

RESEARCH ARTICLE

Maximal indexes of flag varieties for spin groups

Rostislav A. Devyatov¹, Nikita A. Karpenko² and Alexander S. Merkurjev³

¹Max-Planck-Institut für Mathematik, Bonn, Germany; E-mail: deviatov@mccme.ru.

²Mathematical and Statistical Sciences, University of Alberta, Edmonton, Canada; E-mail: karpenko@ualberta.ca.

³Department of Mathematics, University of California, Los Angeles, USA; E-mail: merkurjev@math.ucla.edu.

Received: 25 November 2020; Accepted: 23 March 2021

2020 Mathematics Subject Classification: Primary – 20G15; Secondary – 14C25

Abstract

We establish the sharp upper bounds on the indexes for most of the twisted flag varieties under the spin groups $\text{Spin}(n)$.

Part I

Odd dimension

1. Introduction

Let F be any field and let V be an F -vector space of finite odd dimension $2n+1$ for some integer $n > 0$. Let $q: V \rightarrow F$ be a *nondegenerate* quadratic form (see [6, Definition 7.17]). For any $m = 1, \dots, n$, the m th orthogonal Grassmannian X_m of q is defined as the variety of m -dimensional totally isotropic subspaces in V . Thus, X_m is a closed subvariety inside the usual m -Grassmannian of the vector space V . The two extremes here are studied the most in the literature: the projective quadric X_1 and the highest orthogonal Grassmannian X_n (see, e.g., [16, 17], or [6] for a more exhaustive list of results and references).

For quadratic forms of even dimension, similar varieties and the question formulated later will be considered in the second part of this paper.

The variety X_m has a rational point if and only if the Witt index $i_W(q)$ of the quadratic form q is at least m . The index $i(X_m)$ of the variety X_m (or of any other algebraic variety) is defined as the greatest common divisor of the degrees of its closed points. Since q can be completely split by a multiquadratic field extension, $i(X_m)$ is a power of 2. Namely, $i(X_m)$ is the maximal 2-power dividing the degree of every finite field extension L/F satisfying $i_W(q_L) \geq m$.

The question considered in this paper is as follows: Given n and m , what is the maximal value of $i(X_m)$ when F , V and q vary (more precisely, we let F vary over all field extensions of a fixed field) and the Clifford invariant of q (i.e., the Brauer class $[C_0(q)]$ of the even Clifford algebra $C_0(q)$) is trivial? Since this maximal value is known to be realised at any generic quadratic form with trivial Clifford invariant (defined later), what we want is just to determine $i(X_m)$ in the case of a generic q . Note that if we drop the condition on triviality of the Clifford invariant, the answer to the modified question becomes 2^m for any m . Clearly, this is an upper bound for the original question.

Multiplication of q by a nonzero element of F does not change the varieties X_m . In the case of trivial Clifford invariant, the similarity class of the quadratic form q is given by a $\text{Spin}(2n+1)$ -torsor E over F . Instead of the similarity class, one sometimes prefers to speak about its (unique up to isomorphism) discriminant 1 representative (see [6] for definition of a discriminant in arbitrary characteristic). And any $\text{Spin}(2n+1)$ -torsor over F yields a similarity class of a $(2n+1)$ -dimensional quadratic form with Brauer trivial even Clifford algebra. (The ‘generic q ’ we mentioned earlier means it is given by a *generic torsor* defined as the generic fibre of the quotient morphism $\text{GL}(N) \rightarrow \text{GL}(N)/\text{Spin}(2n+1)$ for an embedding $\text{Spin}(2n+1) \hookrightarrow \text{GL}(N)$ with some $N \geq 1$.)

Moreover, one has $X_m \simeq E/P$ for an appropriate parabolic subgroup $P \subset \text{Spin}(2n+1)$. In other terms, X_m is the flag variety $\text{Spin}(2n+1)/P$ twisted by E . For an arbitrary proper parabolic subgroup $P \subset \text{Spin}(2n+1)$, the twisted flag variety E/P is the variety of flags of totally isotropic subspaces in V of some dimensions $1 \leq m_1 < \dots < m_r \leq n$ (with some $r \geq 1$). The index of this variety coincides with $i(X_{m_r})$ and therefore does not require additional investigation.

The case of the highest orthogonal Grassmannian X_n has been studied [15]. (Note that $i(X_n) = i(X_{n-1}) = i(X_{n-2})$ if defined.) The answer and the proof there are quite complicated. For generic q , the integer $i(X_n)$ is equal to 2^t , where t is approximately $n - 2 \log_2(n)$.

