1

Sentential Logic

1.1. Deductive Reasoning and Logical Connectives

As we saw in the introduction, proofs play a central role in mathematics, and deductive reasoning is the foundation on which proofs are based. Therefore, we begin our study of mathematical reasoning and proofs by examining how deductive reasoning works.

Example 1.1.1. Here are three examples of deductive reasoning:

1. It will either rain or snow tomorrow.
   It's too warm for snow.
   Therefore, it will rain.
2. If today is Sunday, then I don't have to go to work today.
   Today is Sunday.
   Therefore, I don't have to go to work today.
3. I will go to work either tomorrow or today.
   I'm going to stay home today.
   Therefore, I will go to work tomorrow.

In each case, we have arrived at a *conclusion* from the assumption that some other statements, called *premises*, are true. For example, the premises in argument 3 are the statements "I will go to work either tomorrow or today" and "I'm going to stay home today." The conclusion is "I will go to work tomorrow," and it seems to be forced on us somehow by the premises.

But is this conclusion really correct? After all, isn't it possible that I'll stay home today, and then wake up sick tomorrow and end up staying home again? If that happened, the conclusion would turn out to be false. But notice that in that case the first premise, which said that I would go to work either tomorrow

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or today, would be false as well! Although we have no guarantee that the conclusion is true, it can only be false if at least one of the premises is also false. *If* both premises are true, we can be sure that the conclusion is also true. This is the sense in which the conclusion is forced on us by the premises, and this is the standard we will use to judge the correctness of deductive reasoning. We will say that an argument is *valid* if the premises cannot all be true without the conclusion being true as well. All three of the arguments in our example are valid arguments.

Here's an example of an invalid deductive argument:

Either the butler is guilty or the maid is guilty.
Either the maid is guilty or the cook is guilty.
Therefore, either the butler is guilty or the cook is guilty.

The argument is invalid because the conclusion could be false even if both premises are true. For example, if the maid were guilty, but the butler and the cook were both innocent, then both premises would be true and the conclusion would be false.

We can learn something about what makes an argument valid by comparing the three arguments in Example 1.1.1. On the surface it might seem that arguments 2 and 3 have the most in common, because they're both about the same subject: attendance at work. But in terms of the reasoning used, arguments 1 and 3 are the most similar. They both introduce two possibilities in the first premise, rule out the second one with the second premise, and then conclude that the first possibility must be the case. In other words, both arguments have the form:

P or Q.
not Q.
Therefore, P.

It is this form, and not the subject matter, that makes these arguments valid. You can see that argument 1 has this form by thinking of the letter P as standing for the statement "It will rain tomorrow," and Q as standing for "It will snow tomorrow." For argument 3, P would be "I will go to work tomorrow," and Q would be "I will go to work today."

Replacing certain statements in each argument with letters, as we have in stating the form of arguments 1 and 3, has two advantages. First, it keeps us from being distracted by aspects of the arguments that don't affect their validity. You don't need to know anything about weather forecasting or work habits to recognize that arguments 1 and 3 are valid. That's because both arguments have the form shown earlier, and you can tell that this argument form is valid without

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even knowing what P and Q stand for. If you don't believe this, consider the following argument:

> Either the framger widget is misfiring, or the wrompal mechanism is out of alignment.
> I've checked the alignment of the wrompal mechanism, and it's fine.
> Therefore, the framger widget is misfiring.

If a mechanic gave this explanation after examining your car, you might still be mystified about why the car won't start, but you'd have no trouble following his logic!

Perhaps more important, our analysis of the forms of arguments 1 and 3 makes clear what *is* important in determining their validity: the words *or* and *not*. In most deductive reasoning, and in particular in mathematical reasoning, the meanings of just a few words give us the key to understanding what makes a piece of reasoning valid or invalid. (Which are the important words in argument 2 in Example 1.1.1?) The first few chapters of this book are devoted to studying those words and how they are used in mathematical writing and reasoning.

In this chapter, we'll concentrate on words used to combine statements to form more complex statements. We'll continue to use letters to stand for statements, but only for unambiguous statements that are either true or false. Questions, exclamations, and vague statements will not be allowed. It will also be useful to use symbols, sometimes called *connective symbols*, to stand for some of the words used to combine statements. Here are our first three connective symbols and the words they stand for:

Symbol	Meaning
$\vee$	or
$\wedge$	and
$\neg$	not

Thus, if P and Q stand for two statements, then we'll write $P \vee Q$ to stand for the statement "P or Q," $P \wedge Q$ for "P and Q," and $\neg P$ for "not P" or "P is false." The statement $P \vee Q$ is sometimes called the *disjunction* of P and Q, $P \wedge Q$ is called the *conjunction* of P and Q, and $\neg P$ is called the *negation* of P.

Example 1.1.2. Analyze the logical forms of the following statements:

1. Either John went to the store, or we're out of eggs.
2. Joe is going to leave home and not come back.
3. Either Bill is at work and Jane isn't, or Jane is at work and Bill isn't.

Solutions

1. If we let P stand for the statement "John went to the store" and Q stand for "We're out of eggs," then this statement could be represented symbolically as $P \vee Q$.
2. If we let P stand for the statement "Joe is going to leave home" and Q stand for "Joe is not going to come back," then we could represent this statement symbolically as $P \wedge Q$. But this analysis misses an important feature of the statement, because it doesn't indicate that Q is a negative statement. We could get a better analysis by letting R stand for the statement "Joe is going to come back" and then writing the statement Q as $\neg R$. Plugging this into our first analysis of the original statement, we get the improved analysis $P \wedge \neg R$.
3. Let B stand for the statement "Bill is at work" and J for the statement "Jane is at work." Then the first half of the statement, "Bill is at work and Jane isn't," can be represented as $B \wedge \neg J$. Similarly, the second half is $J \wedge \neg B$. To represent the entire statement, we must combine these two with *or*, forming their disjunction, so the solution is $(B \wedge \neg J) \vee (J \wedge \neg B)$.

Notice that in analyzing the third statement in the preceding example, we added parentheses when we formed the disjunction of $B \wedge \neg J$ and $J \wedge \neg B$ to indicate unambiguously which statements were being combined. This is like the use of parentheses in algebra, in which, for example, the product of $a + b$ and $a - b$ would be written $(a + b) \cdot (a - b)$, with the parentheses serving to indicate unambiguously which quantities are to be multiplied. As in algebra, it is convenient in logic to omit some parentheses to make our expressions shorter and easier to read. However, we must agree on some conventions about how to read such expressions so that they are still unambiguous. One convention is that the symbol $\neg$ always applies only to the statement that comes immediately after it. For example, $\neg P \wedge Q$ means $(\neg P) \wedge Q$ rather than $\neg(P \wedge Q)$. We'll see some other conventions about parentheses later.

Example 1.1.3. What English sentences are represented by the following expressions?

1. $(\neg S \wedge L) \vee S$, where S stands for "John is stupid" and L stands for "John is lazy."
2. $\neg S \wedge (L \vee S)$, where S and L have the same meanings as before.
3. $\neg(S \wedge L) \vee S$, with S and L still as before.

Solutions

1. Either John isn't stupid and he is lazy, or he's stupid.
2. John isn't stupid, and either he's lazy or he's stupid. Notice how the placement of the word *either* in English changes according to where the parentheses are.
3. Either John isn't both stupid and lazy, or John is stupid. The word *both* in English also helps distinguish the different possible positions of parentheses.

It is important to keep in mind that the symbols $\wedge$, $\vee$, and $\neg$ don't really correspond to all uses of the words *and*, *or*, and *not* in English. For example, the symbol $\wedge$ could not be used to represent the use of the word *and* in the sentence "John and Bill are friends," because in this sentence the word *and* is not being used to combine two statements. The symbols $\wedge$ and $\vee$ can only be used *between two statements*, to form their conjunction or disjunction, and the symbol $\neg$ can only be used *before a statement*, to negate it. This means that certain strings of letters and symbols are simply meaningless. For example, $P\neg \wedge Q$, $P \wedge/\vee Q$, and $P\neg Q$ are all "ungrammatical" expressions in the language of logic. "Grammatical" expressions, such as those in Examples 1.1.2 and 1.1.3, are sometimes called *well-formed formulas* or just *formulas*. Once again, it may be helpful to think of an analogy with algebra, in which the symbols $+$, $-$, $\cdot$, and $\div$ can be used *between two numbers*, as operators, and the symbol $-$ can also be used *before a number*, to negate it. These are the only ways that these symbols can be used in algebra, so expressions such as $x - \div y$ are meaningless.

Sometimes, words other than *and*, *or*, and *not* are used to express the meanings represented by $\wedge$, $\vee$, and $\neg$. For example, consider the first statement in Example 1.1.3. Although we gave the English translation "Either John isn't stupid and he is lazy, or he's stupid," an alternative way of conveying the same information would be to say "Either John isn't stupid *but* he is lazy, or he's stupid." Often, the word *but* is used in English to mean *and*, especially when there is some contrast or conflict between the statements being combined. For a more striking example, imagine a weather forecaster ending his forecast with the statement "Rain and snow are the only two possibilities for tomorrow's weather." This is just a roundabout way of saying that it will either rain or snow tomorrow. Thus, even though the forecaster has used the word *and*, the meaning expressed by his statement is a disjunction. The lesson of these examples is that to determine the logical form of a statement you must think about what the statement means, rather than just translating word by word into symbols.

Sometimes logical words are hidden within mathematical notation. For example, consider the statement $3 \leq \pi$. Although it appears to be a simple statement that contains no words of logic, if you read it out loud you will hear the word *or*. If we let P stand for the statement $3 < \pi$ and Q for the statement $3 = \pi$, then the statement $3 \leq \pi$ would be written $P \vee Q$. In this example the statements represented by the letters P and Q are so short that it hardly seems worthwhile to abbreviate them with single letters. In cases like this we will sometimes not bother to replace the statements with letters, so we might also write this statement as $(3 < \pi) \vee (3 = \pi)$.

For a slightly more complicated example, consider the statement $3 \leq \pi < 4$. This statement means $3 \leq \pi$ *and* $\pi < 4$, so once again a word of logic has been hidden in mathematical notation. Filling in the meaning that we just worked out for $3 \leq \pi$, we can write the whole statement as $[(3 < \pi) \vee (3 = \pi)] \wedge (\pi < 4)$. Knowing that the statement has this logical form might be important in understanding a piece of mathematical reasoning involving this statement.

Exercises

*1. Analyze the logical forms of the following statements:
   (a) We'll have either a reading assignment or homework problems, but we won't have both homework problems and a test.
   (b) You won't go skiing, or you will and there won't be any snow.
   (c) $\sqrt{7} \not\leq 2$.
2. Analyze the logical forms of the following statements:
   (a) Either John and Bill are both telling the truth, or neither of them is.
   (b) I'll have either fish or chicken, but I won't have both fish and mashed potatoes.
   (c) 3 is a common divisor of 6, 9, and 15.
3. Analyze the logical forms of the following statements:
   (a) Alice and Bob are not both in the room.
   (b) Alice and Bob are both not in the room.
   (c) Either Alice or Bob is not in the room.
   (d) Neither Alice nor Bob is in the room.
4. Which of the following expressions are well-formed formulas?
   (a) $\neg(\neg P \vee \neg\neg R)$.
   (b) $\neg(P, Q, \wedge R)$.
   (c) $P \wedge \neg P$.
   (d) $(P \wedge Q)(P \vee R)$.

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*5. Let P stand for the statement "I will buy the pants" and S for the statement "I will buy the shirt." What English sentences are represented by the following expressions?
   (a) $\neg(P \wedge \neg S)$.
   (b) $\neg P \wedge \neg S$.
   (c) $\neg P \vee \neg S$.
6. Let S stand for the statement "Steve is happy" and G for "George is happy." What English sentences are represented by the following expressions?
   (a) $(S \vee G) \wedge (\neg S \vee \neg G)$.
   (b) $[S \vee (G \wedge \neg S)] \vee \neg G$.
   (c) $S \vee [G \wedge (\neg S \vee \neg G)]$.
7. Identify the premises and conclusions of the following deductive arguments and analyze their logical forms. Do you think the reasoning is valid? (Although you will have only your intuition to guide you in answering this last question, in the next section we will develop some techniques for determining the validity of arguments.)
   (a) Jane and Pete won't both win the math prize. Pete will win either the math prize or the chemistry prize. Jane will win the math prize. Therefore, Pete will win the chemistry prize.
   (b) The main course will be either beef or fish. The vegetable will be either peas or corn. We will not have both fish as a main course and corn as a vegetable. Therefore, we will not have both beef as a main course and peas as a vegetable.
   (c) Either John or Bill is telling the truth. Either Sam or Bill is lying. Therefore, either John is telling the truth or Sam is lying.
   (d) Either sales will go up and the boss will be happy, or expenses will go up and the boss won't be happy. Therefore, sales and expenses will not both go up.

