AN N-PARAMETER CHEBYSHEV SET WHICH IS NOT A SUN

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Recently, Dunham has given examples for 1-parameter and 2-parameter Chebyshev sets which are not suns. In this note n-parameter sets with these properties are described.

1. Introduction. When studying the old problem whether Chebyshev sets are always convex, Klee [10] introduced certain sets which were called suns by Efimov and Stechkin [7]. Recently, in two shorts notes Dunham [4, 5] has given examples of 1-parameter- and 2-parameter-sets which are Chebyshev sets but not suns (cf. also [3]). The examples refer to Chebyshev sets in $C[0, 1]$ containing an isolated point.

Combining Dunham's idea with some more advanced techniques, in this note we will construct Chebyshev sets in $C[0, 1]$ which are the union of an n-dimensional manifold with boundary and an isolated point. Since every sun is a connected set [4], the constructed set is not a sun.

2. The underlying set. The construction is started by introducing the following convex cone in $C[0, 1]$: \[
K = \left\{ h : h(x) = \sum_{j=1}^{n} a_j x^j, \quad a_j \geq 0, \quad j = 1, 2, \ldots, n \right\}.
\]

Observe that $K \setminus \{0\}$ belongs to the set of positive functions:
\[
C^+ = \{ h \in C[0, 1] : h(x) > 0, \quad x \in [0, 1] \}.
\]

Moreover, the cone $K$ has the Haar property [1].

Definition. Let $u_1, u_2, \ldots, u_n \in C[0, 1]$ and $0 \leq m \leq n$. The convex cone \[
\left\{ h(x) = \sum_{j=1}^{n} a_j u_j(x) ; \quad a_j \in \mathbb{R}, \quad j = 1, 2, \ldots, m; \quad a_j \geq 0, \quad j = m+1, \ldots, n \right\}
\]
has the Haar property, if the functions $\{u_j\}_j$ span a Haar subspace whenever \[
\{1, 2, \ldots, m\} \subset J \subset \{1, 2, \ldots, n\}.
\]

More generally, we get cones with the Haar property contained in $C^+ \cup \{0\}$. 

Received by the editors July 29, 1974 and, in revised form, December 16, 1974.

Key words and phrases: Convexity of Chebyshev sets.

MOS classification: 41 A 65, 41 A 50.
when in (2.1) the terms \((x+j)^{-1}\) are replaced by \(\gamma(j, x)\) with \(\gamma\) being an arbitrary totally positive kernel [9].

The function

\[
\varphi(x, y) = e^{y-(x/y)}, \quad 0 \leq x \leq 1, \ y > 0,
\]

is strictly increasing in \(y\), if \(x\) is considered fixed. Hence, \(\varphi\) induces a continuous mapping:

\[
\varphi : C^+ \to C^+, \\
(\varphi h)(x) = \varphi(x, h(x)).
\]

We will consider the approximation in the transformed family

\[
G = \varphi(K \setminus \{0\}) \cup \{0\}.
\]

Since \(g(0) > 1\) for each \(g \in G\), \(g \neq 0\), zero is an isolated point in \(G\).

3. Existence. Let \(\mathcal{C}[0, 1]\) be endowed with the uniform norm:

\[
\|f\| = \sup\{|f(x)| : x \in [0, 1]\}.
\]

An element \(g^*\) in a non-void subset \(G \subset \mathcal{C}[0, 1]\) is called a best approximation to \(f\) in \(G\), if \(\|f-g\| \geq \|f-g^*\|\) for all \(g \in G\).

To prove that there is a best approximation in \(G\) to each \(f \in \mathcal{C}[0, 1]\) consider a minimizing sequence \(\{g_v\}\) satisfying

\[
\lim_{v \to \infty} \|f-g_v\| = \eta = \inf\{\|f-g\| : g \in G\}.
\]

Without loss of generality we may assume \(g_v \neq 0\). Let \(g_v = \varphi(h_v)\). By standard arguments \(\{g_v\}\) is bounded. This implies boundedness of \(g_v(0)\) and \(h_v(0)\). From the representation (2.1) of the elements in \(K\) it follows that \(\|h_v\|\) is also bounded. Select a subsequence of \(\{h_v\}\) which converges to some \(h^* \in K\). If \(h^* \neq 0\), then the corresponding subsequence of \(\{g_v\}\) converges uniformly to \(g^* = \varphi(h^*)\), which is a best approximation. If on the other hand \(h^* = 0\), then the subsequence converges to \(g^* = 0\) uniformly on each compact subinterval of \((0, 1)\). This implies optimality of \(g^*\) by simple arguments (cf. [5]).