For the other extreme, the projective quadric X_1 , the answer is much simpler: it is just $2 = 2^m$ if we look only at the forms q of dimension at least 7. In other terms, the evident upper bound $i(X_m) \leq 2^m$ turns out to be sharp here. Note that this answer is equivalent to the following assertion: there are anisotropic forms q with trivial Clifford invariant in every dimension $\dim q \geq 7$. (We do not know a simple proof for this statement. Several different proofs are presented in Section 8.)

In this paper we show for arbitrary $m \leq n$ that the evident upper bound 2^m is sharp, provided that $2^{n-m} > \dim X_m$ (see Theorem 4.2). In fact, the number m only needs to be slightly smaller than t . For instance, for $n = 2,021$ the equality $i(X_m) = 2^m$ holds for generic q if $m \leq 2,000$ and $t = 2,001$, by [15, Theorem 0.1]. Since $i(X_m) \leq i(X_n)$ for any m , the formula $\max i(X_m) = 2^m$ fails for $m > 2,001$. So $m = 2,001$ is the only value of m for which Theorem 4.2 and [15, Theorem 0.1] together do not determine whether it satisfies the formula.

Actually, an overwhelming majority of the natural numbers n have the same property: the formula $i(X_m) = 2^m$ holds for $m < t$ and fails for $m > t$. More precisely, the proportion of such $n < N$ tends to 1 when $N \rightarrow \infty$. Note that for such numbers n , we know that $m \mapsto \log_2 i(X_m)$ is an integer-valued increasing function with $j \mapsto j$ for $j = 1, \dots, t - 1$ and $n \mapsto t$, but we do not know where exactly this function jumps from $t - 1$ to t .

A conjecture in the direction of Theorem 4.2 has been suggested for quadratic forms of even dimension in [9] as a possible enhancement of some results from [2]. Extended to quadratic forms of odd dimension, it would affirm that (outside of small n) the maximal value of $i(X_m)$ is 2^m in the case of $m \leq n/2$. Our main result here not only confirms this for $n \geq 13$ but also drastically extends the range of m for larger n .

In the next section we suggest a method of bounding the indexes of twisted flag varieties under any split reductive algebraic group. On the example of the spin groups, we see then that this method is capable to provide interesting results.

2. General considerations

Let G be a split reductive group over a field F . Let $T \subset B \subset P \subset G$ be a split maximal torus, a Borel subgroup and a parabolic subgroup of G . There is a canonical homomorphism of graded rings

$$f_P: \text{CH}(BP) \rightarrow \text{CH}(G/P).$$

The ring on the left is the Chow ring of the classifying space of P , originally defined in [14], which coincides with the P -equivariant Chow ring $\text{CH}_P(\text{Spec } F)$ of the point (see [5]). And the homomorphism f_P is simply the pullback $\text{CH}_P(\text{Spec } F) \rightarrow \text{CH}_P(G) = \text{CH}(G/P)$ with respect to the structure morphism of the F -variety G .

For $d := \dim G/P$, we have $\text{CH}^d(G/P) = \mathbb{Z}$ and the cokernel of $\text{CH}^d(BP) \rightarrow \text{CH}^d(G/P)$ has a finite order $i_P \geq 1$. It was shown in [12, Theorem 6.4] (generalising a theorem of Grothendieck) that the integer i_P thus defined is the index of the variety E/P , where E is any generic G -torsor (over a field extension of F). So for G a spin group, this will be the integer of our interest.

In fact, for any (not necessarily generic) E , there are canonical homomorphisms

$$\text{CH}(BP) \rightarrow \text{CH}(E/P) \rightarrow \text{CH}(G/P)$$

with composition f_P . The first one is defined similarly to f_P . The second one is defined as the composition

$$\text{CH}(E/P) \rightarrow \text{CH}(E/P)_K \xrightarrow{\sim} \text{CH}(G/P)_K = \text{CH}(G/P),$$

where K/F is an extension field such that the G -torsor E_K is trivial; it is independent of the choice of K as well as of the choice of trivialisation for E_K (see [8, Corollary 4.2]). For generic E , by homotopy invariance and the localisation property of equivariant Chow groups, the map $\text{CH}(BP) \rightarrow \text{CH}(E/P)$ is surjective. Thus we get the following:

Proposition 2.1 ([12, Theorem 6.4]). *The image of f_P is contained in the image of $\text{CH}(E/P) \rightarrow \text{CH}(G/P)$ for any E . For generic E , the two images coincide.*

If the parabolic subgroup P is special, the ring $\text{CH}(BP)$ is identified with $\text{CH}(BT)^W = S(\hat{T})^W$, where W is the Weyl group of P , \hat{T} is the group of characters of T and S is the symmetric ring functor (see [5, Proposition 6], together with [11, Proof of Proposition 6.1]). This makes it possible to compute the index i_P (and constitutes the starting point in obtaining the result of [15]).