1.2. Truth Tables

We saw in Section 1.1 that an argument is valid if the premises cannot all be true without the conclusion being true as well. Thus, to understand how words such as *and*, *or*, and *not* affect the validity of arguments, we must see how they contribute to the truth or falsity of statements containing them.

When we evaluate the truth or falsity of a statement, we assign to it one of the labels *true* or *false*, and this label is called its *truth value*. It is clear how the word *and* contributes to the truth value of a statement containing it. A statement of the form $P \wedge Q$ can only be true if both P and Q are true; if either P or Q is false, then $P \wedge Q$ will be false too. Because we have assumed that P and

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P	Q	$P \wedge Q$
F	F	F
F	T	F
T	F	F
T	T	T

Figure 1

Q both stand for statements that are either true or false, we can summarize all the possibilities with the table shown in Figure 1. This is called a *truth table* for the formula $P \wedge Q$. Each row in the truth table represents one of the four possible combinations of truth values for the statements P and Q. Although these four possibilities can appear in the table in any order, it is best to list them systematically so we can be sure that no possibilities have been skipped. The truth table for $\neg P$ is also quite easy to construct because for $\neg P$ to be true, P must be false. The table is shown in Figure 2.

P	$\neg P$
F	T
T	F

Figure 2

The truth table for $P \vee Q$ is a little trickier. The first three lines should certainly be filled in as shown in Figure 3, but there may be some question about the last line. Should $P \vee Q$ be true or false in the case in which P and Q are both true? In other words, does $P \vee Q$ mean "P or Q, or both" or does it mean "P or Q but not both"? The first way of interpreting the word *or* is called the *inclusive or* (because it *includes* the possibility of both statements being true), and the second is called the *exclusive or*. In mathematics, *or* always means inclusive or, unless specified otherwise, so we will interpret $\vee$ as inclusive or. We therefore complete the truth table for $P \vee Q$ as shown in Figure 4. See exercise 3 for more about the exclusive or.

P	Q	$P \vee Q$
F	F	F
F	T	T
T	F	T
T	T	?

Figure 3

P	Q	$P \vee Q$
F	F	F
F	T	T
T	F	T
T	T	T

Figure 4

Using the rules summarized in these truth tables, we can now work out truth tables for more complex formulas. All we have to do is work out the truth values of the component parts of a formula, starting with the individual letters and working up to more complex formulas a step at a time.

Example 1.2.1. Make a truth table for the formula $\neg(P \vee \neg Q)$.

Solution

P	Q	$\neg Q$	$P \vee \neg Q$	$\neg(P \vee \neg Q)$
F	F	T	T	F
F	T	F	F	T
T	F	T	T	F
T	T	F	T	F

The first two columns of this table list the four possible combinations of truth values of P and Q. The third column, listing truth values for the formula $\neg Q$, is found by simply negating the truth values for Q in the second column. The fourth column, for the formula $P \vee \neg Q$, is found by combining the truth values for P and $\neg Q$ listed in the first and third columns, according to the truth value rule for $\vee$ summarized in Figure 4. According to this rule, $P \vee \neg Q$ will be false only if both P and $\neg Q$ are false. Looking in the first and third columns, we see that this happens only in row two of the table, so the fourth column contains an F in the second row and T's in all other rows. Finally, the truth values for the formula $\neg(P \vee \neg Q)$ are listed in the fifth column, which is found by negating the truth values in the fourth column. (Note that these columns had to be worked out in order, because each was used in computing the next.)

Example 1.2.2. Make a truth table for the formula $\neg(P \wedge Q) \vee \neg R$.

Solution

P	Q	R	$P \wedge Q$	$\neg(P \wedge Q)$	$\neg R$	$\neg(P \wedge Q) \vee \neg R$
F	F	F	F	T	T	T
F	F	T	F	T	F	T
F	T	F	F	T	T	T
F	T	T	F	T	F	T
T	F	F	F	T	T	T
T	F	T	F	T	F	T
T	T	F	T	F	T	T
T	T	T	T	F	F	F

Note that because this formula contains three letters, it takes eight lines to list all possible combinations of truth values for these letters. (If a formula contains n different letters, how many lines will its truth table have?)

Here's a way of making truth tables more compactly. Instead of using separate columns to list the truth values for the component parts of a formula, just list those truth values below the corresponding connective symbol in the original formula. This is illustrated in Figure 5, for the formula from Example 1.2.1.

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In the first step, we have listed the truth values for P and Q below these letters where they appear in the formula. In step two, the truth values for $\neg Q$ have been added under the $\neg$ symbol for $\neg Q$. In the third step, we have combined the truth values for P and $\neg Q$ to get the truth values for $P \vee \neg Q$, which are listed under the $\vee$ symbol. Finally, in the last step, these truth values are negated and listed under the initial $\neg$ symbol. The truth values added in the last step give the truth value for the entire formula, so we will call the symbol under which they are listed (the first $\neg$ symbol in this case) the *main connective* of the formula. Notice that the truth values listed under the main connective in this case agree with the values we found in Example 1.2.1.

Step 1

P	Q	$\neg$	$(P$	$\vee$	$\neg$	$Q)$
F	F		**F**			**F**
F	T		**F**			**T**
T	F		**T**			**F**
T	T		**T**			**T**

Step 2

P	Q	$\neg$	$(P$	$\vee$	$\neg$	$Q)$
F	F		F		**T**	F
F	T		F		**F**	T
T	F		T		**T**	F
T	T		T		**F**	T

Step 3

P	Q	$\neg$	$(P$	$\vee$	$\neg$	$Q)$
F	F		F	**T**	T	F
F	T		F	**F**	F	T
T	F		T	**T**	T	F
T	T		T	**T**	F	T

Step 4

P	Q	$\neg$	$(P$	$\vee$	$\neg$	$Q)$
F	F	**F**	F	T	T	F
F	T	**T**	F	F	F	T
T	F	**F**	T	T	T	F
T	T	**F**	T	T	F	T

Figure 5

Now that we know how to make truth tables for complex formulas, we're ready to return to the analysis of the validity of arguments. Consider again our first example of a deductive argument:

It will either rain or snow tomorrow.
It's too warm for snow.
Therefore, it will rain.

As we have seen, if we let P stand for the statement "It will rain tomorrow" and Q for the statement "It will snow tomorrow," then we can represent the argument symbolically as follows:

$$\begin{array}{l} P \vee Q \\ \underline{\neg Q \quad\;\;} \\ \therefore P \end{array}$$

(The symbol $\therefore$ means *therefore*.)

We can now see how truth tables can be used to verify the validity of this argument. Figure 6 shows a truth table for both premises and the conclusion of the argument. Recall that we decided to call an argument valid if the

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premises cannot all be true without the conclusion being true as well. Looking at Figure 6 we see that the only row of the table in which both premises come out true is row three, and in this row the conclusion is also true. Thus, the truth table confirms that if the premises are all true, the conclusion must also be true, so the argument is valid.

		Premises		Conclusion
P	Q	$P \vee Q$	$\neg Q$	P
F	F	F	T	F
F	T	T	F	F
T	F	T	T	T
T	T	T	F	T

Figure 6

Example 1.2.3. Determine whether the following arguments are valid.

1. Either John isn't stupid and he is lazy, or he's stupid.
   John is stupid.
   Therefore, John isn't lazy.
2. The butler and the cook are not both innocent.
   Either the butler is lying or the cook is innocent.
   Therefore, the butler is either lying or guilty.

Solutions

1. As in Example 1.1.3, we let S stand for the statement "John is stupid" and L stand for "John is lazy." Then the argument has the form:

$$\begin{array}{l} (\neg S \wedge L) \vee S \\ \underline{S \qquad\qquad\quad} \\ \therefore \neg L \end{array}$$

Now we make a truth table for both premises and the conclusion. (You should work out the intermediate steps in deriving column three of this table to confirm that it is correct.)

		Premises		Conclusion
S	L	$(\neg S \wedge L) \vee S$	S	$\neg L$
F	F	F	F	T
F	T	T	F	F
T	F	T	T	T
T	T	T	T	F

Both premises are true in lines three and four of this table. The conclusion is also true in line three, but it is false in line four. Thus, it is possible for both premises to be true and the conclusion false, so the argument is invalid. In fact, the table shows us exactly why the argument is invalid. The problem occurs in the fourth line of the table, in which S and L are both true – in other words, John is both stupid and lazy. Thus, if John is both stupid and lazy, then both premises will be true but the conclusion will be false, so it would be a mistake to infer that the conclusion must be true from the assumption that the premises are true.

2. Let B stand for the statement "The butler is innocent," C for the statement "The cook is innocent," and L for the statement "The butler is lying." Then the argument has the form:

$$\begin{array}{l} \neg(B \wedge C) \\ \underline{L \vee C} \\ \therefore L \vee \neg B \end{array}$$

Here is the truth table for the premises and conclusion:

			Premises		Conclusion
B	C	L	$\neg(B \wedge C)$	$L \vee C$	$L \vee \neg B$
F	F	F	T	F	T
F	F	T	T	T	T
F	T	F	T	T	T
F	T	T	T	T	T
T	F	F	T	F	F
T	F	T	T	T	T
T	T	F	F	T	F
T	T	T	F	T	T

The premises are both true only in lines two, three, four, and six, and in each of these cases the conclusion is true as well. Therefore, the argument is valid.

If you expected the first argument in Example 1.2.3 to turn out to be valid, it's probably because the first premise confused you. It's a rather complicated statement, which we represented symbolically with the formula $(\neg S \wedge L) \vee S$. According to our truth table, this formula is false if S and L are both false, and true otherwise. But notice that this is exactly the same as the truth table for the simpler formula $L \vee S$! Because of this, we say that the formulas $(\neg S \wedge L) \vee S$ and $L \vee S$ are *equivalent*. Equivalent formulas always have the same truth value no matter what statements the letters in them stand for and no matter what the truth values of those statements are. The equivalence of the premise $(\neg S \wedge L) \vee S$ and the simpler formula $L \vee S$ may help you understand why

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the argument is invalid. Translating the formula $L \vee S$ back into English, we see that the first premise could have been stated more simply as "John is either lazy or stupid (or both)." But from this premise and the second premise (that John is stupid), it clearly doesn't follow that he's not lazy, because he might be both stupid and lazy.

Example 1.2.4. Which of these formulas are equivalent?

$$\neg(P \wedge Q), \qquad \neg P \wedge \neg Q, \qquad \neg P \vee \neg Q.$$

Solution

Here's a truth table for all three statements. (You should check it yourself!)

P	Q	$\neg(P \wedge Q)$	$\neg P \wedge \neg Q$	$\neg P \vee \neg Q$
F	F	T	T	T
F	T	T	F	T
T	F	T	F	T
T	T	F	F	F

The third and fifth columns in this table are identical, but they are different from the fourth column. Therefore, the formulas $\neg(P \wedge Q)$ and $\neg P \vee \neg Q$ are equivalent, but neither is equivalent to the formula $\neg P \wedge \neg Q$. This should make sense if you think about what all the symbols mean. For example, suppose P stands for the statement "The Yankees won last night" and Q stands for "The Red Sox won last night." Then $\neg(P \wedge Q)$ would mean "The Yankees and the Red Sox did not both win last night," and $\neg P \vee \neg Q$ would mean "Either the Yankees or the Red Sox lost last night"; these statements clearly convey the same information. On the other hand, $\neg P \wedge \neg Q$ would mean "The Yankees and the Red Sox both lost last night," which is an entirely different statement.

You can check for yourself by making a truth table that the formula $\neg P \wedge \neg Q$ from Example 1.2.4 is equivalent to the formula $\neg(P \vee Q)$. (To see that this equivalence makes sense, notice that the statements "Both the Yankees and the Red Sox lost last night" and "Neither the Yankees nor the Red Sox won last night" mean the same thing.) This equivalence and the one discovered in Example 1.2.4 are called *DeMorgan's laws*.

In analyzing deductive arguments and the statements that occur in them it is helpful to be familiar with a number of equivalences that come up often. Verify the equivalences in the following list yourself by making truth tables, and check that they make sense by translating the formulas into English, as we did in Example 1.2.4.

DeMorgan's laws

$$\neg(P \wedge Q) \text{ is equivalent to } \neg P \vee \neg Q.$$
$$\neg(P \vee Q) \text{ is equivalent to } \neg P \wedge \neg Q.$$

Commutative laws

$$P \wedge Q \text{ is equivalent to } Q \wedge P.$$
$$P \vee Q \text{ is equivalent to } Q \vee P.$$

Associative laws

$$P \wedge (Q \wedge R) \text{ is equivalent to } (P \wedge Q) \wedge R.$$
$$P \vee (Q \vee R) \text{ is equivalent to } (P \vee Q) \vee R.$$

Idempotent laws

$$P \wedge P \text{ is equivalent to } P.$$
$$P \vee P \text{ is equivalent to } P.$$

Distributive laws

$$P \wedge (Q \vee R) \text{ is equivalent to } (P \wedge Q) \vee (P \wedge R).$$
$$P \vee (Q \wedge R) \text{ is equivalent to } (P \vee Q) \wedge (P \vee R).$$

Absorption laws

$$P \vee (P \wedge Q) \text{ is equivalent to } P.$$
$$P \wedge (P \vee Q) \text{ is equivalent to } P.$$

Double Negation law

$$\neg\neg P \text{ is equivalent to } P.$$

Notice that because of the associative laws we can leave out parentheses in formulas of the forms $P \wedge Q \wedge R$ and $P \vee Q \vee R$ without worrying that the resulting formula will be ambiguous, because the two possible ways of filling in the parentheses lead to equivalent formulas.

