander–Reiten quiver of the subcategory $\widetilde{\mathcal{K}_S[i]}$ of $\mathrm{coh}(\mathbf{P}^n)$, where $n > 1$, has one preprojective component, and the remaining components are all full subquivers of a quiver of the type $\mathbf{Z}A_\infty$.*

In the case when $n = 1$, we can use the ideas from the proof of Theorem 2.5 to show that $\mathrm{coh}(\mathbf{P}^1)$ has left Auslander–Reiten sequences. Namely, we use the fact that over the exterior algebra in two variables applying any positive power of the syzygy functor Ω to the sequence

$$0 \longrightarrow \mathcal{E}(C[i]) \longrightarrow \mathcal{E}(B[i]) \longrightarrow \mathcal{E}(A[i]) \longrightarrow 0$$

yields (a shift) of an Auslander–Reiten sequence in $\mathcal{K}_R$, hence the maps $\mathcal{E}(h)$ in the proof of the previous theorem can be always lifted. This means that we can transport Auslander–Reiten sequences from $\mathcal{K}_R$ over $\mathrm{coh}(\mathbf{P}^1)$. Thus, in the category $\mathrm{coh}(\mathbf{P}^1)$ we have tubes of rank one containing the torsion sheaves, and also a component that comes from the preinjective component over the linear R-modules. We can say more in this case: each sheaf not lying on a tube is invertible and thus locally free. If $\mathcal{F}$ is such a sheaf, look at its dual sheaf $\mathcal{F}^*$. Using what we have obtained so far, there exists a *left* Auslander–Reiten sequence starting from $\mathcal{F}^*$ in $\mathrm{coh}(\mathbf{P}^1)$ and each indecomposable sheaf appearing in this sequence must be torsion free. Dualizing this sequence yields a *right* Auslander–Reiten sequence ending at $\mathcal{F}$ in the category of coherent sheaves. This means that the 'preprojective'-like component becomes a transjective component, and as an easy application, we have just recaptured the following well-known result.

Corollary 2.7. *The category $\mathrm{coh}(\mathbf{P}^1)$ has Auslander–Reiten sequences.*

3. Remarks on locally free sheaves

In this section we study the subcategory of locally free sheaves over the projective space, and we use the information developed in the previous section to construct a family of indecomposable locally free sheaves, whose ranks are given by the Chebysheff polynomials of the second kind. We also address the possible location of locally free sheaves of small ranks in the Auslander–Reiten quiver of each subcategory $\widetilde{\mathcal{K}_S[i]}$ of $\mathrm{coh}(\mathbf{P}^n)$. Observe first, that if $\mathcal{M}$ is an indecomposable locally free sheaf, and if we write $\mathcal{M} = \mathbf{G}^{(-i)}(M)$ for some indecomposable R-linear module M, then $\mathbf{G}(\Omega M[1])$, $\mathbf{G}(M_{\geqslant k}[k])$, and thus $\mathbf{G}(\sigma M)$ are also locally free sheaves. This implies that if $\mathcal{M}$ is an indecomposable locally free sheaf, and if

$$0 \to \mathcal{M} \longrightarrow \mathcal{B}_1 \oplus \cdots \oplus \mathcal{B}_i \longrightarrow \delta^{-1}\mathcal{M} \longrightarrow 0$$

is the Auslander–Reiten sequence starting at $\mathcal{M}$ in some $\widetilde{\mathcal{K}_S[i]}$, then each term $\mathcal{B}_i$, and $\delta^{-1}\mathcal{M}$ are also locally free. This means that the successors in the Auslander–Reiten quiver of each locally free sheaf are also locally free. We have the following immediate consequence of this fact.