For instance, B is a special parabolic subgroup with trivial Weyl group so that we have

$$f_B: S(\hat{T}) \rightarrow \text{CH}(G/B).$$

Unfortunately, for arbitrary (not necessarily special) P , a computation of $\text{CH}(BP)$ is sometimes (or rather most of the time) out of reach. However, there still is a canonical homomorphism of graded rings $\text{CH}(BP) \rightarrow S(\hat{T})^W$. (It is neither injective nor surjective in general, but becomes an isomorphism after tensoring with \mathbb{Q} .) Indeed, the restriction of action yields a homomorphism $\text{CH}(BP) \rightarrow \text{CH}(BT) = S(\hat{T})$ whose image consists of W -invariant elements.

Lemma 2.2. *There is one and unique homomorphism of graded rings*

$$f'_P: S(\hat{T})^W \rightarrow \text{CH}(G/P)$$

such that the composition

$$\text{CH}(BP) \longrightarrow S(\hat{T})^W \xrightarrow{f'_P} \text{CH}(G/P)$$

is f_P . The square

$$\begin{array}{ccc} S(\hat{T}) & \xrightarrow{f_B=f'_B} & \text{CH}(G/B) \\ \uparrow & & \uparrow \\ S(\hat{T})^W & \xrightarrow{f'_P} & \text{CH}(G/P) \end{array}$$

commutes.

Proof. The group $\text{CH}(G/P)$ is (torsion) free. The cokernel (as well as the kernel) of $\text{CH}(BP) \rightarrow S(\hat{T})^W$ is torsion [5, Proposition 6] (reduction to the Levi subgroup of P is explained in [11, Proof of Proposition 6.1]). This implies unicity of f'_P .

By [1, Proposition 20.5], for any extension field K/F , the map $G(K) \rightarrow (G/P)(K)$ of the sets of K -points is surjective. Applying this property to the function field of the variety G/P , one sees that the P -torsor given by the generic fibre of the quotient map $G \rightarrow G/P$ is trivial. In particular, the generic fibre of the projection $\pi: G/B \rightarrow G/P$ has a rational point. The class $x \in \text{CH}(G/B)$ of its closure in G/B satisfies $\pi_*(x) = 1$. By the projection formula, for any $y \in \text{CH}(G/P)$ we have $\pi_*(\pi^*(y) \cdot x) = y \cdot \pi_*(x) = y$. It follows that

$$\pi^*: \text{CH}(G/P) \rightarrow \text{CH}(G/B)$$

is a split monomorphism. In particular, its cokernel is (torsion) free (see also [7, Remark 3.3]).

Given any

$$x \in S(\hat{T})^W \subset S(\hat{T}),$$

we can find a nonzero integer n with nx in the image of $\text{CH}(BP)$. The element $f_B(x) \in \text{CH}(G/B)$ has then the property $nf_B(x) = f_B(nx) \in \text{CH}(G/P)$. Consequently, $f_B(x) \in \text{CH}(G/P) \subset \text{CH}(G/B)$. So the restriction of the map f_B to $S(\hat{T})^W$ yields the required map f'_P . \square

Remark 2.3. By [14, Theorem 1.3], the ring $\text{CH}(BP)$ can be viewed as the ring of all assignments to every P -torsor over a smooth variety X of an element in $\text{CH}(X)$ (natural in X). The ring $S(\hat{T})^W$ has a similar interpretation with ‘ P -torsor’ replaced by ‘Zariski locally trivial P -torsor’ ([4, Theorem 1]). In this interpretation, the homomorphism $\text{CH}(BP) \rightarrow S(\hat{T})^W$ is given by restriction of assignments. The homomorphisms f_P and f'_P are given by evaluation of assignments at the P -torsor $G \rightarrow G/P$; here f'_P is well defined because this P -torsor is Zariski locally trivial.

It follows that the order of $\text{Coker}(S^d(\hat{T})^W \rightarrow \text{CH}^d(G/P))$ is a lower bound on i_P . For G a spin group, this lower bound is going to satisfy our needs.