Many of the equivalences in the list should remind you of similar rules involving $+$, $\cdot$, and $-$ in algebra. As in algebra, these rules can be applied to more complex formulas, and they can be combined to work out more complicated equivalences. Any of the letters in these equivalences can be replaced by more complicated formulas, and the resulting equivalence will still be true. For example, by replacing P in the double negation law with the formula $Q \vee \neg R$, you can see that $\neg\neg(Q \vee \neg R)$ is equivalent to $Q \vee \neg R$. Also, if two formulas are equivalent, you can always substitute one for the other in any expression and the results will be equivalent. For example, since $\neg\neg P$ is equivalent to

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P, if $\neg\neg P$ occurs in any formula, you can always replace it with P and the resulting formula will be equivalent to the original.

Example 1.2.5. Find simpler formulas equivalent to these formulas:

1. $\neg(P \vee \neg Q)$.
2. $\neg(Q \wedge \neg P) \vee P$.

Solutions

1. $\neg(P \vee \neg Q)$

is equivalent to	$\neg P \wedge \neg\neg Q$	(DeMorgan's law),
which is equivalent to	$\neg P \wedge Q$	(double negation law).

You can check that this equivalence is right by making a truth table for $\neg P \wedge Q$ and seeing that it is the same as the truth table for $\neg(P \vee \neg Q)$ found in Example 1.2.1.

2. $\neg(Q \wedge \neg P) \vee P$

is equivalent to	$(\neg Q \vee \neg\neg P) \vee P$	(DeMorgan's law),
which is equivalent to	$(\neg Q \vee P) \vee P$	(double negation law),
which is equivalent to	$\neg Q \vee (P \vee P)$	(associative law),
which is equivalent to	$\neg Q \vee P$	(idempotent law).

Some equivalences are based on the fact that certain formulas are either always true or always false. For example, you can verify by making a truth table that the formula $Q \wedge (P \vee \neg P)$ is equivalent to just Q. But even before you make the truth table, you can probably see why they are equivalent. In every line of the truth table, $P \vee \neg P$ will come out true, and therefore $Q \wedge (P \vee \neg P)$ will come out true when Q is also true, and false when Q is false. Formulas that are always true, such as $P \vee \neg P$, are called *tautologies*. Similarly, formulas that are always false are called *contradictions*. For example, $P \wedge \neg P$ is a contradiction.

Example 1.2.6. Are these statements tautologies, contradictions, or neither?

$$P \vee (Q \vee \neg P), \quad P \wedge \neg(Q \vee \neg Q), \quad P \vee \neg(Q \vee \neg Q).$$

Solution

First we make a truth table for all three statements.

P	Q	$P \vee (Q \vee \neg P)$	$P \wedge \neg(Q \vee \neg Q)$	$P \vee \neg(Q \vee \neg Q)$
F	F	T	F	F
F	T	T	F	F
T	F	T	F	T
T	T	T	F	T

From the truth table it is clear that the first formula is a tautology, the second a contradiction, and the third neither. In fact, since the last column is identical to the first, the third formula is equivalent to P.

We can now state a few more useful laws involving tautologies and contradictions. You should be able to convince yourself that all of these laws are correct by thinking about what the truth tables for the statements involved would look like.

Tautology laws

$$P \wedge (\text{a tautology}) \text{ is equivalent to } P.$$
$$P \vee (\text{a tautology}) \text{ is a tautology}.$$
$$\neg(\text{a tautology}) \text{ is a contradiction}.$$

Contradiction laws

$$P \wedge (\text{a contradiction}) \text{ is a contradiction}.$$
$$P \vee (\text{a contradiction}) \text{ is equivalent to } P.$$
$$\neg(\text{a contradiction}) \text{ is a tautology}.$$

Example 1.2.7. Find simpler formulas equivalent to these formulas:

1. $P \vee (Q \wedge \neg P)$.
2. $\neg(P \vee (Q \wedge \neg R)) \wedge Q$.

Solutions

1. $P \vee (Q \wedge \neg P)$
   is equivalent to $(P \vee Q) \wedge (P \vee \neg P)$ (distributive law),
   which is equivalent to $P \vee Q$ (tautology law).
   The last step uses the fact that $P \vee \neg P$ is a tautology.
2. $\neg(P \vee (Q \wedge \neg R)) \wedge Q$
   is equivalent to $(\neg P \wedge \neg(Q \wedge \neg R)) \wedge Q$ (DeMorgan's law),
   which is equivalent to $(\neg P \wedge (\neg Q \vee \neg\neg R)) \wedge Q$ (DeMorgan's law),
   which is equivalent to $(\neg P \wedge (\neg Q \vee R)) \wedge Q$ (double negation law),
   which is equivalent to $\neg P \wedge ((\neg Q \vee R) \wedge Q)$ (associative law),
   which is equivalent to $\neg P \wedge (Q \wedge (\neg Q \vee R))$ (commutative law),
   which is equivalent to $\neg P \wedge ((Q \wedge \neg Q) \vee (Q \wedge R))$
   (distributive law),
   which is equivalent to $\neg P \wedge (Q \wedge R)$ (contradiction law).

   The last step uses the fact that $Q \wedge \neg Q$ is a contradiction. Finally, by the associative law for $\wedge$ we can remove the parentheses without making the formula ambiguous, so the original formula is equivalent to the formula $\neg P \wedge Q \wedge R$.

Exercises

*1. Make truth tables for the following formulas:
   (a) $\neg P \vee Q$.
   (b) $(S \vee G) \wedge (\neg S \vee \neg G)$.
2. Make truth tables for the following formulas:
   (a) $\neg[P \wedge (Q \vee \neg P)]$.
   (b) $(P \vee Q) \wedge (\neg P \vee R)$.
3. In this exercise we will use the symbol $+$ to mean *exclusive or*. In other words, $P + Q$ means "P or Q, but not both."
   (a) Make a truth table for $P + Q$.
   (b) Find a formula using only the connectives $\wedge$, $\vee$, and $\neg$ that is equivalent to $P + Q$. Justify your answer with a truth table.
4. Find a formula using only the connectives $\wedge$ and $\neg$ that is equivalent to $P \vee Q$. Justify your answer with a truth table.
*5. Some mathematicians use the symbol $\downarrow$ to mean *nor*. In other words, $P \downarrow Q$ means "neither P nor Q."
   (a) Make a truth table for $P \downarrow Q$.
   (b) Find a formula using only the connectives $\wedge$, $\vee$, and $\neg$ that is equivalent to $P \downarrow Q$.
   (c) Find formulas using only the connective $\downarrow$ that are equivalent to $\neg P$, $P \vee Q$, and $P \wedge Q$.
6. Some mathematicians write $P \mid Q$ to mean "P and Q are not both true." (This connective is called *nand*, and is used in the study of circuits in computer science.)
   (a) Make a truth table for $P \mid Q$.
   (b) Find a formula using only the connectives $\wedge$, $\vee$, and $\neg$ that is equivalent to $P \mid Q$.
   (c) Find formulas using only the connective $\mid$ that are equivalent to $\neg P$, $P \vee Q$, and $P \wedge Q$.
*7. Use truth tables to determine whether or not the arguments in exercise 7 of Section 1.1 are valid.
8. Use truth tables to determine which of the following formulas are equivalent to each other:
   (a) $(P \wedge Q) \vee (\neg P \wedge \neg Q)$.
   (b) $\neg P \vee Q$.
   (c) $(P \vee \neg Q) \wedge (Q \vee \neg P)$.
   (d) $\neg(P \vee Q)$.
   (e) $(Q \wedge P) \vee \neg P$.
*9. Use truth tables to determine which of these statements are tautologies, which are contradictions, and which are neither:

(a) $(P \lor Q) \land (\neg P \lor \neg Q)$.
(b) $(P \lor Q) \land (\neg P \land \neg Q)$.
(c) $(P \lor Q) \lor (\neg P \lor \neg Q)$.
(d) $[P \land (Q \lor \neg R)] \lor (\neg P \lor R)$.

10. Use truth tables to check these laws:
    (a) The second DeMorgan's law. (The first was checked in the text.)
    (b) The distributive laws.

*11. Use the laws stated in the text to find simpler formulas equivalent to these formulas. (See Examples 1.2.5 and 1.2.7.)
    (a) $\neg(\neg P \land \neg Q)$.
    (b) $(P \land Q) \lor (P \land \neg Q)$.
    (c) $\neg(P \land \neg Q) \lor (\neg P \land Q)$.

12. Use the laws stated in the text to find simpler formulas equivalent to these formulas. (See Examples 1.2.5 and 1.2.7.)
    (a) $\neg(\neg P \lor Q) \lor (P \land \neg R)$.
    (b) $\neg(\neg P \land Q) \lor (P \land \neg R)$.
    (c) $(P \land R) \lor [\neg R \land (P \lor Q)]$.

13. Use the first DeMorgan's law and the double negation law to derive the second DeMorgan's law.

*14. Note that the associative laws say only that parentheses are unnecessary when combining *three* statements with $\land$ or $\lor$. In fact, these laws can be used to justify leaving parentheses out when more than three statements are combined. Use associative laws to show that $[P \land (Q \land R)] \land S$ is equivalent to $(P \land Q) \land (R \land S)$.

15. How many lines will there be in the truth table for a statement containing n letters?

*16. Find a formula involving the connectives $\land$, $\lor$, and $\neg$ that has the following truth table:

P	Q	???
F	F	T
F	T	F
T	F	T
T	T	T

17. Find a formula involving the connectives $\land$, $\lor$, and $\neg$ that has the following truth table:

P	Q	???
F	F	F
F	T	T
T	F	T
T	T	F

18. Suppose the conclusion of an argument is a tautology. What can you conclude about the validity of the argument? What if the conclusion is a contradiction? What if one of the premises is either a tautology or a contradiction?

1.3. Variables and Sets

In mathematical reasoning it is often necessary to make statements about objects that are represented by letters called *variables*. For example, if the variable x is used to stand for a number in some problem, we might be interested in the statement "x is a prime number." Although we may sometimes use a single letter, say P, to stand for this statement, at other times we will revise this notation slightly and write $P(x)$, to stress that this is a statement *about* x. The latter notation makes it easy to talk about substituting some number for x in the statement. For example, $P(7)$ would represent the statement "7 is a prime number," and $P(a+b)$ would mean "$a+b$ is a prime number." If a statement contains more than one variable, our abbreviation for the statement will include a list of all the variables involved. For example, we might represent the statement "p is divisible by q" by $D(p, q)$. In this case, $D(12, 4)$ would mean "12 is divisible by 4."

Although you have probably seen variables used most often to stand for numbers, they can stand for anything at all. For example, we could let $M(x)$ stand for the statement "x is a man," and $W(x)$ for "x is a woman." In this case, we are using the variable x to stand for a person. A statement might even contain several variables that stand for different kinds of objects. For example, in the statement "x has y children," the variable x stands for a person, and y stands for a number.

Statements involving variables can be combined using connectives, just like statements without variables.

Example 1.3.1. Analyze the logical forms of the following statements:

1. x is a prime number, and either y or z is divisible by x.
2. x is a man and y is a woman and x likes y, but y doesn't like x.

Solutions

1. We could let P stand for the statement "x is a prime number," D for "y is divisible by x," and E for "z is divisible by x." The entire statement would then be represented by the formula $P \wedge (D \vee E)$. But this analysis, though not incorrect, fails to capture the relationship between the statements

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D and E. A better analysis would be to let $P(x)$ stand for "x is a prime number" and $D(y, x)$ for "y is divisible by x." Then $D(z, x)$ would mean "z is divisible by x," so the entire statement would be $P(x) \wedge (D(y, x) \vee D(z, x))$.

2. Let $M(x)$ stand for "x is a man," $W(y)$ for "y is a woman," and $L(x, y)$ for "x likes y." Then $L(y, x)$ would mean "y likes x." (Notice that the order of the variables after the L makes a difference!) The entire statement would then be represented by the formula $M(x) \wedge W(y) \wedge L(x, y) \wedge \neg L(y, x)$.