Proposition 3.1. *The preprojective component of the Auslander–Reiten quiver of each subcategory $\widetilde{\mathcal{K}_S[i]}$ of $\mathrm{coh}(\mathbf{P}^n)$ consists entirely of locally free sheaves.*

Proof. The sheaves $\mathbf{G}(K)$ and $\mathbf{G}(\mathrm{soc}^2 R[2])$ are locally free, as by applying the Koszul duality, K corresponds to the S-module S, and $\mathrm{soc}^2 R[2]$ to a syzygy of the trivial S-module K. As the preprojective component consists only of the orbits of $\mathbf{G}^{(-i)}(K)$ and $\mathbf{G}^{(-i)}(\mathrm{soc}^2 R[2])$, the result follows immediately. □

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It is possible to compute the ranks of the locally free sheaves in $\mathrm{coh}(\mathbf{P}^n)$ by doing an easy computation in the category of linear R-modules. First, if A is a finitely generated linear S-module, then its sheafication $\widetilde{A}$ is locally free if and only if each localization $A_{\mathfrak{p}}$ is free over $S_{\mathfrak{p}}$ for each maximal relevant homogeneous ideal $\mathfrak{p}$ of S. Recall now that the *Euler number* of A is $\chi(A) = \sum_{i=0}^{t}(-1)^i \operatorname{rank} F_i$, where

$$0 \to F_t \to F_{t-1} \to \cdots \to F_1 \to F_0 \to A \to 0$$

is a minimal free resolution of A. As A is linear, its Euler number equals the rank of $\widetilde{A}$. If $M = M_0 \oplus M_1 \oplus \cdots \oplus M_p$ is a linear R-module such that $\mathcal{E}(M) = A$ (so that we have $\mathbf{G}(M) = \widetilde{A}$), we define the *sheaf rank of* M as the alternating sum $\sum_{i=0}^{p}(-1)^i \dim M_i$. From the Koszul duality, it is clear that the sheaf rank of M equals the rank of $\mathbf{G}(M)$. In the case that $\mathcal{F}$ lies in one of the preprojective components then we may assume that up to shift, $\mathcal{F}$ is obtained by sheafifying a linear S-module of projective dimension one. Then the rank of $\mathcal{F}$ equals the sheaf rank of a corresponding linear R-module $M = M_0 \oplus M_1$ that is in the preinjective component of $\mathcal{K}_R$. Thus, the rank of $\mathcal{F}$ equals $\dim M_0 - \dim M_1$. It is rather easy to compute dimension vectors for the modules lying in the preinjective component of $\mathcal{K}_R$, and we have the following.

Proposition 3.2. *Let $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2 \dots$ be the locally free sheaves lying in the preprojective component of the subcategory $\widetilde{\mathcal{K}_S[i]}$ of $\mathrm{coh}(\mathbf{P}^n)$. Denote by*

$$T_k(x) = \sum_{m=0}^{[k/2]} (-1)^m \binom{k-m}{m} (2x)^{k-2m}$$

the Chebysheff polynomials of the second kind. Then for each $k \geqslant 1$, we have

$$\operatorname{rank} \mathcal{F}_k = T_k\left(\frac{n+1}{2}\right) - T_{k-1}\left(\frac{n+1}{2}\right).$$

In addition, if $n > 1$, then for each k, $\operatorname{rank} \mathcal{F}_{k+1} > \operatorname{rank} \mathcal{F}_k$.

Proof. For a module $M = M_0 \oplus M_1$ in the preinjective component of $\mathcal{K}_R$, we use the structure of the preinjective component of R/J^2 to describe the preinjective component of $\mathcal{K}_R$ and to compute $\dim M_0 - \dim M_1$. Denote by $Z_0, Z_1, \dots, Z_k, \dots$ the modules in the preinjective component of $\mathcal{K}_R$. Then, for each $k \geqslant 0$, we have Auslander–Reiten sequences

$$0 \to Z_{k+2} \to Z_{k+1}^{n+1} \to Z_k \to 0$$

where $\dim(Z_0)_0 = 1$, $\dim(Z_0)_1 = 0$, and $\dim(Z_1)_0 = n+1$, $\dim(Z_1)_1 = 1$. An easy computation shows that for each integer $k \geqslant 1$, we have

$$\dim(Z_{k+1})_0 = (n+1)\dim(Z_k)_0 - \dim(Z_{k-1})_0$$

and also that $\dim(Z_s)_1 = \dim(Z_{s-1})_0$. It is obvious that all these dimensions are polynomials in $n+1$, and that by putting $2x = n+1$, we obtain that for each integer $k \geqslant 1$, the $\dim(Z_k)_0$ satisfies the usual recurrence formula for the Chebysheff polynomials of the second kind

$$T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x).$$

The inequality $\operatorname{rank} \mathcal{F}_{k+1} > \operatorname{rank} \mathcal{F}_k$ also follows immediately. □

If $n = 1$, the only locally free sheaves are those in the transjective, or flat component, and they all have rank one as long known. By specializing to the projective plane, we obtain the following corollary.

