On the Quasi-periodic Solutions of Mathieu's Differential Equation.

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Mathieu's differential equation

$$\frac{d^2y}{dz^2} + (a + k^2\cos^2 z)y = 0 \qquad (1)$$

is the equation which arises out of those two-dimensional problems in Mathematical Physics in which the boundary is an ellipse, such problems, for example, as the vibrations of an elliptic membrane, which was first discussed by Mathieu,* and the scattering of electromagnetic waves by a wire of elliptic cross-section. A different use of the same equation is found in Celestial Mechanics in the treatment of perturbations and oscillations about periodic orbits,† and, in a more mundane connection, it has been shown to be the differential equation of the variety artiste who holds an assistant poised on a pole above his head while he himself is standing on a spherical ball rolling on the ground!

The coefficient $a + k^2\cos^2 z$ is a periodic function of z, and hence the equation (1) belongs to the class of differential equations with periodic coefficients, a class which has been discussed by Floquet,‡ who gave what may be termed the Fuchsian theory, and, recently, by Hamel,§ who has investigated certain properties affecting the "stability" of the solutions.

* Mathieu: Liouville's Journal (2), XIII., p. 137–203, 1868.

† Bruns: Ast. Nach. No. 2533, S. 193–204, 1883, and No. 2553, S. 129–132, 1884.

‡ Floquet: Annales de l'Ecole Normale Supérieure (2), T. 12, p. 47–88, 1883.

§ Hamel: Mathematische Annalen, B. 73, S. 371–412, 1913.

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From Floquet's theory we know that the general solution of equation (1) is of the form

$$y = \mathrm{A}e^{cz}p_1(z) + \mathrm{B}e^{-cz}p_2(z) \quad \text{.........................(2)}$$

where $p_1(z)$ and $p_2(z)$ are periodic functions having the same period as the coefficient of (1),

c is a constant depending on the constants of the original differential equation,

and A and B are the arbitrary constants of the solution.

Such a solution as (2) is called *quasi-periodic*, inasmuch as after the lapse of a period it does not repeat itself exactly but contains multiplying factors, thus

$$y(z+2\pi) = e^{2\pi c}\mathrm{A}e^{cz}p_1(z) + e^{-2\pi c}\mathrm{B}e^{-cz}p_2(z).$$

When the constants a and k of the original differential equation are such that $c=0$, the above existence theorem fails to give the general solution. In this case one solution is purely-periodic, and the other solution is related to it in a way analogous to that in which, for example, the Bessel's functions of the first and second kinds are connected. These periodic solutions, which in Professor Whittaker's * notation, are written

$$ce_0(z),\ ce_1(z),\ ce_2(z),\ \ldots$$
$$se_1(z),\ se_2(z),\ \ldots$$

are infinite in number, and as Hilbert † has shown that they may be regarded as the *eigenfunktionen* (or autofunctions) of a certain integral equation, of which the corresponding *eigenwerte* (or auto-values) are the values of a concerned, we may agree to call these solutions *eigenfunktionen*, and these values of a, *eigenwerte of the differential equation.*

For values of a other than these *eigenwerte* the solution is quasi-periodic, and it is the main object of this paper to show how these quasi-periodic solutions change their nature as a is varied (k being regarded as a constant parameter), and how the quasi-periodicity

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* Whittaker: Cambridge International Congress, Vol. I., 1912.

† Hilbert: Göttingen Nachrichten, 1904, S. 213–234. The Hilbert integral equation is essentially different from that given by Whittaker (*loc. cit.*). Whittaker's has a continuous *kern*, whereas Hilbert's *kern* has a discontinuity.

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merges into pure periodicity in the neighbourhood of the *eigenwerte.* With this in view we cannot make much progress by means of the method, used by Lindemann* and Maclaurin,† of transforming the equation into an algebraic equation by the substitution of $x = \cos z$ and then expanding in Taylor's series round certain points in the x-plane. Accordingly we shall confine ourselves to solutions of the form (2).