When studying statements that do not contain variables, we can easily talk about their truth values, since each statement is either true or false. But if a statement contains variables, we can no longer describe the statement as being simply true or false. Its truth value might depend on the values of the variables involved. For example, if $P(x)$ stands for the statement "x is a prime number," then $P(x)$ would be true if $x = 23$, but false if $x = 22$. To solve this problem, we will define *truth sets* for statements containing variables. Before giving this definition, though, it might be helpful to review some basic definitions from set theory.

A *set* is a collection of objects. The objects in the collection are called the *elements* of the set. The simplest way to specify a particular set is to list its elements between braces. For example, $\{3, 7, 14\}$ is the set whose elements are the three numbers 3, 7, and 14. We use the symbol $\in$ to mean *is an element of*. For example, if we let A stand for the set $\{3, 7, 14\}$, then we could write $7 \in A$ to say that 7 is an element of A. To say that 11 is not an element of A, we write $11 \notin A$.

A set is completely determined once its elements have been specified. Thus, two sets that have exactly the same elements are always equal. Also, when a set is defined by listing its elements, all that matters is *which* objects are in the list of elements, not the order in which they are listed. An element can even appear more than once in the list. Thus, $\{3, 7, 14\}$, $\{14, 3, 7\}$, and $\{3, 7, 14, 7\}$ are three different names for the same set.

It may be impractical to define a set that contains a very large number of elements by listing all of its elements, and it would be impossible to give such a definition for a set that contains infinitely many elements. Often this problem can be overcome by listing a few elements with an ellipsis (...) after them, if it is clear how the list should be continued. For example, suppose we define a set B by saying that $B = \{2, 3, 5, 7, 11, 13, 17, \ldots\}$. Once you recognize that the numbers listed in the definition of B are the prime numbers, then you know that, for example, $23 \in B$, even though it wasn't listed explicitly when we defined B. But this method requires recognition of the pattern in the list of numbers in the definition of B, and this requirement introduces an element of ambiguity and subjectivity into our notation that is best avoided in mathematical writing. It is therefore usually better to define such a set by spelling out the pattern that determines the elements of the set.

In this case we could be explicit by defining B as follows:

$$B = \{x \mid x \text{ is a prime number}\}.$$

This is read "$B =$ the set of all x such that x is a prime number," and it means that the elements of B are the values of x that make the statement "x is a prime number" come out true. You should think of the statement "x is a prime number" as an *elementhood test* for the set. Any value of x that makes this statement come out true passes the test and is an element of the set. Anything else fails the test and is not an element. Of course, in this case the values of x that make the statement true are precisely the prime numbers, so this definition says that B is the set whose elements are the prime numbers, exactly as before.

Example 1.3.2. Rewrite these set definitions using elementhood tests:

1. $E = \{2, 4, 6, 8, \ldots\}$.
2. P = {George Washington, John Adams, Thomas Jefferson, James Madison, ... }.

Solutions

Although there might be other ways of continuing these lists of elements, probably the most natural ones are given by the following definitions:

1. $E = \{n \mid n \text{ is a positive even integer}\}$.
2. $P = \{z \mid z \text{ was a president of the United States}\}$.

If a set has been defined using an elementhood test, then that test can be used to determine whether or not something is an element of the set. For example, consider the set $\{x \mid x^2 < 9\}$. If we want to know if 5 is an element of this set, we simply apply the elementhood test in the definition of the set – in other words, we check whether or not $5^2 < 9$. Since $5^2 = 25 > 9$, it fails the test, so $5 \notin \{x \mid x^2 < 9\}$. On the other hand, $(-2)^2 = 4 < 9$, so $-2 \in \{x \mid x^2 < 9\}$. The same reasoning would apply to any other number. For any number y, to determine whether or not $y \in \{x \mid x^2 < 9\}$, we just check whether or not $y^2 < 9$. In fact, we could think of the statement $y \in \{x \mid x^2 < 9\}$ as just a roundabout way of saying $y^2 < 9$.

Notice that because the statement $y \in \{x \mid x^2 < 9\}$ means the same thing as $y^2 < 9$, it is a statement about y, but not x! To determine whether or not $y \in \{x \mid x^2 < 9\}$ you need to know what y is (so you can compare its square to 9), but not what x is. We say that in the statement $y \in \{x \mid x^2 < 9\}$, y is a *free* variable,

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whereas x is a *bound* variable (or a *dummy* variable). The free variables in a statement stand for objects that the statement says something about. Plugging in different values for a free variable affects the meaning of a statement and may change its truth value. The fact that you can plug in different values for a free variable means that it is free to stand for anything. Bound variables, on the other hand, are simply letters that are used as a convenience to help express an idea and should not be thought of as standing for any particular object. A bound variable can always be replaced by a new variable without changing the meaning of the statement, and often the statement can be rephrased so that the bound variables are eliminated altogether. For example, the statements $y \in \{x \mid x^2 < 9\}$ and $y \in \{w \mid w^2 < 9\}$ mean the same thing, because they both mean "y is an element of the set of all numbers whose squares are less than 9." In this last statement, all bound variables have been eliminated, and the only variable mentioned is the free variable y.

Note that x is a bound variable in the statement $y \in \{x \mid x^2 < 9\}$ even though it is a free variable in the statement $x^2 < 9$. This last statement is a statement about x that would be true for some values of x and false for others. It is only when this statement is used inside the elementhood test notation that x becomes a bound variable. We could say that the notation $\{x \mid \ldots\}$ *binds* the variable x.

Everything we have said about the set $\{x \mid x^2 < 9\}$ would apply to any set defined by an elementhood test. In general, the statement $y \in \{x \mid P(x)\}$ means the same thing as $P(y)$, which is a statement about y but not x. Similarly, $y \notin \{x \mid P(x)\}$ means the same thing as $\neg P(y)$. Of course, the expression $\{x \mid P(x)\}$ is not a statement at all; it is a name for a set. As you learn more mathematical notation, it will become increasingly important to make sure you are careful to distinguish between expressions that are mathematical statements and expressions that are names for mathematical objects.

Example 1.3.3. What do these statements mean? What are the free variables in each statement?

1. $a + b \notin \{x \mid x \text{ is an even number}\}$.
2. $y \in \{x \mid x \text{ is divisible by } w\}$.
3. $2 \in \{w \mid 6 \notin \{x \mid x \text{ is divisible by } w\}\}$.

Solutions

1. This statement says that $a + b$ is not an element of the set of all even numbers, or in other words, $a + b$ is not an even number. Both a and b are free variables, but x is a bound variable. The statement will be true for some values of a and b and false for others.

2. This statement says that y is divisible by w. Both y and w are free variables, but x is a bound variable. The statement is true for some values of y and w and false for others.
3. This looks quite complicated, but if we go a step at a time, we can decipher it. First, note that the statement $6 \notin \{x \mid x \text{ is divisible by } w\}$, which appears inside the given statement, means the same thing as "6 is not divisible by w." Substituting this into the given statement, we find that the original statement is equivalent to the simpler statement $2 \in \{w \mid 6 \text{ is not divisible by } w\}$. But this just means the same thing as "6 is not divisible by 2." Thus, the statement has no free variables, and both x and w are bound variables. Because there are no free variables, the truth value of the statement doesn't depend on the values of any variables. In fact, since 6 *is* divisible by 2, the statement is false.

Perhaps you have guessed by now how we can use set theory to help us understand truth values of statements containing free variables. As we have seen, a statement, say $P(x)$, containing a free variable x, may be true for some values of x and false for others. To distinguish the values of x that make $P(x)$ true from those that make it false, we could form the set of values of x for which $P(x)$ is true. We will call this set the *truth set* of $P(x)$.

Definition 1.3.4. The *truth set* of a statement $P(x)$ is the set of all values of x that make the statement $P(x)$ true. In other words, it is the set defined by using the statement $P(x)$ as an elementhood test:

$$\text{Truth set of } P(x) = \{x \mid P(x)\}.$$

Note that we have defined truth sets only for statements containing *one* free variable. We will discuss truth sets for statements with more than one free variable in Chapter 4.

Example 1.3.5. What are the truth sets of the following statements?

1. Shakespeare wrote x.
2. n is an even prime number.

Solutions

1. $\{x \mid \text{Shakespeare wrote } x\} = \{\text{Hamlet, Macbeth, Twelfth Night}, \ldots\}$.
2. $\{n \mid n \text{ is an even prime number}\}$. Because the only even prime number is 2, this is the set $\{2\}$. Note that 2 and $\{2\}$ are not the same thing! The first is a number, and the second is a set whose only element is a number. Thus, $2 \in \{2\}$, but $2 \neq \{2\}$.

Suppose A is the truth set of a statement $P(x)$. According to the definition of truth set, this means that $A = \{x \mid P(x)\}$. We've already seen that for any object y, the statement $y \in \{x \mid P(x)\}$ means the same thing as $P(y)$. Substituting in A for $\{x \mid P(x)\}$, it follows that $y \in A$ means the same thing as $P(y)$. Thus, we see that in general, if A is the truth set of $P(x)$, then to say that $y \in A$ means the same thing as saying $P(y)$.

When a statement contains free variables, it is often clear from context that these variables stand for objects of a particular kind. The set of all objects of this kind – in other words, the set of all possible values for the variables – is called the *universe of discourse* for the statement, and we say that the variables *range over* this universe. For example, in most contexts the universe for the statement $x^2 < 9$ would be the set of all real numbers; the universe for the statement "x is a man" might be the set of all people.

Certain sets come up often in mathematics as universes of discourse, and it is convenient to have fixed names for them. Here are a few of the most important ones:

> $\mathbb{R} = \{x \mid x \text{ is a real number}\}$.
> $\mathbb{Q} = \{x \mid x \text{ is a rational number}\}$.
> (Recall that a *real* number is any number on the number line, and a *rational* number is a number that can be written as a fraction p/q, where p and q are integers.)
> $\mathbb{Z} = \{x \mid x \text{ is an integer}\} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.
> $\mathbb{N} = \{x \mid x \text{ is a natural number}\} = \{0, 1, 2, 3, \ldots\}$.
> (Some books include 0 as a natural number and some don't. In this book, we consider 0 to be a natural number.)

The letters $\mathbb{R}$, $\mathbb{Q}$, and $\mathbb{Z}$ can be followed by a superscript $+$ or $-$ to indicate that only positive or negative numbers are to be included in the set. For example, $\mathbb{R}^+ = \{x \mid x \text{ is a positive real number}\}$, and $\mathbb{Z}^- = \{x \mid x \text{ is a negative integer}\}$.

Although the universe of discourse can usually be determined from context, it is sometimes useful to identify it explicitly. Consider a statement $P(x)$ with a free variable x that ranges over a universe U. Although we have written the truth set of $P(x)$ as $\{x \mid P(x)\}$, if there were any possibility of confusion about what the universe was, we could specify it explicitly by writing $\{x \in U \mid P(x)\}$; this is read "the set of all x in U such that $P(x)$." This notation indicates that only elements of U are to be considered for elementhood in this truth set, and among elements of U, only those that pass the elementhood test $P(x)$ will actually be in the truth set. For example, consider again the statement $x^2 < 9$. If the universe of discourse for this statement were the set of all real numbers, then its truth set would be $\{x \in \mathbb{R} \mid x^2 < 9\}$, or in other words, the set of all real numbers between -3 and 3. But if the universe were the set of all integers, then the truth set would be $\{x \in \mathbb{Z} \mid x^2 < 9\} = \{-2, -1, 0, 1, 2\}$. Thus, for example, $1.58 \in \{x \in \mathbb{R} \mid x^2 < 9\}$ but $1.58 \notin \{x \in \mathbb{Z} \mid x^2 < 9\}$. Clearly, the choice of universe can sometimes make a difference!

Sometimes this explicit notation is used not to specify the universe of discourse but to restrict attention to just a part of the universe. For example, in the case of the statement $x^2 < 9$, we might want to consider the universe of discourse to be the set of all real numbers, but in the course of some reasoning involving this statement we might want to temporarily restrict our attention to only positive real numbers. We might then be interested in the set $\{x \in \mathbb{R}^+ \mid x^2 < 9\}$. As before, this notation indicates that only positive real numbers will be considered for elementhood in this set, and among positive real numbers, only those whose square is less than 9 will be in the set. Thus, for a number to be an element of this set, it must pass two tests: it must be a positive real number, and its square must be less than 9. In other words, the statement $y \in \{x \in \mathbb{R}^+ \mid x^2 < 9\}$ means the same thing as $y \in \mathbb{R}^+ \wedge y^2 < 9$. In general, $y \in \{x \in A \mid P(x)\}$ means the same thing as $y \in A \wedge P(y)$.