Corollary 3.3. *Let $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2 \dots$ be the locally free sheaves lying in the preprojective component of some subcategory $\widetilde{\mathcal{K}_S[i]}$ of $\mathrm{coh}(\mathbf{P}^2)$. Then, for each $k \geqslant 0$, $\operatorname{rank} \mathcal{F}_k = f_{2k}$, where $f_0, f_1, f_2 \dots$ is the Fibonacci numbers sequence.*

Proof. We prove first by induction on k that $\dim(Z_k)_0 = f_{2k+1}$, the cases $k = 0, 1$ being obvious. By looking over the proof of the previous proposition, we note that our recursion formula becomes

$$\begin{aligned}\dim(Z_k)_0 &= 3\dim(Z_{k-1})_0 - \dim(Z_{k-2})_0\\ &= 3f_{2k-1} - f_{2k-3}\\ &= 2f_{2k-1} + (f_{2k-1} - f_{2k-3})\\ &= 2f_{2k-1} + f_{2k-2}\\ &= f_{2k-1} + f_{2k-2} + f_{2k-1}\\ &= f_{2k-1} + f_{2k} = f_{2k+1}.\end{aligned}$$

Finally, we get that $\operatorname{rank}\mathcal{F}_k = \dim(Z_k)_0 - \dim(Z_k)_1 = f_{2k+1} - f_{2k-1} = f_{2k}$. □

Even though not mentioned explicitly, the connection with the Fibonacci numbers was also observed in [Rud89]. In this article, the existence of exceptional vector bundles whose ranks are the Markov numbers was proved using mutations, and the Fibonacci numbers in Corollary 3.3 are Markov numbers. These connections will be explored in a forthcoming article.

Proposition 3.4. *Let $n > 1$ and M be a linear R-module of Loewy length two. Then*

$$\operatorname{rank}\sigma M > n \operatorname{rank} M.$$

Proof. We use the fact that σM is the second socle of the Auslander–Reiten translate τM, and we show that

$$\dim(\sigma M)_0 - \dim(\sigma M)_1 > n(\dim M_0 - \dim M_1).$$

It follows from Theorem 2.1, that we are in the radical square zero case, so we can use the Coxeter matrix. Recall (see [Rin84]) that, in our case, the Coxeter matrix is the matrix

$$\mathbf{\Phi} = \begin{pmatrix} n^2+2n & -n-1\\ n+1 & -1\end{pmatrix}$$

and we can easily relate the dimension vectors of a module $M = M_0 \oplus M_1$ to those of σM. Namely, we have

$$\begin{pmatrix}\dim(\sigma M)_0\\ \dim(\sigma M)_1\end{pmatrix} = \mathbf{\Phi}\begin{pmatrix}\dim M_0\\ \dim M_1\end{pmatrix}.$$

A very easy computation shows that $\dim(\sigma M)_0 - \dim(\sigma M)_1 = (n^2+n-1)\dim M_0 - n\dim M_1 > n(\dim M_0 - \dim M_1)$ and the proof is complete. □

We also have the following.

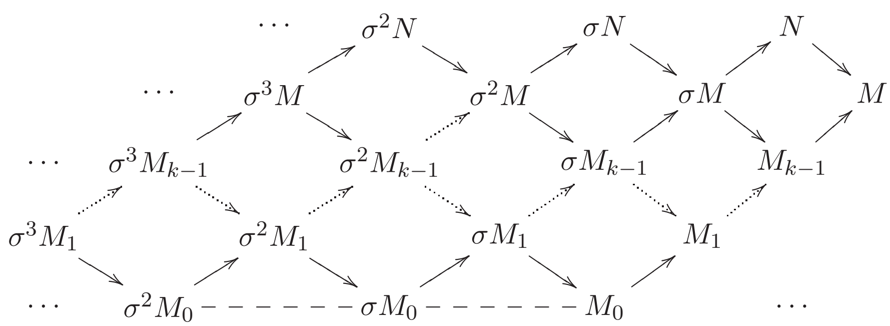
Lemma 3.5. *Let M be a linear R-module. Then the sheaf rank of M is nonnegative.*