§1. *Convenient Form of the differential equation.*

If we write $k^2 = 32q$ in (1), with a view to saving repeated powers of 2 in the expansions, we obtain for our differential equation

$$\frac{d^2y}{dz^2} + (a + 32q\cos^2 z)y = 0$$

or

$$\frac{d^2y}{dz^2} + (a + 16q + 16q\cos 2z)y = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

§2. *Expression of the Quasi-periodic Solution in a Form which reduces to the ce_0-function.*

We know that the elliptic-cylinder function (or periodic solution) of zero order, $ce_0(z)$, is given by

$$y = ce_0(z) = 1 + 4q\cos 2z + 2q^2\cos 4z + q^3\left(\frac{4}{9}\cos 6z - 28\cos 2z\right) + \ldots$$

when $a + 16q = -32q^2 + 224q^4 - \dfrac{29696}{9}q^6 + \ldots$

Let us try to get a solution

$y = \mathrm{A}e^{cz}(\tfrac{1}{2}$ + a periodic function) + $\mathrm{B}e^{-cz}$ ($\tfrac{1}{2}$ + a periodic function)

which will reduce to $ce_0(z)$ when $c = 0$ and $\mathrm{A} = \mathrm{B} = 1$.

The process used is as follows:

Assume $\quad a = a_0 + a_1q + a_2q^2 + a_3q^3 + \ldots$

and

$$y = \mathrm{A}e^{cz}(\tfrac{1}{2} + b_1(z)q + b_2(z)q^2 + b_3(z)q^3 + \ldots) + \mathrm{B}e^{-cz}(\tfrac{1}{2} + d_1(z)q + d_2(z)q^2 + d_3(z)q^3 + \ldots).$$

Substituting in the equation (3), and equating to zero the terms not containing q as a factor, we have

$$\tfrac{1}{2}c^2 + \tfrac{1}{2}a_0 = 0,$$

and

$$\therefore \quad a_0 = -c^2.$$

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* Lindemann: Mathematische Annalen, B. 22, S. 117–123, 1883.

† Maclaurin: Trans. Camb. Phil. Soc., Vol. XVII., p. 41–108, 1899

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Equating to zero the terms with the first power of q as a factor, we obtain

$$b_1''(z) + 2b_1'(z)c + b_1(z)c^2 - b_1(z)c^2 + \tfrac{1}{2}a_1 + 8 + 8\cos 2z = 0,$$
$$d_1''(z) - 2d_1'(z)c + d_1(z)c^2 - d_1(z)c^2 + \tfrac{1}{2}a_1 + 8 + 8\cos 2z = 0.$$

In order that $b_1(z)$ and $d_1(z)$, the integrals of these equations may have only periodic terms, we must make

$$\tfrac{1}{2}a_1 + 8 = 0$$

and $$\therefore \quad a_1 = -16,$$

and we then have

$$b_1(z) = \frac{2\cos 2z - 2c\sin 2z}{1+c^2}.$$
$$d_1(z) = \frac{2\cos 2z + 2c\sin 2z}{1+c^2}.$$

In this way we proceed to determine in succession $a_2, a_3, \ldots$, $b_2(z), b_3(z), \ldots, d_2(z), d_3(z), \ldots$, and thus obtain

$$y = \mathrm{A}e^{cz}\left[\tfrac{1}{2} + \frac{2\cos 2z - 2c\sin 2z}{1+c^2}\cdot q + \frac{(4-2c^2)\cos 4z - 6c\sin 4z}{(1+c^2)(4+c^2)}\cdot q^2\right.$$
$$+ \left\{\cos 2z\left(-\frac{16(1-c^2)}{(1+c^2)^3} + \frac{8(1-2c^2)}{(1+c^2)^2(4+c^2)}\right) + \sin 2z\left(\frac{32c}{(1+c^2)^3} - \frac{4c(5-c^2)}{(1+c^2)^2(4+c^2)}\right.\right.$$
$$\left.\left.+ \cos 6z\frac{8(1-c^2)}{(1+c^2)(4+c^2)(9+c^2)} - \sin 6z\frac{4c(11-c^2)}{3(1+c^2)(4+c^2)(9+c^2)}\right\}q^3 + \ldots\right.$$
$$+ \mathrm{B}e^{-cz}\,[\text{the same function of } -c]$$