When a new mathematical concept has been defined, mathematicians are usually interested in studying any possible extremes of this concept. For example, when we discussed truth tables, the extremes we studied were statements whose truth tables contained only T's (tautologies) or only F's (contradictions). For the concept of the truth set of a statement containing a free variable, the corresponding extremes would be the truth sets of statements that are always true or always false. Suppose $P(x)$ is a statement containing a free variable x that ranges over a universe U. It should be clear that if $P(x)$ comes out true for every value of $x \in U$, then the truth set of $P(x)$ will be the whole universe U. For example, since the statement $x^2 \geq 0$ is true for every real number x, the truth set of this statement is $\{x \in \mathbb{R} \mid x^2 \geq 0\} = \mathbb{R}$. Of course, this is not unrelated to the concept of a tautology. For example, since $P \vee \neg P$ is a tautology, the statement $P(x) \vee \neg P(x)$ will be true for every $x \in U$, no matter what statement $P(x)$ stands for or what the universe U is, and therefore the truth set of the statement $P(x) \vee \neg P(x)$ will be U.

For a statement $P(x)$ that is false for every possible value of x, nothing in the universe can pass the elementhood test for the truth set of $P(x)$, and so this truth set must have no elements. The idea of a set with no elements may sound strange, but it arises naturally when we consider truth sets for statements that are always false. Because a set is completely determined once its elements have been specified, there is only one set that has no elements. It is called the *empty set*, or the *null set*, and is often denoted $\varnothing$. For example, $\{x \in \mathbb{Z} \mid x \neq x\} = \varnothing$.

Since the empty set has no elements, the statement $x \in \varnothing$ is an example of a statement that is always false, no matter what x is.

Another common notation for the empty set is based on the fact that any set can be named by listing its elements between braces. Since the empty set has no elements, we write nothing between the braces, like this: $\varnothing = \{\ \}$. Note that $\{\varnothing\}$ is not correct notation for the empty set. Just as we saw earlier that 2 and $\{2\}$ are not the same thing, $\varnothing$ is not the same as $\{\varnothing\}$. The first is a set with no elements, whereas the second is a set with one element, that one element being $\varnothing$, the empty set.

Exercises

*1. Analyze the logical forms of the following statements:
   (a) 3 is a common divisor of 6, 9, and 15. (Note: You did this in exercise 2 of Section 1.1, but you should be able to give a better answer now.)
   (b) x is divisible by both 2 and 3 but not 4.
   (c) x and y are natural numbers, and exactly one of them is prime.

2. Analyze the logical forms of the following statements:
   (a) x and y are men, and either x is taller than y or y is taller than x.
   (b) Either x or y has brown eyes, and either x or y has red hair.
   (c) Either x or y has both brown eyes and red hair.

*3. Write definitions using elementhood tests for the following sets:
   (a) {Mercury, Venus, Earth, Mars, Jupiter, Saturn, Uranus, Neptune, Pluto}.
   (b) {Brown, Columbia, Cornell, Dartmouth, Harvard, Princeton, University of Pennsylvania, Yale}.
   (c) {Alabama, Alaska, Arizona, ..., Wisconsin, Wyoming}.
   (d) {Alberta, British Columbia, Manitoba, New Brunswick, Newfoundland and Labrador, Northwest Territories, Nova Scotia, Nunavut, Ontario, Prince Edward Island, Quebec, Saskatchewan, Yukon}.

4. Write definitions using elementhood tests for the following sets:
   (a) $\{1, 4, 9, 16, 25, 36, 49, \ldots\}$.
   (b) $\{1, 2, 4, 8, 16, 32, 64, \ldots\}$.
   (c) $\{10, 11, 12, 13, 14, 15, 16, 17, 18, 19\}$.

*5. Simplify the following statements. Which variables are free and which are bound? If the statement has no free variables, say whether it is true or false.
   (a) $-3 \in \{x \in \mathbb{R} \mid 13 - 2x > 1\}$.
   (b) $4 \in \{x \in \mathbb{R}^- \mid 13 - 2x > 1\}$.
   (c) $5 \notin \{x \in \mathbb{R} \mid 13 - 2x > c\}$.

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6. Simplify the following statements. Which variables are free and which are bound? If the statement has no free variables, say whether it is true or false.
   (a) $w \in \{x \in \mathbb{R} \mid 13 - 2x > c\}$.
   (b) $4 \in \{x \in \mathbb{R} \mid 13 - 2x \in \{y \mid y \text{ is a prime number}\}\}$. (It might make this statement easier to read if we let $P = \{y \mid y \text{ is a prime number}\}$; using this notation, we could rewrite the statement as $4 \in \{x \in \mathbb{R} \mid 13 - 2x \in P\}$.)
   (c) $4 \in \{x \in \{y \mid y \text{ is a prime number}\} \mid 13 - 2x > 1\}$. (Using the same notation as in part (b), we could write this as $4 \in \{x \in P \mid 13 - 2x > 1\}$.)

*7. What are the truth sets of the following statements? List a few elements of the truth set if you can.
   (a) Elizabeth Taylor was once married to x.
   (b) x is a logical connective studied in Section 1.1.
   (c) x is the author of this book.

8. What are the truth sets of the following statements? List a few elements of the truth set if you can.
   (a) x is a real number and $x^2 - 4x + 3 = 0$.
   (b) x is a real number and $x^2 - 2x + 3 = 0$.
   (c) x is a real number and $5 \in \{y \in \mathbb{R} \mid x^2 + y^2 < 50\}$.

1.4. Operations on Sets

Suppose A is the truth set of a statement $P(x)$ and B is the truth set of $Q(x)$. What are the truth sets of the statements $P(x) \wedge Q(x)$, $P(x) \vee Q(x)$, and $\neg P(x)$? To answer these questions, we introduce some basic operations on sets.

Definition 1.4.1. The *intersection* of two sets A and B is the set $A \cap B$ defined as follows:

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$

The *union* of A and B is the set $A \cup B$ defined as follows:

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$

The *difference* of A and B is the set $A \setminus B$ defined as follows:

$$A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}.$$

Remember that the statements that appear in these definitions are *elementhood tests*. Thus, for example, the definition of $A \cap B$ says that for an object to be an element of $A \cap B$, it must be an element of both A and B. In other words, $A \cap B$ is the set consisting of the elements that A and B have in common.

Because the word *or* is always interpreted as *inclusive or* in mathematics, anything that is an element of either A or B, or both, will be an element of $A \cup B$. Thus, we can think of $A \cup B$ as the set resulting from throwing all the elements of A and B together into one set. $A \setminus B$ is the set you would get if you started with the set A and removed from it any elements that were also in B.

Example 1.4.2. Suppose $A = \{1, 2, 3, 4, 5\}$ and $B = \{2, 4, 6, 8, 10\}$. List the elements of the following sets:

1. $A \cap B$.
2. $A \cup B$.
3. $A \setminus B$.
4. $(A \cup B) \setminus (A \cap B)$.
5. $(A \setminus B) \cup (B \setminus A)$.

Solutions

1. $A \cap B = \{2, 4\}$.
2. $A \cup B = \{1, 2, 3, 4, 5, 6, 8, 10\}$.
3. $A \setminus B = \{1, 3, 5\}$.
4. We have just computed $A \cup B$ and $A \cap B$ in solutions 1 and 2, so all we need to do is start with the set $A \cup B$ from solution 2 and remove from it any elements that are also in $A \cap B$. The answer is $(A \cup B) \setminus (A \cap B) = \{1, 3, 5, 6, 8, 10\}$.
5. We already have the elements of $A \setminus B$ listed in solution 3, and $B \setminus A = \{6, 8, 10\}$. Thus, their union is $(A \setminus B) \cup (B \setminus A) = \{1, 3, 5, 6, 8, 10\}$. Is it just a coincidence that this is the same as the answer to part 4?

Example 1.4.3. Suppose $A = \{x \mid x \text{ is a man}\}$ and $B = \{x \mid x \text{ has brown hair}\}$. What are $A \cap B$, $A \cup B$, and $A \setminus B$?

Solution

By definition, $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$. As we saw in the last section, the definitions of A and B tell us that $x \in A$ means the same thing as "x is a man," and $x \in B$ means the same thing as "x has brown hair." Plugging this into the definition of $A \cap B$, we find that

$$A \cap B = \{x \mid x \text{ is a man and } x \text{ has brown hair}\}.$$

Similar reasoning shows that

$$A \cup B = \{x \mid \text{either } x \text{ is a man or } x \text{ has brown hair}\}$$

and

$$A \setminus B = \{x \mid x \text{ is a man and } x \text{ does not have brown hair}\}.$$

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Sometimes it is helpful when working with operations on sets to draw pictures of the results of these operations. One way to do this is with diagrams like that in Figure 1. This is called a *Venn diagram*. The interior of the rectangle enclosing the diagram represents the universe of discourse U, and the interiors of the two circles represent the two sets A and B. Other sets formed by combining these sets would be represented by different regions in the diagram. For example, the shaded region in Figure 2 is the region common to the circles representing A and B, and so it represents the set $A \cap B$. Figures 3 and 4 show the regions representing $A \cup B$ and $A \setminus B$, respectively.

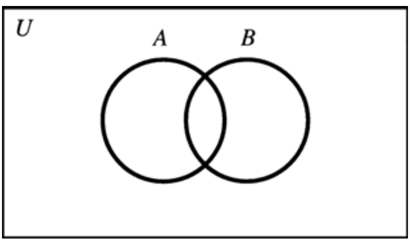
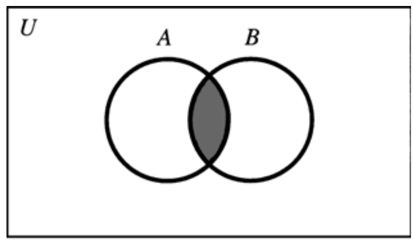
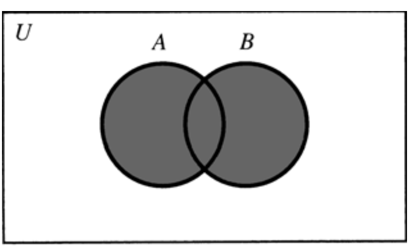


Figure 1



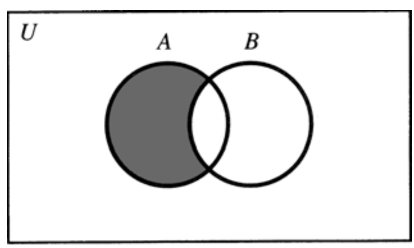
$A \cap B$

Figure 2



$A \cup B$

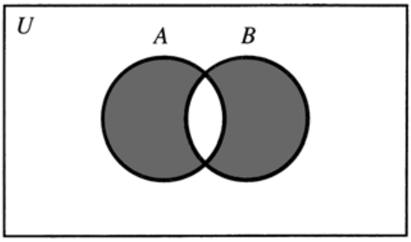
Figure 3



$A \setminus B$

Figure 4

Here's an example of how Venn diagrams can help us understand operations on sets. In Example 1.4.2 the sets $(A \cup B) \setminus (A \cap B)$ and $(A \setminus B) \cup (B \setminus A)$ turned out to be equal, for a particular choice of A and B. You can see by making Venn diagrams for both sets that this was not a coincidence. You'll find that both Venn diagrams look like Figure 5. Thus, these sets will always be equal, no matter what the sets A and B are, because both sets will always be the set of objects that are elements of either A or B but not both. This set is called the *symmetric difference* of A and B and is written $A \triangle B$. In other words, $A \triangle B = (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$. Later in this section we'll see another explanation of why these sets are always equal.



$(A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A)$

Figure 5

Let's return now to the question with which we began this section. If A is the truth set of a statement $P(x)$ and B is the truth set of $Q(x)$, then, as we saw in the last section, $x \in A$ means the same thing as $P(x)$ and $x \in B$ means the same thing as $Q(x)$. Thus, the truth set of $P(x) \wedge Q(x)$ is $\{x \mid P(x) \wedge Q(x)\} = \{x \mid x \in A \wedge x \in B\} = A \cap B$. This should make sense. It just says that the truth set of $P(x) \wedge Q(x)$ consists of those elements that the truth sets of $P(x)$ and $Q(x)$ have in common – in other words, the values of x that make both $P(x)$ and $Q(x)$ come out true. We have already seen an example of this. In Example 1.4.3 the sets A and B were the truth sets of the statements "x is a man" and "x has brown hair," and $A \cap B$ turned out to be the truth set of "x is a man and x has brown hair."

Similar reasoning shows that the truth set of $P(x) \vee Q(x)$ is $A \cup B$. To find the truth set of $\neg P(x)$, we need to talk about the universe of discourse U. The truth set of $\neg P(x)$ will consist of those elements of the universe for which $P(x)$ is false, and we can find this set by starting with U and removing from it those elements for which $P(x)$ is true. Thus, the truth set of $\neg P(x)$ is $U \setminus A$.