Proof. As $\mathcal{E}(M)$ has a finite free resolution, its Euler number is nonnegative, see [Mat04] for instance. Thus, $\operatorname{rank} M = \chi(\mathcal{E}(M)) \geqslant 0$. □

As promised in the previous section, we can be more precise now in our description of the Auslander–Reiten quivers of $\mathcal{K}_R$, $\mathcal{K}_S[i]$, and of $\widetilde{\mathcal{K}_S[i]}$. First we prove the following.

Proposition 3.6. *Let $n > 1$ and M be a linear R-module. Then M has at most finitely many successors in the component of the Auslander–Reiten quiver of $\mathcal{K}_R$ containing it. Consequently, if $\mathcal{M}$ is an indecomposable coherent sheaf lying some $\widetilde{\mathcal{K}_S[i]}$, then $\mathcal{M}$ has finitely many predecessors in the component of the Auslander–Reiten quiver containing it.*

Proof. If the module is in the preinjective component of $\mathcal{K}_R$, then the result is obvious. Assume that our module is in a component that is a subquiver of a $\mathbf{Z}A_\infty$-type quiver. From the sheaf rank inequality $\operatorname{rank}\sigma A > n \operatorname{rank} A$ for linear modules A of Loewy length two and Lemma 3.5, we see that for every linear R-module A there are only finitely many linear modules B with $A = \sigma^i B$ for some $i \geqslant 0$. Let M be a linear R-module having the property that $M \neq \sigma B$ for any linear module B. We show first that if there are no irreducible morphisms in $\mathcal{K}_R$ starting at M, then M must be at the mouth of the component. If not, it is easy to see in the following diagram that, as $\mathcal{K}_R$ is closed under cokernels of monomorphisms, there exists an Auslander–Reiten sequence in $\mathcal{K}_R$ starting at M_0. However, then there is an Auslander–Reiten sequence in $\mathcal{K}_R$ starting at M_1, and we can now inductively prove that there is an Auslander–Reiten sequence in $\mathcal{K}_R$ starting at M contradicting our hypothesis.



Finally, it is easy to see now that if there is no Auslander–Reiten sequence in $\mathcal{K}_R$ starting at M then the module M must lie on a diagonal and the entire component is of type $\mathbf{N}^- A_\infty$. This clearly implies that every linear module has finitely many successors in its component. The second part of the proposition is an immediate translation. □

We can completely describe now the shape of the Auslander–Reiten quivers.

Theorem 3.7. *Let $n > 1$. Then:*

(i) *the Auslander–Reiten quiver of $\mathcal{K}_R$ has a connected component that coincides with the preinjective component of $\operatorname{gr}_0 R/J^2$, a component consisting only of the module R, and all the remaining connected components are of the type $\mathbf{N}^- A_\infty$;*

(ii) *for each integer i, the Auslander–Reiten quiver of $\mathcal{K}_S[i]$ has one preprojective component, and all of the other components are quivers of type $\mathbf{N}A_\infty$;*

(iii) *for each integer i, the Auslander–Reiten quiver of $\widetilde{\mathcal{K}_S[i]}$ has one preprojective component, and all of the other components are quivers of type $\mathbf{N}A_\infty$.*

Now we again turn our attention to locally free sheaves.

Theorem 3.8. *Let $n > 1$ and $\mathcal{F} \in \operatorname{coh}(\mathbf{P}^n)$ be an indecomposable locally free sheaf lying in some $\widetilde{\mathcal{K}_S[i]}$. Then:*

(i) *every successor of $\mathcal{F}$ in the Auslander–Reiten component containing $\mathcal{F}$ is locally free;*

(ii) *let $0 \to \mathcal{F} \to \mathcal{B} \to \delta^{-1}\mathcal{F} \to 0$ be the Auslander–Reiten sequence in $\widetilde{\mathcal{K}_S[i]}$ starting at $\mathcal{F}$. Then*

$$\operatorname{rank} \delta^{-2}\mathcal{F} > n \operatorname{rank} \delta^{-1}\mathcal{F}.$$

Consequently, the ranks increase exponentially in each component of the Auslander–Reiten quiver.