when $$a + 16q = -c^2 - \frac{32}{1+c^2}\cdot q + 128\left[\frac{2(1-c^2)}{(1+c^2)^3} - \frac{(1-2c^2)}{(1+c^2)^2(4+c^2)}\right]q^3 + \ldots.$$

It will be seen that these expansions contain as denominators $(1+c^2)$, $(4+c^2)$, etc., so that they are not convergent near $c = i, 2i, \ldots$. Now these are the values of c which, when substituted in

$$y = \mathrm{A}e^{cz}\,(\text{periodic function}) + \mathrm{B}e^{-cz}\,(\text{periodic function})$$

would give periodic solutions, and hence this expansion for the quasi-periodic solution cannot be used practically for those values of a which are in the neighbourhood of the *eigenwerte* corresponding to the ce_1, $ce_2 \ldots$, se_1, se_2, ... functions. We shall afterwards show that in these neighbourhoods the nature of the quasi-periodicity undergoes changes, and it is interesting to note how these changes take place under cover of the divergence of the expression, a circum-

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stance in many ways analogous to the well-known property, which the constants of an asymptotic expansion have, of changing their character in the "haziness" that arises in certain regions of the argument.

§ 3. *Expression of the Quasi-periodic Solution in Forms which reduce to the Elliptic-cylinder Functions other than the ce_0-function.*

The method of the last section was used to obtain an expression which would reduce to $ce_0(z)$ for $c=0$. It is natural to try to get an expansion on similar lines which would reduce to $ce_1(z)$ or to any of the other periodic solutions, and Professor Whittaker has pointed out to me an elegant method by which we deduce an expansion reducing to ce_1 and se_1, and which can be easily extended to provide expansions reducing to any required periodic solution.*

The ce_1 solution is

$$ce_1(z) = \cos z + q\cos 3z + q^2(-\cos 3z + \tfrac{1}{3}\cos 5z) + \ldots$$

when $$a + 16q = 1 - 8q - 8q^2 + 8q^3 - \ldots$$

and the se_1 solution is

$$se_1(z) = \sin z + q\sin 3z + q^2(\sin 3z + \tfrac{1}{3}\sin 5z) + \ldots$$

when $$a + 16q = 1 + 8q - 8q^2 - 8q^3 - \ldots;$$

and Professor Whittaker's solution shows that these are simply particular cases (corresponding to $\sigma = -\dfrac{\pi}{2}$ and to $\sigma = 0$) of the theorem that a quasi-periodic solution is

$$y = e^{4qz\sin 2\sigma}[\sin(z-\sigma) + q\sin(3z-\sigma) + q^2(3\sin 2\sigma\cos(3z-\sigma) + \cos 2\sigma\sin(3z-\sigma) + \tfrac{1}{3}\sin(5z-\sigma)) + \ldots]$$

when $$a + 16q = 1 + 8\cos 2\sigma \,.\, q - 8(1 + 2\sin^2 2\sigma)q^2 + \ldots$$

This expansion is obtained by assuming the solution

$$y = e^{Nqz\sin 2\sigma}[\sin(z-\sigma) + qb_1(z) + q^2b_2(\sigma) + \ldots]$$

along with $$a + 16q = 1 + qa_1(\sigma) + q^2a_2(\sigma) + \ldots,$$

and finding $a_1, a_2, \ldots$ in succession such that $b_1, b_2, b_3, \ldots$ are periodic functions. It is found in this case that $N = 4$.