These observations about truth sets illustrate the fact that the set theory operations $\cap$, $\cup$, and $\setminus$ are related to the logical connectives $\wedge$, $\vee$, and $\neg$. This shouldn't be surprising, since after all the words *and*, *or*, and *not* appear in their definitions. (The word *not* doesn't appear explicitly, but it's there, hidden in the mathematical symbol $\notin$ in the definition of the difference of two sets.) It is important to remember, though, that although the set theory operations and logical connectives are related, they are not interchangeable. The logical connectives can only be used to combine *statements*, whereas the set theory operations must be used to combine *sets*. For example, if A is the truth set of $P(x)$ and B is the truth set of $Q(x)$, then we can say that $A \cap B$ is the truth set of $P(x) \wedge Q(x)$, but expressions such as $A \wedge B$ or $P(x) \cap Q(x)$ are completely meaningless and should never be used.

The relationship between set theory operations and logical connectives also becomes apparent when we analyze the logical forms of statements about intersections, unions, and differences of sets. For example, according to the definition of intersection, to say that $x \in A \cap B$ means that $x \in A \wedge x \in B$. Similarly, to say that $x \in A \cup B$ means that $x \in A \vee x \in B$, and $x \in A \setminus B$ means $x \in A \wedge x \notin B$, or in other words $x \in A \wedge \neg(x \in B)$. We can combine these rules when analyzing statements about more complex sets.

Example 1.4.4. Analyze the logical forms of the following statements:

1. $x \in A \cap (B \cup C)$.
2. $x \in A \setminus (B \cap C)$.
3. $x \in (A \cap B) \cup (A \cap C)$.

Solutions

1. $x \in A \cap (B \cup C)$
   is equivalent to $x \in A \wedge x \in (B \cup C)$ (definition of $\cap$),
   which is equivalent to $x \in A \wedge (x \in B \vee x \in C)$ (definition of $\cup$).
2. $x \in A \setminus (B \cap C)$
   is equivalent to $x \in A \wedge \neg(x \in B \cap C)$ (definition of $\setminus$),
   which is equivalent to $x \in A \wedge \neg(x \in B \wedge x \in C)$ (definition of $\cap$).
3. $x \in (A \cap B) \cup (A \cap C)$
   is equivalent to $x \in (A \cap B) \vee x \in (A \cap C)$ (definition of $\cup$),
   which is equivalent to $(x \in A \wedge x \in B) \vee (x \in A \wedge x \in C)$
   (definition of $\cap$).

Look again at the solutions to parts 1 and 3 of Example 1.4.4. You should recognize that the statements we ended up with in these two parts are equivalent. (If you don't, look back at the distributive laws in Section 1.2.) This equivalence means that the statements $x \in A \cap (B \cup C)$ and $x \in (A \cap B) \cup (A \cap C)$ are equivalent. In other words, the objects that are elements of the set $A \cap (B \cup C)$ will be precisely the same as the objects that are elements of $(A \cap B) \cup (A \cap C)$, no matter what the sets A, B, and C are. But recall that sets with the same elements are equal, so it follows that for any sets A, B, and C, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. Another way to see this is with the Venn diagram in Figure 6. Our earlier Venn diagrams had two circles, because in previous examples only two sets were being combined. This Venn diagram has three circles, which represent the three sets A, B, and C that are being combined in this case. Although it is possible to create Venn diagrams for more than three sets, it is rarely done, because it cannot be done with overlapping circles. For more on Venn diagrams for more than three sets, see exercise 10.

Thus, we see that a distributive law for logical connectives has led to a distributive law for set theory operations. You might guess that because there

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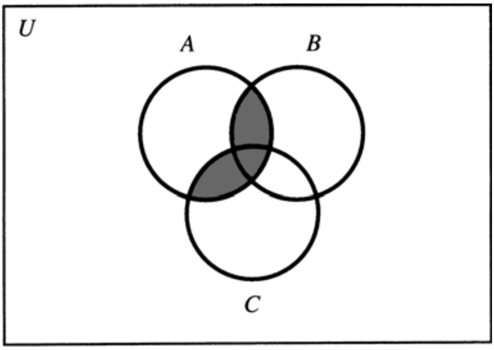

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Figure 6

were *two* distributive laws for the logical connectives, with $\wedge$ and $\vee$ playing opposite roles in the two laws, there might be two distributive laws for set theory operations too. The second distributive law for sets should say that for any sets A, B, and C, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. You can verify this for yourself by writing out the statements $x \in A \cup (B \cap C)$ and $x \in (A \cup B) \cap (A \cup C)$ using logical connectives and verifying that they are equivalent, using the second distributive law for the logical connectives $\wedge$ and $\vee$. Another way to see it is to make a Venn diagram.

We can derive another set theory identity by finding a statement equivalent to the statement we ended up with in part 2 of Example 1.4.4:

$x \in A \setminus (B \cap C)$
is equivalent to $x \in A \wedge \neg(x \in B \wedge x \in C)$ (Example 1.4.4),
which is equivalent to $x \in A \wedge (x \notin B \vee x \notin C)$ (DeMorgan's law),
which is equivalent to $(x \in A \wedge x \notin B) \vee (x \in A \wedge x \notin C)$
(distributive law),
which is equivalent to $(x \in A \setminus B) \vee (x \in A \setminus C)$ (definition of $\setminus$),
which is equivalent to $x \in (A \setminus B) \cup (A \setminus C)$ (definition of $\cup$).

Thus, we have shown that for any sets A, B, and C, $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$. Once again, you can verify this with a Venn diagram as well.

Earlier we promised an alternative way to check the identity $(A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A)$. You should see now how this can be done. First, we write out the logical forms of the statements $x \in (A \cup B) \setminus (A \cap B)$ and $x \in (A \setminus B) \cup (B \setminus A)$:

$x \in (A \cup B) \setminus (A \cap B)$ means $(x \in A \vee x \in B) \wedge \neg(x \in A \wedge x \in B)$;
$x \in (A \setminus B) \cup (B \setminus A)$ means $(x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A)$.

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You can now check, using equivalences from Section 1.2, that these statements are equivalent. An alternative way to check the equivalence is with a truth table. To simplify the truth table, let's use P and Q as abbreviations for the statements $x \in A$ and $x \in B$. Then we must check that the formulas $(P \vee Q) \wedge \neg(P \wedge Q)$ and $(P \wedge \neg Q) \vee (Q \wedge \neg P)$ are equivalent. The truth table in Figure 7 shows this.

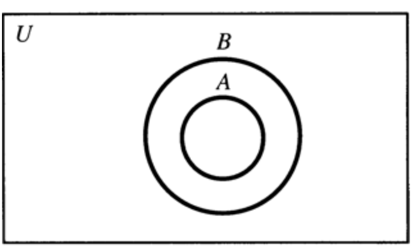
P	Q	$(P \vee Q) \wedge \neg(P \wedge Q)$	$(P \wedge \neg Q) \vee (Q \wedge \neg P)$
F	F	F	F
F	T	T	T
T	F	T	T
T	T	F	F

Figure 7

Definition 1.4.5. Suppose A and B are sets. We will say that A is a *subset* of B if every element of A is also an element of B. We write $A \subseteq B$ to mean that A is a subset of B. A and B are said to be *disjoint* if they have no elements in common. Note that this is the same as saying that the set of elements they have in common is the empty set, or in other words $A \cap B = \varnothing$.

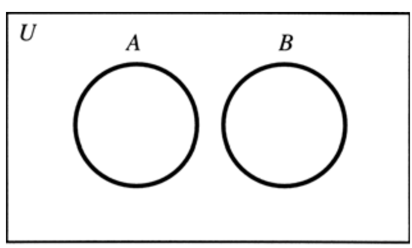
Example 1.4.6. Suppose $A = \{\text{red, green}\}$, $B = \{\text{red, yellow, green, purple}\}$, and $C = \{\text{blue, purple}\}$. Then the two elements of A, red and green, are both also in B, and therefore $A \subseteq B$. Also, $A \cap C = \varnothing$, so A and C are disjoint.

If we know that $A \subseteq B$, or that A and B are disjoint, then we might draw a Venn diagram for A and B differently to reflect this. Figures 8 and 9 illustrate this.



$A \subseteq B$

Figure 8



$A \cap B = \varnothing$

Figure 9

Just as we earlier derived identities showing that certain sets are always equal, it is also sometimes possible to show that certain sets are always disjoint, or that one set is always a subset of another. For example, you can see in a Venn

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diagram that the sets $A \cap B$ and $A \setminus B$ do not overlap, and therefore they will always be disjoint for any sets A and B. Another way to see this would be to write out what it means to say that $x \in (A \cap B) \cap (A \setminus B)$:

$$x \in (A \cap B) \cap (A \setminus B) \text{ means } (x \in A \wedge x \in B) \wedge (x \in A \wedge x \notin B),$$
$$\text{which is equivalent to } x \in A \wedge (x \in B \wedge x \notin B).$$

But this last statement is clearly a contradiction, so the statement $x \in (A \cap B) \cap (A \setminus B)$ will always be false, no matter what x is. In other words, nothing can be an element of $(A \cap B) \cap (A \setminus B)$, so it must be the case that $(A \cap B) \cap (A \setminus B) = \varnothing$. Therefore, $A \cap B$ and $A \setminus B$ are disjoint.

The next theorem gives another example of a general fact about set operations. The proof of this theorem illustrates that the principles of deductive reasoning we have been studying are actually used in mathematical proofs.

Theorem 1.4.7. *For any sets A and B, $(A \cup B) \setminus B \subseteq A$.*

Proof. We must show that if something is an element of $(A \cup B) \setminus B$, then it must also be an element of A, so suppose that $x \in (A \cup B) \setminus B$. This means that $x \in A \cup B$ and $x \notin B$, or in other words $x \in A \vee x \in B$ and $x \notin B$. But notice that these statements have the logical form $P \vee Q$ and $\neg Q$, and this is precisely the form of the premises of our very first example of a deductive argument in Section 1.1! As we saw in that example, from these premises we can conclude that $x \in A$ must be true. Thus, anything that is an element of $(A \cup B) \setminus B$ must also be an element of A, so $(A \cup B) \setminus B \subseteq A$. □

You might think that such a careful application of logical laws is not needed to understand why Theorem 1.4.7 is correct. The set $(A \cup B) \setminus B$ could be thought of as the result of starting with the set A, adding in the elements of B, and then removing them again. Common sense suggests that the result will just be the original set A; in other words, it appears that $(A \cup B) \setminus B = A$. However, as you are asked to show in exercise 9, this conclusion is incorrect. This illustrates that in mathematics, you must not allow imprecise reasoning to lead you to jump to conclusions. Applying laws of logic carefully, as we did in our proof of Theorem 1.4.7, may help you to avoid jumping to unwarranted conclusions.

Exercises

*1. Let $A = \{1, 3, 12, 35\}$, $B = \{3, 7, 12, 20\}$, and $C = \{x \mid x \text{ is a prime number}\}$. List the elements of the following sets. Are any of the sets

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below disjoint from any of the others? Are any of the sets below subsets of any others?
(a) $A \cap B$.
(b) $(A \cup B) \setminus C$.
(c) $A \cup (B \setminus C)$.

2. Let $A = \{$United States, Germany, China, Australia$\}$, $B = \{$Germany, France, India, Brazil$\}$, and $C = \{x \mid x$ is a country in Europe$\}$. List the elements of the following sets. Are any of the sets below disjoint from any of the others? Are any of the sets below subsets of any others?
(a) $A \cup B$.
(b) $(A \cap B) \setminus C$.
(c) $(B \cap C) \setminus A$.
3. Verify that the Venn diagrams for $(A \cup B) \setminus (A \cap B)$ and $(A \setminus B) \cup (B \setminus A)$ both look like Figure 5, as stated in this section.
*4. Use Venn diagrams to verify the following identities:
(a) $A \setminus (A \cap B) = A \setminus B$.
(b) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
5. Verify the identities in exercise 4 by writing out (using logical symbols) what it means for an object x to be an element of each set and then using logical equivalences.
6. Use Venn diagrams to verify the following identities:
(a) $(A \cup B) \setminus C = (A \setminus C) \cup (B \setminus C)$.
(b) $A \cup (B \setminus C) = (A \cup B) \setminus (C \setminus A)$.
7. Verify the identities in exercise 6 by writing out (using logical symbols) what it means for an object x to be an element of each set and then using logical equivalences.
*8. For each of the following sets, write out (using logical symbols) what it means for an object x to be an element of the set. Then determine which of these sets must be equal to each other by determining which statements are equivalent.
(a) $(A \setminus B) \setminus C$.
(b) $A \setminus (B \setminus C)$.
(c) $(A \setminus B) \cup (A \cap C)$.
(d) $(A \setminus B) \cap (A \setminus C)$.
(e) $A \setminus (B \cup C)$.
9. It was shown in this section that for any sets A and B, $(A \cup B) \setminus B \subseteq A$. Give an example of two sets A and B for which $(A \cup B) \setminus B \neq A$.
*10. It is claimed in this section that you cannot make a Venn diagram for four sets using overlapping circles.