4. Varisolvency of transformed Haar subspaces. Assume that \(u_1, u_2, \ldots, u_d \in \mathcal{C}[0, 1]\) span a \(d\)-dimensional subspace. With these functions a mapping

\[
F : \mathbb{R}^d \to \mathcal{C}[0, 1], \\
F(a_1, a_2, \ldots, a_d) = \sum_i a_i u_i(x)
\]

is defined. Let \(A\) be an open subset of \(\mathbb{R}^d\) such that \(H = F(A)\) is contained in \(C^+\), the set of positive functions. Then \(V = \varphi(H)\) is a well defined family which will be investigated now.

Let \(h_1, h_2 \in H\), \(h_1 \neq h_2\). By the Haar condition \(h_1 - h_2\) has at most \(d-1\) zeros in \([0, 1]\). It follows from the monotonicity of \(\varphi(x, h)\) that \(\varphi(h_1) - \varphi(h_2)\) has as many
zeros as $h_1 - h_2$. Consequently, for each pair $g_1, g_2 \in V$ the difference $g_1 - g_2$ has at most $d - 1$ zeros.

Let $x_1 < x_2 < \cdots < x_d$ be $d$ distinct points in $[0, 1]$. We introduce the restriction mapping

$$R : \mathbb{C}[0, 1] \to \mathbb{R}^d$$

$$R \cdot f = (f(x_1), f(x_2), \ldots, f(x_d)).$$

The preceding discussion shows that $R : V \to \mathbb{R}^d$ is a one-one mapping. Consequently the product map $R \circ \psi : A \to R(V) \subset \mathbb{R}^d$ is a homeomorphism. By virtue of Brouwer’s theorem on the invariance of the domain \cite{8}, $R(V)$ is open in $\mathbb{R}^d$. This means that the set of vectors $(y_1, y_2, \ldots, y_d)$, for which the interpolation problem

$$g(x_i) = y_i, \quad i = 1, 2, \ldots, d$$

has a solution $g \in V$, is open in $d$-space. Moreover, the solution is determined by the continuous mapping $R^{-1} = \psi \circ A \circ (R \circ \psi \circ A)^{-1}$. Hence, $V$ is varisolvent \cite[p. 3]{12} with constant degree $d$.

Rice’s theory of varisolvent families establishes that there is at most one best approximation in $V$. The gap in his theory discovered by Dunham \cite{6}, does not matter in this case, because the degree is a constant \cite{2}.

Finally, we notice that $V$ is asymptotically convex \cite[p. 163]{11} and is an Haar embedded manifold \cite{13}. The construction of sets with these properties from Haar subspaces in \cite{11} and \cite{13} is very similar.

5. **Uniqueness.** Now we are ready to prove uniqueness of the best approximation in the set $G$ introduced in Section 2. Formally the proof is similar to the proof of uniqueness for cones with the Haar property \cite{1}.

Assume that $g_1 = \psi(h_1) \neq 0$, $i = 1, 2$, are two best approximations to $f$ in $G$. Put $h^* = (h_1 + h_2)/2$ and observe that $g^* = \psi(h^*)$ is another best approximation, because the monotonicity of $\psi$ implies that $h^*(x)$ lies between $h_1(x)$ and $h_2(x)$ for each $x \in [0, 1]$. Write $h^*(x) = \sum_{j=1}^{n} a_j^* \cdot (x+j)^{-1}$ and set $J = \{j : 1 \leq j \leq n, a_j^* > 0\}$

The manifold

$$H = \left\{ h = \sum_{j \in J} a_j(x+j)^{-1} : a_j \in \mathbb{R} \right\} \cap C^+$$

is a subset of a Haar subspace and satisfies the conditions specified in the last section. Hence, there is at most one best approximation in the varisolvent family $\psi(H)$. Since $g_1, g_2 \in \psi(H)$, we have $g_1 = g_2$. This proves uniqueness in $G \setminus \{0\}$.

Assume that $g_1 \neq \psi(h_1) \neq 0$ and $g_2 = 0$ are two best approximations. Put $h_3 = h_1/2$. From $g_2(x) = 0 < \psi(h_3)(x) < \psi(h_1)(x)$ we conclude that $g_3 = \psi(h_3) \in G$ is another best approximation. This contradicts uniqueness in $G \setminus \{0\}$.

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