3. Computation of invariants

Let $R = \mathbb{Z}[x_1, \dots, x_m, y_1, \dots, y_l]$ be the polynomial ring over \mathbb{Z} in variables x_1, \dots, x_m and y_1, \dots, y_l with some $m, l \geq 1$. We consider the R -algebra $R[z]$ with a generator z subject to the relation

$$2z = x_1 + \dots + x_m + y_1 + \dots + y_l.$$

Let $A := (\mathbb{Z}/2\mathbb{Z})^{\times l}$ be the direct product of l copies of the group $\mathbb{Z}/2\mathbb{Z}$ acting on R as follows: for any $i = 1, \dots, l$, the i th copy of $\mathbb{Z}/2\mathbb{Z}$ acts by changing the sign of y_i and trivially on the remaining variables. Note that R^A is the subring in R generated by x_1, \dots, x_m and the squares y_1^2, \dots, y_l^2 .

The action of A on R extends uniquely to $R[z]$: the 1 of the i th copy of $\mathbb{Z}/2\mathbb{Z}$ maps z to $z - y_i$.

The orbit of the element z under this action consists of 2^l elements $z - \sum_{i \in I} y_i$, where I runs over all subsets of $\{1, \dots, l\}$. We write $\tilde{z} \in R[z]^A$ for the product of the elements in the orbit of z .

We construct some more A -invariant elements f_k (for $k \geq 0$) in $R[z]$. We set

$$f_0 := 2z - y_1 - \dots - y_l = x_1 + \dots + x_m \in R^A.$$

Assume that for some $k = 0, \dots, l - 2$ the element f_k is already constructed and has the shape

$$f_k = 2z \cdot g_k + a_1 + \dots + a_s, \tag{3.1}$$

where g_k is a monic polynomial in the independent variable z over the ring $R' = \mathbb{Z}[y_1, \dots, y_l]$ and where a_1, \dots, a_s for some $s \geq 0$ are monomials in y_1, \dots, y_l . Then we define f_{k+1} as one half of the difference

$$f_k^2 - (a_1^2 + \dots + a_s^2) = 2(2z(zg_k^2 + (a_1 + \dots + a_s)g_k) + \sum_{i \neq j} a_i a_j).$$

Note that the new polynomial f_{k+1} has the shape (3.1), allowing us to continue the procedure.

We will also consider the induced action of A on the quotient ring $R[z]/2R[z]$. This quotient is the polynomial ring $S[z]$ in the variable z (which is the class of z already discussed but subject to no relations) over the ring

$$S = \mathbb{F}_2[x_1, \dots, x_m, y_1, \dots, y_l]/(x_1 + \dots + x_m + y_1 + \dots + y_l),$$

which is itself a polynomial ring (over the field \mathbb{F}_2 in $m + l - 1$ variables). Note that the action of A on S is trivial and that the element $\tilde{z} \in S[z]$ (the class modulo 2 of the $\tilde{z} \in R[z]$ already discussed) is (still) A -invariant. In other terms, $S[z]^A \supset S[\tilde{z}]$.

Lemma 3.2. $S[z]^A = S[\tilde{z}]$.

Proof. Given any A -invariant $f \in S[z]$, viewed as a polynomial in z , we remove its constant term. Then f is divisible by z and therefore by every factor in the definition of \tilde{z} . Since all these factors are distinct primes of the factorial ring $S[z]$, f is divisible by \tilde{z} . Since S is an integral domain, the quotient f/\tilde{z} is A -invariant as well and so – by induction on degree – is a polynomial in \tilde{z} . Therefore, $f \in S[\tilde{z}]$. \square

Proposition 3.3. The R^A -algebra $R[z]^A$ is generated by the l elements

$$\tilde{z}, f_1, \dots, f_{l-1}.$$

Proof. It suffices to prove the result for $m = 1$. Indeed, we can view $R[z]$ as the polynomial ring over $\mathbb{Z}[z, y_1, \dots, y_l]$ in the variables x_2, \dots, x_m . A polynomial here is A -invariant if and only if all its coefficients are. Moreover, our l potential generators are in the coefficient ring $\mathbb{Z}[z, y_1, \dots, y_l]$.

So we work with the case $m = 1$. From now on, we view the ring $R[z]$ as the ring of polynomials over $R' := \mathbb{Z}[y_1, \dots, y_l]$ in the (independent) variable z . We are going to prove that the R'^A -algebra $R'[z]^A$ is generated by the indicated elements.

Let $f \in R'[z]$ be A -invariant. We prove that f is in the R'^A -subalgebra generated by $\tilde{z}, f_1, \dots, f_{l-1}$ using induction on $\deg f$.

If $\deg f \leq 0$, then $f \in R'^A$. We assume that $\deg f > 0$.