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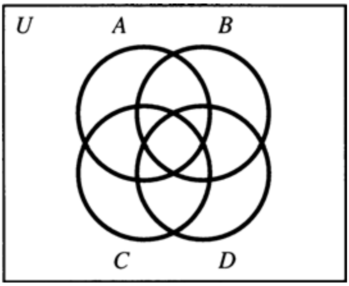
(a) What's wrong with the following diagram? (Hint: Where's the set $(A \cap D) \setminus (B \cup C)$?)



(b) Can you make a Venn diagram for four sets using shapes other than circles?

11. (a) Make Venn diagrams for the sets $(A \cup B) \setminus C$ and $A \cup (B \setminus C)$. What can you conclude about whether one of these sets is necessarily a subset of the other?
    (b) Give an example of sets A, B, and C for which $(A \cup B) \setminus C \neq A \cup (B \setminus C)$.

*12. Use Venn diagrams to show that the associative law holds for symmetric difference; that is, for any sets A, B, and C, $A \triangle (B \triangle C) = (A \triangle B) \triangle C$.

13. Use any method you wish to verify the following identities:
    (a) $(A \triangle B) \cup C = (A \cup C) \triangle (B \setminus C)$.
    (b) $(A \triangle B) \cap C = (A \cap C) \triangle (B \cap C)$.
    (c) $(A \triangle B) \setminus C = (A \setminus C) \triangle (B \setminus C)$.

14. Use any method you wish to verify the following identities:
    (a) $(A \cup B) \triangle C = (A \triangle C) \triangle (B \setminus A)$.
    (b) $(A \cap B) \triangle C = (A \triangle C) \triangle (A \setminus B)$.
    (c) $(A \setminus B) \triangle C = (A \triangle C) \triangle (A \cap B)$.

15. Fill in the blanks to make true identities:
    (a) $(A \triangle B) \cap C = (C \setminus A) \triangle$ ________.
    (b) $C \setminus (A \triangle B) = (A \cap C) \triangle$ ________.
    (c) $(B \setminus A) \triangle C = (A \triangle C) \triangle$ ________.

1.5. The Conditional and Biconditional Connectives

It is time now to return to a question we left unanswered in Section 1.1. We have seen how the reasoning in the first and third arguments in Example 1.1.1 can be understood by analyzing the connectives $\vee$ and $\neg$. But what about the

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reasoning in the second argument? Recall that the argument went like this:

> If today is Sunday, then I don't have to go to work today.
> Today is Sunday.
> Therefore, I don't have to go to work today.

What makes this reasoning valid?

It appears that the crucial words here are *if* and *then*, which occur in the first premise. We therefore introduce a new logical connective, $\rightarrow$, and write $P \rightarrow Q$ to represent the statement "If P then Q." This statement is sometimes called a *conditional* statement, with P as its *antecedent* and Q as its *consequent*. If we let P stand for the statement "Today is Sunday" and Q for the statement "I don't have to go to work today," then the logical form of the argument would be

$$\begin{array}{l} P \rightarrow Q \\ \underline{P \qquad\quad} \\ \therefore Q \end{array}$$

Our analysis of the new connective $\rightarrow$ should lead to the conclusion that this argument is valid.

Example 1.5.1. Analyze the logical forms of the following statements:

1. If it's raining and I don't have my umbrella, then I'll get wet.
2. If Mary did her homework, then the teacher won't collect it, and if she didn't, then he'll ask her to do it on the board.

Solutions

1. Let R stand for the statement "It's raining," U for "I have my umbrella," and W for "I'll get wet." Then statement 1 would be represented by the formula $(R \wedge \neg U) \rightarrow W$.
2. Let H stand for "Mary did her homework," C for "The teacher will collect it," and B for "The teacher will ask Mary to do the homework on the board." Then the given statement means $(H \rightarrow \neg C) \wedge (\neg H \rightarrow B)$.

To analyze arguments containing the connective $\rightarrow$ we must work out the truth table for the formula $P \rightarrow Q$. Because $P \rightarrow Q$ is supposed to mean that if P is true then Q is also true, we certainly want to say that if P is true and Q is false then $P \rightarrow Q$ is false. If P is true and Q is also true, then it seems reasonable to say that $P \rightarrow Q$ is true. This gives us the last two lines of the truth table in Figure 1. The remaining two lines of the truth table are harder to fill in, although some people might say that if P and Q are both false then

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$P \to Q$ should be considered true. Thus, we can sum up our conclusions so far with the table in Figure 1.

P	Q	$P \to Q$
F	F	T?
F	T	?
T	F	F
T	T	T

Figure 1

To help us fill in the undetermined lines in this truth table, let's look at an example. Consider the statement "If $x > 2$ then $x^2 > 4$," which we could represent with the formula $P(x) \to Q(x)$, where $P(x)$ stands for the statement $x > 2$ and $Q(x)$ stands for $x^2 > 4$. Of course, the statements $P(x)$ and $Q(x)$ contain x as a free variable, and each will be true for some values of x and false for others. But surely, no matter what the value of x is, we would say it is true that *if* $x > 2$ then $x^2 > 4$, so the conditional statement $P(x) \to Q(x)$ should be true. Thus, the truth table should be completed in such a way that no matter what value we plug in for x, this conditional statement comes out true.

For example, suppose $x = 3$. In this case $x > 2$ and $x^2 = 9 > 4$, so $P(x)$ and $Q(x)$ are both true. This corresponds to line four of the truth table in Figure 1, and we've already decided that the statement $P(x) \to Q(x)$ should come out true in this case. But now consider the case $x = 1$. Then $x < 2$ and $x^2 = 1 < 4$, so $P(x)$ and $Q(x)$ are both false, corresponding to line one in the truth table. We have tentatively placed a T in this line of the truth table, and now we see that this tentative choice must be right. If we put an F there, then the statement $P(x) \to Q(x)$ would come out false in the case $x = 1$, and we've already decided that it should be true for all values of x.

Finally, consider the case $x = -5$. Then $x < 2$, so $P(x)$ is false, but $x^2 = 25 > 4$, so $Q(x)$ is true. Thus, in this case we find ourselves in the second line of the truth table, and once again, if the conditional statement $P(x) \to Q(x)$ is to be true in this case, we must put a T in this line. So it appears that all the questionable lines in the truth table in Figure 1 must be filled in with T's, and the completed truth table for the connective $\to$ must be as shown in Figure 2.

P	Q	$P \to Q$
F	F	T
F	T	T
T	F	F
T	T	T

Figure 2

Of course, there are many other values of x that could be plugged into our statement "If $x > 2$ then $x^2 > 4$"; but if you try them, you'll find that they all lead to line one, two, or four of the truth table, as our examples $x = 1, -5$, and 3 did. No value of x will lead to line three, because you could never have $x > 2$ but $x^2 \leq 4$. After all, that's why we said that the statement "If $x > 2$ then $x^2 > 4$" was always true, no matter what x was! The point of saying that this conditional statement is always true is simply to say that you will never find a value of x such that $x > 2$ and $x^2 \leq 4$ – in other words, there is no value of x for which $P(x)$ is true but $Q(x)$ is false. Thus, it should make sense that in the truth table for $P \to Q$, the only line that is false is the line in which P is true and Q is false.

As the truth table in Figure 3 shows, the formula $\neg P \vee Q$ is also true in every case except when P is true and Q is false. Thus, if we accept the truth table in Figure 2 as the correct truth table for the formula $P \to Q$, then we will be forced to accept the conclusion that the formulas $P \to Q$ and $\neg P \vee Q$ are equivalent. Is this consistent with the way the words *if* and *then* are used in ordinary language? It may not seem to be at first, but, at least for some uses of the words *if* and *then*, it is.

P	Q	$\neg P \vee Q$
F	F	T
F	T	T
T	F	F
T	T	T

Figure 3

For example, imagine a teacher saying to a class, in a threatening tone of voice, "You won't neglect your homework, or you'll fail the course." Grammatically, this statement has the form $\neg P \vee Q$, where P is the statement "You will neglect your homework" and Q is "You'll fail the course." But what message is the teacher trying to convey with this statement? Clearly the intended message is "If you neglect your homework, then you'll fail the course," or in other words $P \to Q$. Thus, in this example, the statements $\neg P \vee Q$ and $P \to Q$ seem to mean the same thing.

There is a similar idea at work in the first statement from Example 1.1.2, "Either John went to the store, or we're out of eggs." In Section 1.1 we represented this statement by the formula $P \vee Q$, with P standing for "John went to the store" and Q for "We're out of eggs." But someone who made this statement would probably be trying to express the idea that if John didn't go to the store, then we're out of eggs, or in other words $\neg P \to Q$. Thus, this example suggests that $\neg P \to Q$ means the same thing as $P \vee Q$. In fact, we can derive this equivalence from the previous one by substituting $\neg P$ for P. Because $P \to Q$

is equivalent to $\neg P \lor Q$, it follows that $\neg P \to Q$ is equivalent to $\neg\neg P \lor Q$, which is equivalent to $P \lor Q$ by the double negation law.

We can derive another useful equivalence as follows:

$$\begin{aligned} \neg P \lor Q \qquad & \text{is equivalent to } \neg P \lor \neg\neg Q \quad \text{(double negation law)}, \\ & \text{which is equivalent to } \neg(P \land \neg Q) \quad \text{(DeMorgan's law)}. \end{aligned}$$

Thus, $P \to Q$ is also equivalent to $\neg(P \land \neg Q)$. In fact, this is precisely the conclusion we reached earlier when discussing the statement "If $x > 2$ then $x^2 > 4$." We decided then that the reason this statement is true for every value of x is that there is no value of x for which $x > 2$ and $x^2 \leq 4$. In other words, the statement $P(x) \land \neg Q(x)$ is never true, where as before $P(x)$ stands for $x > 2$ and $Q(x)$ for $x^2 > 4$. But that's the same as saying that the statement $\neg(P(x) \land \neg Q(x))$ is always true. Thus, to say that $P(x) \to Q(x)$ is always true means the same thing as saying that $\neg(P(x) \land \neg Q(x))$ is always true.

For another example of this equivalence, consider the statement "If it's going to rain, then I'll take my umbrella." Of course, this statement has the form $P \to Q$, where P stands for the statement "It's going to rain" and Q stands for "I'll take my umbrella." But we could also think of this statement as a declaration that I won't be caught in the rain without my umbrella – in other words, $\neg(P \land \neg Q)$.

To summarize, so far we have discovered the following equivalences involving conditional statements:

Conditional laws

$$P \to Q \text{ is equivalent to } \neg P \lor Q.$$
$$P \to Q \text{ is equivalent to } \neg(P \land \neg Q).$$

In case you're still not convinced that the truth table in Figure 2 is right, we give one more reason. We know that, using this truth table, we can now analyze the validity of deductive arguments involving the words *if* and *then*. We'll find, when we analyze a few simple arguments, that the truth table in Figure 2 leads to reasonable conclusions about the validity of these arguments. But if we were to make any changes in the truth table, we would end up with conclusions that are clearly incorrect. For example, let's return to the argument form with which we started this section:

$$\begin{array}{l} P \to Q \\ \underline{P \qquad\quad} \\ \therefore Q \end{array}$$

We have already decided that this form of argument should be valid, and the truth table in Figure 4 confirms this. The premises are both true only in line four of the table, and in this line the conclusion is true as well.

		Premises		Conclusion
P	Q	$P \rightarrow Q$	P	Q
F	F	T	F	F
F	T	T	F	T
T	F	F	T	F
T	T	T	T	T

Figure 4

You can also see from Figure 4 that both premises are needed to make this argument valid. But if we were to change the truth table for the conditional statement to make $P \rightarrow Q$ false in the first line of the table, then the second premise of this argument would no longer be needed. We would end up with the conclusion that, just from the single premise $P \rightarrow Q$, we could infer that Q must be true, since in the two lines of the truth table in which the premise $P \rightarrow Q$ would still be true, lines two and four, the conclusion Q is true too. But this doesn't seem right. Just knowing that *if* P is true then Q is true, but not knowing that P *is* true, it doesn't seem reasonable that we should be able to conclude that Q is true. For example, suppose we know that the statement "If John didn't go to the store then we're out of eggs" is true. Unless we also know whether or not John has gone to the store, we can't reach any conclusion about whether or not we're out of eggs. Thus, changing the first line of the truth table for $P \rightarrow Q$ would lead to an incorrect conclusion about the validity of an argument.

Changing the second line of the truth table would also lead to unacceptable conclusions about the validity of arguments. To see this, consider the argument form:

$$\begin{array}{l} P \rightarrow Q \\ \underline{Q\qquad\quad} \\ \therefore P \end{array}$$

This should *not* be considered a valid form of reasoning. For example, consider the following argument, which has this form:

If Jones was convicted of murdering Smith, then he will go to jail.
Jones will go to jail.
Therefore, Jones was convicted of murdering Smith.