Proof. For the first part we may assume without loss of generality that $\mathcal{F} = \widetilde{A}$ for some linear S-module A, and let $0 \to A \to B \to C \to 0$ be the Auslander–Reiten sequence in $\mathcal{K}_S$ starting at A.

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However, using the Koszul duality, we see that $C = \Omega^{n-1}J^2A[-n-1]$, so its sheafification $\widetilde{C}$ is also locally free. Using the fact that being locally free is closed under extensions we get that all the successors of $\mathcal{F}$ in its component are locally free. Note also that as the locally free sheaves are also closed under kernels of epimorphisms we can dualize the proof of Proposition 3.6 to obtain that all the locally free sheaves in the component lie on a $\mathbf{N}A_\infty$-slice as well. The second part follows trivially from Proposition 3.4. □

The previous results give us information on the possible location of indecomposable locally free sheaves of small ranks in $\mathbf{P}^n$; see [Har79]. Namely, unless the rank is equal to one, these sheaves cannot belong to a preprojective component so they must lie in a $\mathbf{N}A_\infty$-type component. Moreover, if $\operatorname{rank}\mathcal{F} < n$, then $\mathcal{F}$ must lie very close to the starting edge of the Auslander–Reiten component that contains it. In particular Theorem 3.7 tells us that $\mathcal{F}$ cannot be of the form $\delta^{-2}\mathcal{G}$ for some indecomposable locally free sheaf $\mathcal{G}$.

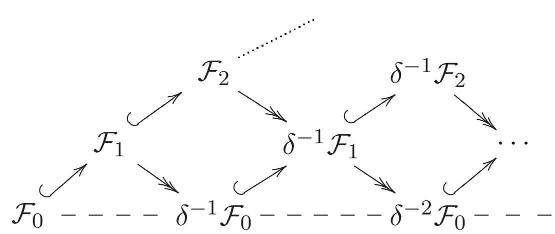
We end with the following theorem.

Theorem 3.9. *Let $n > 1$. For each integer i, each Auslander–Reiten component of $\widetilde{\mathcal{K}_S[i]}$ contains at most one locally free sheaf of rank less than n.*

Proof. If the component is a preprojective component then the result is an immediate consequence of Proposition 3.2, so assume that we are dealing with a component of type $\mathbf{N}A_\infty$. Assume first that $M \neq R$ is an indecomposable linear R-module. We make use of the fact that σM has Loewy length precisely two. By using the Koszul duality we see that $\mathcal{E}(\sigma M)$ has projective dimension one, hence its depth is n by the Auslander–Buchsbaum formula. As a consequence the S-module $\mathcal{E}(\sigma M)$ satisfies Serre's (S_n) condition, and the Evans–Griffith theorem [EG81] tells us that the rank (and thus the Euler number) of $\mathcal{E}(\sigma M)$ is at least n. As an immediate consequence we have that if $\mathcal{F}$ is an indecomposable locally free sheaf and

$$0 \to \mathcal{F} \to \mathcal{B} \to \delta^{-1}\mathcal{F} \to 0$$

is the Auslander–Reiten sequence in $\widetilde{\mathcal{K}_S[i]}$ starting at $\mathcal{F}$, then $\operatorname{rank}\delta^{-1}\mathcal{F} \geqslant n$.



Denote by $\mathcal{F}_0, \mathcal{F}_1, \ldots$ the locally free sheaves in the component lying on the leftmost diagonal. It follows from the above that $\operatorname{rank}\delta^{-i}\mathcal{F}_k \geqslant n$ for each $i \geqslant 1$ and $k \geqslant 0$. As each morphism

$$\delta^{-i}\mathcal{F}_k \longrightarrow \delta^{-i-1}\mathcal{F}_{k-1}$$

is an epimorphism and the rank is additive, it follows that the only possible locally free sheaf having rank less than n is $\mathcal{F}_0$. The proof is now complete. □

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