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* An account of this method is given in a note by Professor Whittaker in the present volume.

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It is clear that the general solution of the equation may be written

$$\begin{aligned} y = {} & \mathrm{A}e^{4qz\sin 2\sigma}[\sin(z-\sigma) + q\sin(3z-\sigma) + \ldots] \\ & + \mathrm{B}e^{-4qz\sin 2\sigma}[\sin(z+\sigma) + q\sin(3z+\sigma) + \ldots] \end{aligned}$$

when $\quad a + 16q = 1 + 8\cos 2\sigma \,.\, q - 8(1 + 2\sin^2 2\sigma) \,.\, q^2 + \ldots.$

When we aim at getting a solution which will reduce to, say, ce_n and se_n, we assume as solution

$$y = e^{\mathrm{N}q^n z\sin 2\sigma}[\sin(nz-\sigma) + qb_1(z) + q^2 b_2(z) + \ldots],$$

along with $\quad a + 16q = n^2 + qa_1(\sigma) + q^2 a_2(\sigma) + \ldots,$

the only change being the substitution of q^n for q in the exponential factor and in taking $\sin(nz - \sigma)$ as the solution when $q = 0$.

The case $n = 2$ gives

$$\begin{aligned} y = \mathrm{A}e^{-4q^2 z\sin 2\sigma}[& \sin(2z-\sigma) + (2\sin\sigma + \tfrac{2}{3}\sin(4z-\sigma))q + \tfrac{1}{6}\sin(6z-\sigma) \,.\, q^2 \\ & + \{\tfrac{8}{3}\sin\sigma - 16\sin^3\sigma - \tfrac{16}{9}\sin 2\sigma\cos(4z-\sigma) \\ & \quad + (-\tfrac{5}{3} + \tfrac{16}{9}\sin^2\sigma)\sin(4z-\sigma) + \tfrac{1}{45}\sin(8z-\sigma)\}q^3 + \ldots] \\ + \mathrm{B}e^{+4q^2 z\sin 2\sigma} & [\text{the same function of } -\sigma], \end{aligned}$$

when $a + 16q = 4 - (\tfrac{16}{3} - 32\sin^2\sigma)q^2 + \ldots.$

This solution reduces to $ce_2(z)$ for $\sigma = -\dfrac{\pi}{2}$,

and to $\qquad se_2(z)$ for $\sigma = 0$.

The case $n = 3$ gives similarly

$$\begin{aligned} y = \mathrm{A}e^{\frac{4}{3}q^3 z\sin 2\sigma}[& \sin(3z-\sigma) + (-\sin(z-\sigma) + \tfrac{1}{2}\sin(5z-\sigma))q \\ & + (-\sin(z+\sigma) + \tfrac{1}{10}\sin(7z-\sigma))q^2 \\ & + (-\tfrac{1}{2}\sin(z-\sigma) + \tfrac{7}{40}\sin(5z-\sigma) + \tfrac{1}{90}\sin(9z-\sigma))q^3 + \ldots] \\ + \mathrm{B}e^{-\frac{4}{3}q^3 z\sin 2\sigma} & [\text{the same function of } -\sigma] \end{aligned}$$

when $\qquad a + 16q = 9 + 4q^2 + 8\cos 2\sigma \,.\, q + \ldots.$

This solution reduces to $ce_3(z)$ for $\sigma = -\dfrac{\pi}{2}$

and to $\qquad se_3(z)$ for $\sigma = 0$.

With these solutions we can explore beyond $\sigma = 0$ and $\sigma = \dfrac{\pi}{2}$ by using imaginary values of σ, but from later considerations it will be evident that any one of these solutions cannot provide an expansion for all values of the parameter a, and use will accordingly be made of these solutions only in the vicinity of the *eigenwerte* of the periodic solutions to which they naturally reduce.