Even if the premises of this argument are true, the conclusion that Jones was convicted of murdering Smith doesn't follow. Maybe the reason he will go to jail is that he robbed a bank or cheated on his income tax. Thus, the conclusion of this argument could be false even if the premises were true, so the argument isn't valid.

The truth table analysis in Figure 5 agrees with this conclusion. In line two of the table, the conclusion P is false, but both premises are true, so the argument is invalid. But notice that if we were to change the truth table for $P \to Q$ and make it false in line two, then the truth table analysis would say that the argument is valid. Thus, the analysis of this argument seems to support our decision to put a T in the second line of the truth table for $P \to Q$.

		Premises		Conclusion
P	Q	$P \to Q$	Q	P
F	F	T	F	F
F	T	T	T	F
T	F	F	F	T
T	T	T	T	T

Figure 5

The last example shows that from the premises $P \to Q$ and Q it is incorrect to infer P. But it would certainly be correct to infer P from the premises $Q \to P$ and Q. This shows that the formulas $P \to Q$ and $Q \to P$ do *not* mean the same thing. You can check this by making a truth table for both and verifying that they are not equivalent. For example, a person might believe that, in general, the statement "If you are a convicted murderer then you are untrustworthy" is true, without believing that the statement "If you are untrustworthy then you are a convicted murderer" is generally true. The formula $Q \to P$ is called the *converse* of $P \to Q$. It is very important to make sure you never confuse a conditional statement with its converse.

The *contrapositive* of $P \to Q$ is the formula $\neg Q \to \neg P$, and it *is* equivalent to $P \to Q$. This may not be obvious at first, but you can verify it with a truth table. For example, the statements "If John cashed the check I wrote then my bank account is overdrawn" and "If my bank account isn't overdrawn then John hasn't cashed the check I wrote" are equivalent. Both would be true in exactly the same circumstances – namely, if the check I wrote was for more money than I had in my account. The equivalence of conditional statements and their contrapositives is used often in mathematical reasoning. We add it to our list of important equivalences:

Contrapositive law

$$P \to Q \text{ is equivalent to } \neg Q \to \neg P.$$

Example 1.5.2. Which of the following statements are equivalent?

1. If it's either raining or snowing, then the game has been canceled.
2. If the game hasn't been canceled, then it's not raining and it's not snowing.

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3. If the game has been canceled, then it's either raining or snowing.
4. If it's raining then the game has been canceled, and if it's snowing then the game has been canceled.
5. If it's neither raining nor snowing, then the game hasn't been canceled.

Solution

We translate all of the statements into the notation of logic, using the following abbreviations: R stands for the statement "It's raining," S stands for "It's snowing," and C stands for "The game has been canceled."

1. $(R \vee S) \rightarrow C$.
2. $\neg C \rightarrow (\neg R \wedge \neg S)$. By one of DeMorgan's laws, this is equivalent to $\neg C \rightarrow \neg(R \vee S)$. This is the contrapositive of statement 1, so they are equivalent.
3. $C \rightarrow (R \vee S)$. This is the converse of statement 1, which is *not* equivalent to it. You can verify this with a truth table, or just think about what the statements mean. Statement 1 says that rain or snow would result in cancelation of the game. Statement 3 says that these are the *only* circumstances in which the game will be canceled.
4. $(R \rightarrow C) \wedge (S \rightarrow C)$. This is also equivalent to statement 1, as the following reasoning shows:

$$
\begin{aligned}
&(R \rightarrow C) \wedge (S \rightarrow C) \\
&\qquad\text{is equivalent to } (\neg R \vee C) \wedge (\neg S \vee C) && \text{(conditional law),} \\
&\qquad\text{which is equivalent to } (\neg R \wedge \neg S) \vee C && \text{(distributive law),} \\
&\qquad\text{which is equivalent to } \neg(R \vee S) \vee C && \text{(DeMorgan's law),} \\
&\qquad\text{which is equivalent to } (R \vee S) \rightarrow C && \text{(conditional law).}
\end{aligned}
$$

   You should read statements 1 and 4 again and see if it makes sense to you that they're equivalent.
5. $\neg(R \vee S) \rightarrow \neg C$. This is the contrapositive of statement 3, so they are equivalent. It is not equivalent to statements 1, 2, and 4.

Statements that mean $P \rightarrow Q$ come up very often in mathematics, but sometimes they are not written in the form "If P then Q." Here are a few other ways of expressing the idea $P \rightarrow Q$ that are used often in mathematics:

P implies Q.
Q, if P.
P only if Q.
P is a sufficient condition for Q.
Q is a necessary condition for P.

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Some of these may require further explanation. The second expression, "Q, if P," is just a slight rearrangement of the statement "If P then Q," so it should make sense that it means $P \to Q$. As an example of a statement of the form "P only if Q," consider the sentence "You can run for president only if you are a citizen." In this case, P is "You can run for president" and Q is "You are a citizen." What the statement means is that if you're not a citizen, then you can't run for president, or in other words $\neg Q \to \neg P$. But by the contrapositive law, this is equivalent to $P \to Q$.

Think of "P is a sufficient condition for Q" as meaning "The truth of P suffices to guarantee the truth of Q," and it should make sense that this should be represented by $P \to Q$. Finally, "Q is a necessary condition for P" means that in order for P to be true, it is necessary for Q to be true also. This means that if Q isn't true, then P can't be true either, or in other words, $\neg Q \to \neg P$. Once again, by the contrapositive law we get $P \to Q$.

Example 1.5.3. Analyze the logical forms of the following statements:

1. If at least ten people are there, then the lecture will be given.
2. The lecture will be given only if at least ten people are there.
3. The lecture will be given if at least ten people are there.
4. Having at least ten people there is a sufficient condition for the lecture being given.
5. Having at least ten people there is a necessary condition for the lecture being given.

Solutions

Let T stand for the statement "At least ten people are there" and L for "The lecture will be given."

1. $T \to L$.
2. $L \to T$. The given statement means that if there are not at least ten people there, then the lecture will not be given, or in other words $\neg T \to \neg L$. By the contrapositive law, this is equivalent to $L \to T$.
3. $T \to L$. This is just a rephrasing of statement 1.
4. $T \to L$. The statement says that having at least ten people there suffices to guarantee that the lecture will be given, and this means that if there are at least ten people there, then the lecture will be given.
5. $L \to T$. This statement means the same thing as statement 2: If there are not at least ten people there, then the lecture will not be given.

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We have already seen that a conditional statement $P \to Q$ and its converse $Q \to P$ are not equivalent. Often in mathematics we want to say that both $P \to Q$ and $Q \to P$ are true, and it is therefore convenient to introduce a new connective symbol, $\leftrightarrow$, to express this. You can think of $P \leftrightarrow Q$ as just an abbreviation for the formula $(P \to Q) \land (Q \to P)$. A statement of the form $P \leftrightarrow Q$ is called a *biconditional* statement, because it represents two conditional statements. By making a truth table for $(P \to Q) \land (Q \to P)$ you can verify that the truth table for $P \leftrightarrow Q$ is as shown in Figure 6. Note that, by the contrapositive law, $P \leftrightarrow Q$ is also equivalent to $(P \to Q) \land (\neg P \to \neg Q)$.

P	Q	$P \leftrightarrow Q$
F	F	T
F	T	F
T	F	F
T	T	T

Figure 6

Because $Q \to P$ can be written "P if Q" and $P \to Q$ can be written "P only if Q," $P \leftrightarrow Q$ means "P if Q and P only if Q," and this is often written "P if and only if Q." The phrase *if and only if* occurs so often in mathematics that there is a common abbreviation for it, *iff*. Thus, $P \leftrightarrow Q$ is usually written "P iff Q." Another statement that means $P \leftrightarrow Q$ is "P is a necessary and sufficient condition for Q."

Example 1.5.4. Analyze the logical forms of the following statements:

1. The game will be canceled iff it's either raining or snowing.
2. Having at least ten people there is a necessary and sufficient condition for the lecture being given.
3. If John went to the store then we have some eggs, and if he didn't then we don't.

Solutions

1. Let C stand for "The game will be canceled," R for "It's raining," and S for "It's snowing." Then the statement would be represented by the formula $C \leftrightarrow (R \lor S)$.
2. Let T stand for "There are at least ten people there" and L for "The lecture will be given." Then the statement means $T \leftrightarrow L$.
3. Let S stand for "John went to the store" and E for "We have some eggs." Then a literal translation of the given statement would be $(S \to E) \land (\neg S \to \neg E)$. This is equivalent to $S \leftrightarrow E$.

One of the reasons it's so easy to confuse a conditional statement with its converse is that in everyday speech we sometimes use a conditional statement when what we mean to convey is actually a biconditional. For example, you probably wouldn't say "The lecture will be given if at least ten people are there" unless it was also the case that if there were fewer than ten people, the lecture wouldn't be given. After all, why mention the number ten at all if it's not the minimum number of people required? Thus, the statement actually suggests that the lecture will be given *iff* there are at least ten people there. For another example, suppose a child is told by his parents, "If you don't eat your dinner, you won't get any dessert." The child certainly expects that if he *does* eat his dinner, he *will* get dessert, although that's not literally what his parents said. In other words, the child interprets the statement as meaning "Eating your dinner is a necessary *and sufficient* condition for getting dessert."

Such a blurring of the distinction between *if* and *iff* is never acceptable in mathematics. Mathematicians always use a phrase such as *iff* or *necessary and sufficient condition* when they want to express a biconditional statement. You should never interpret an if-then statement in mathematics as a biconditional statement, the way you might in everyday speech.

Exercises

*1. Analyze the logical forms of the following statements:
   (a) If this gas either has an unpleasant smell or is not explosive, then it isn't hydrogen.
   (b) Having both a fever and a headache is a sufficient condition for George to go to the doctor.
   (c) Both having a fever and having a headache are sufficient conditions for George to go to the doctor.
   (d) If $x \neq 2$, then a necessary condition for x to be prime is that x be odd.
2. Analyze the logical forms of the following statements:
   (a) Mary will sell her house only if she can get a good price and find a nice apartment.
   (b) Having both a good credit history and an adequate down payment is a necessary condition for getting a mortgage.
   (c) John will kill himself, unless someone stops him. (Hint: First try to rephrase this using the words *if* and *then* instead of *unless*.)
   (d) If x is divisible by either 4 or 6, then it isn't prime.
3. Analyze the logical form of the following statement:
   (a) If it is raining, then it is windy and the sun is not shining.

Now analyze the following statements. Also, for each statement determine whether the statement is equivalent to either statement (a) or its converse.

(b) It is windy and not sunny only if it is raining.
(c) Rain is a sufficient condition for wind with no sunshine.
(d) Rain is a necessary condition for wind with no sunshine.
(e) It's not raining, if either the sun is shining or it's not windy.
(f) Wind is a necessary condition for it to be rainy, and so is a lack of sunshine.
(g) Either it is windy only if it is raining, or it is not sunny only if it is raining.

*4. Use truth tables to determine whether or not the following arguments are valid:

(a) Either sales or expenses will go up. If sales go up, then the boss will be happy. If expenses go up, then the boss will be unhappy. Therefore, sales and expenses will not both go up.
(b) If the tax rate and the unemployment rate both go up, then there will be a recession. If the GNP goes up, then there will not be a recession. The GNP and taxes are both going up. Therefore, the unemployment rate is not going up.
(c) The warning light will come on if and only if the pressure is too high and the relief valve is clogged. The relief valve is not clogged. Therefore, the warning light will come on if and only if the pressure is too high.

5. (a) Show that $P \leftrightarrow Q$ is equivalent to $(P \wedge Q) \vee (\neg P \wedge \neg Q)$.
(b) Show that $(P \rightarrow Q) \vee (P \rightarrow R)$ is equivalent to $P \rightarrow (Q \vee R)$.

*6. (a) Show that $(P \rightarrow R) \wedge (Q \rightarrow R)$ is equivalent to $(P \vee Q) \rightarrow R$.
(b) Formulate and verify a similar equivalence involving $(P \rightarrow R) \vee (Q \rightarrow R)$.

7. (a) Show that $(P \rightarrow Q) \wedge (Q \rightarrow R)$ is equivalent to $(P \rightarrow R) \wedge [(P \leftrightarrow Q) \vee (R \leftrightarrow Q)]$.
(b) Show that $(P \rightarrow Q) \vee (Q \rightarrow R)$ is a tautology.

*8. Find a formula involving only the connectives $\neg$ and $\rightarrow$ that is equivalent to $P \wedge Q$.

9. Find a formula involving only the connectives $\neg$ and $\rightarrow$ that is equivalent to $P \leftrightarrow Q$.

10. Which of the following formulas are equivalent?

(a) $P \rightarrow (Q \rightarrow R)$.
(b) $Q \rightarrow (P \rightarrow R)$.
(c) $(P \rightarrow Q) \wedge (P \rightarrow R)$.
(d) $(P \wedge Q) \rightarrow R$.
(e) $P \rightarrow (Q \wedge R)$.

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