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§ 4. *Discussion of the variation of c.*

We shall return now to a consideration of the Floquet form of solution $y = Ae^{cz}p_1(z) + Be^{-cz}p_2(z)$, and proceed to discuss the variation of the parameter a.

When q is small—as usually happens in astronomical applications—it is evident that a first approximation to the solution of (1) is

$$y = Ae^{\sqrt{-a}\,.\,z} + Be^{-\sqrt{-a}\,.\,z},$$

so that a first approximation to the value of c is $i\sqrt{a}$.

Also if $y = Ae^{cz}p_1(z) + Be^{-cz}p_2(z)$ is a solution, we have solutions of the form

$$y = Ae^{(c \pm ni)z}p_1'(z) + Be^{-(c \pm ni)z}p_2'(z),$$

where $n = 1, 2, 3, \ldots$.

Thus c is not a single-valued quantity, since $c \pm ni$ would act equally well, but it will be convenient to select that value of c which would reduce to $i\sqrt{a}$ if q were taken to be zero. With this convention $c = 0, 1, 2, \ldots, n, \ldots$ will correspond to the periodic solutions

$$ce_0 \; \left.\begin{matrix} ce_1 \\ se_1 \end{matrix}\right\} \left.\begin{matrix} ce_2 \\ se_2 \end{matrix}\right\} \ldots \left.\begin{matrix} ce_n \\ se_n \end{matrix}\right\} \ldots.$$

In § 2 it is shown that, when the parameter a is not in the neighbourhood of any of the *eigenwerte* other than that of the ce_0-function, the relation between c and a is

$$a + 16q = -c^2 - \frac{32}{1+c^2} \cdot q^2 + \left(\frac{256(1-c^2)}{(1+c^2)^3} - \frac{128(1-2c^2)}{(1+c^2)^2(4+c^2)}\right)q^4 + \ldots\ldots$$

When the denominators are expanded, this becomes

$$\begin{aligned} a + 16q = &-c^2 - 32(1 - c^2 + c^4 - c^6 + \ldots)q^2 \\ &+ (224 - 888c^2 + 2046c^4 - 3712{\cdot}5c^6 + \ldots)q^4 \\ &- \left(\frac{29696}{9} + \ldots\right)q^6 + \ldots \end{aligned}$$

or

$$\begin{aligned} a + 16q = &\left(-32q^2 + 224q^4 - \frac{29696}{9}q^6 + \ldots\right) \\ &- c^2(1 - 32q^2 + 888q^4 - \ldots) \\ &+ c^4(-32q^2 + 2046q^4 - \ldots) \\ &- c^6(-32q^2 + 3712{\cdot}5q^4 - \ldots). \end{aligned}$$

For any particular value of q (sufficiently small to make the series in the brackets convergent), this formula gives a for any

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value of c, and by a simple reversion of this equation we can derive a formula for c in terms of a.

Since general formulae have not been obtained, we must have resort to numerical calculation for further knowledge, and the result is, that when we substitute *real* values of c in the formula, values of a are obtained which are less than the *eigenwerte* of the ce_0-function, and, when we substitute *imaginary* values of c, the corresponding values of a are all found to be greater than the *eigenwerte* of ce_0.

As we noted above, this formula cannot be used near the *eigenwerte* of the other periodic solutions, and in their vicinity we must have recourse to the results of § 3.

§ 5. *Note on the Application of G. W. Hill's Method.*

In his celebrated memoir "On the Mean Motion of the Lunar Perigee," Hill * reduced the problem to the discussion of the equation

$$\frac{d^2y}{dz^2}+(\Theta_0+2\Theta_1\cos 2z+2\Theta_2\cos 4z+\ldots)z=0$$

and when $\Theta_2\Theta_3\ldots$ are taken to be zero, the equation becomes the Mathieu equation. We may thus make use of Hill's results, but the formula which he arrived at for c in terms of the Θ's suffers from the same disability as the formula of the last section, in that it cannot be used in the regions near any of the *eigenwerte*, and therefore does not give any extra information.

§ 6. *The value of c near the eigenwerte.*

The results of § 3 are shown in the table annexed, in which are arranged the values of $a+16q$ and of c corresponding to the region of ce_1 and se_1, of ce_2 and se_2, …

In the neighbourhood of the *eigenwerte* of	c	$a+16q$
ce_1 and se_1	$4q\sin 2\sigma$	$1+8\cos 2\sigma\,.\,q-8(1+2\sin^2 2\sigma)q^2+\ldots$
ce_2 and se_2	$-4q^2\sin 2\sigma$	$4-(\frac{16}{3}-32\sin^2\sigma)q^2+\ldots$
ce_3 and se_3	$\frac{4}{3}q^3\sin 2\sigma$	$9+4q^2+8\cos 2\sigma\,.\,q^3+\ldots$
....	....	....

* Hill: Acta Mathematica, Vol. VIII., pp. 1-36, 1886.

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It will be seen that $a+16q$ is *real* for real and imaginary values of σ, but that c is real or imaginary according as σ is real or imaginary.

Now for the real values σ, $a+16q$ lies between the *eigenwerte* corresponding to $\sigma=0$ and $\sigma=-\frac{\pi}{2}$ and hence we see that *c is real for values of a intermediate between eigenwerte of any pair of the periodic functions which are of the same order, e.g. ce_1 and se_1, ce_2 and se_2, etc.*

§ 7. *Variation of c over whole region of the parameter a.*

Combining the results of §§ 4 and 6 we have the following theorem :—

The values of a corresponding to the periodic solutions (or elliptic cylinder functions) serve to mark out regions in the whole range of values of a, in which the corresponding value of c is either always real or always imaginary. In particular

from $a=-\infty$ to the *eigenwerte* of ce_0, c is real;

from the *eigenwerte* of ce_0 to the *eigenwerte* of ce_1, c is imaginary;

" " " " ce_1 " " " se_1, c is real;

" " " " se_1 " " " se_2, c is imaginary;

" " " " se_2 " " " ce_2, c is real;

and so on.

It will be noticed from the last column in the table of § 6 that, since the earliest term of the series containing a function of σ is that in q^n, the interval between the *eigenwerte* belonging to the ce_n-function, and that belonging to the se_n-function depends on q^2, and hence, if q be small, decreases rapidly with increasing values of the order n of the periodic solutions. From the second column c is likewise seen to depend on q^n.

§ 8. *Stability and Instability of the Solutions.*

In most of the physical applications of Mathieu's differential equation it is the periodic solution that is wanted, but in the astronomical questions it is the quasi-periodic, and, more especially, it is the factor of quasi-periodicity, c, that is of importance.

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The equation

$$\frac{d^2y}{dz^2}+(a+16q+16q\cos 2z)y=0$$

may be regarded as specifying the effect of perturbative forces on a system which is moving in a periodic orbit, y being the variable expressing the variation from the periodic trajectory and z being the measure of the time. To each pair of values of a and q there will be an orbit from which the differential equation expresses the variation, and from the nature of the solution

$$y=\mathrm{A}e^{cz}p_1(z)+\mathrm{B}e^{-cz}p_2(z)$$

we see that the periodic orbit corresponding to any pair of values of a and q will be stable or unstable according as c is imaginary or real. Confining our attention to one fixed value of q, we have an infinite family of periodic orbits corresponding to all possible values of a, and the results of last paragraph show how the instability or stability can be determined and how the property changes at each periodic solution of the equation of perturbation. The constant c may be regarded as giving the period of the small oscillation from the periodic orbit which is caused by the disturbing force; this period of small oscillation will gradually change as we proceed from one periodic orbit to another until we come to one for which the period of small oscillation is an aliquot part of the period of the orbit. Beyond this particular orbit there is instability until we arrive at another coincidence of the periods of the small oscillations and the ordinary orbit.

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