If $\deg f \geq 2^l$, then we let h be the highest power of \tilde{z} with $\deg h \leq \deg f$ and we divide f by h with remainder. The division goes through because the leading coefficient of h is 1. Since f and h are A -invariant, so are the partial quotient and the remainder. In addition, their degrees are smaller than $\deg f$.

We are left with the case $0 < \deg f < 2^l$. By Lemma 3.2, all coefficients of f besides the constant term are even (i.e., divisible by 2). We divide f with remainder by f_k with the highest $k \in \{0, 1, \dots, l-1\}$ such that $2^k = \deg f_k \leq \deg f$. By equation (3.1) and Lemma 3.2, all coefficients of f_k besides the constant term are also even. Moreover, the leading coefficient of f_k is 2. Since the degree of f_k is higher than half of the degree of f , the division with remainder goes through. The partial quotient and the remainder are A -invariant and have degrees smaller than $\deg f$. \square

For $n := m + l$, let $c_1, \dots, c_n \in R$ be the elementary symmetric polynomials in $x_1, \dots, x_m, y_1, \dots, y_l$. For convenience, we additionally set $c_i := 0$ for $i > n$. Let I be the ideal of the ring $R[z]$ consisting of the elements such that after multiplication by an appropriate nonzero integer they are in the ideal of $R[z]$ generated by the elements

$$2 \left(c_{2i} - c_1 c_{2i-1} + \dots + (-1)^{i-1} c_{i-1} c_{i+1} \right) + (-1)^i c_i^2 \tag{3.4}$$

with $i = 1, \dots, n$.

Lemma 3.5. For every $k = 0, \dots, l-1$, the element $f_k \in R[z]$ is congruent modulo I to an element of R .

Proof. Inducting on k , we will prove the following (stronger) statement: the element $2zg_k$ is congruent modulo I to an element of $C[y_1, \dots, y_l] \subset R$, where C is the ideal of the polynomial ring $\mathbb{Z}[c_1, \dots, c_n]$ consisting of the polynomials without constant term.

The element $2zg_0 = 2z = c_1$ is in $C[y_1, \dots, y_l]$. Assume that for some $k = 0, \dots, n - 2$, the element $2zg_k$ is congruent modulo I to an element of $C[y_1, \dots, y_l]$. Then $(2zg_k)^2$ is congruent modulo I to a square of an element of $C[y_1, \dots, y_l]$. Note that the square of any element of C is congruent modulo I to an element of $2C$ (see formula (3.4) and [15, discussion after Lemma 4.1]). Therefore $2z^2g_k^2 = (2zg_k)^2/2$ is congruent modulo I to an element of $C[y_1, \dots, y_l]$, and it follows that the element

$$2zg_{k+1} = 2z \left(zg_k^2 + (a_1 + \dots + a_s)g_k \right)$$

satisfies the same property. Indeed, recall that a_1, \dots, a_s are monomials in y_1, \dots, y_l . So if $2zg_k$ is congruent modulo I to some $h \in C[y_1, \dots, y_l]$, then $(a_1 + \dots + a_s)h$ is also in $C[y_1, \dots, y_l]$ and is congruent modulo I to $2z(a_1 + \dots + a_s)g_k$. □

4. Main result

In this section, G is the split spin group $\text{Spin}(2n + 1)$ for some $n \geq 1$ over an arbitrary field F . As in [12, §8.2], we construct G out of a split quadratic form q defined on a vector space V with a basis given by a vector g and vectors $e_i, f_i, i = 1, \dots, n$, where e_i, f_i are pairwise orthogonal hyperbolic pairs orthogonal to g . Let us fix some $m \in \{1, \dots, n\}$, consider the m -dimensional totally isotropic subspace generated by e_1, \dots, e_m and let $P \subset G$ be its stabiliser. Then P is a parabolic subgroup in G and the variety G/P is the m th orthogonal Grassmannian X_m of q .

We take for $T \subset P$ the split maximal torus, mapped under the isogeny $G \rightarrow G/\mu_2 = \text{O}^+(2n + 1)$ (with the special orthogonal group) to the split maximal torus $T' := \mathbb{G}_m^n \hookrightarrow \text{O}^+(2n + 1)$ given by $t(e_i) = t_i e_i, t(f_i) = t_i^{-1} f_i$ and $t(g) = g$, where $t = (t_1, \dots, t_n) \in \mathbb{G}_m^n(F)$. We have an exact sequence

$$1 \rightarrow \mu_2 \rightarrow T \rightarrow T' \rightarrow 1.$$

Writing x_1, \dots, x_n for the standard basis of $\mathbb{Z}^n = \hat{T}'$, we therefore have $\hat{T} = \hat{T}' + \mathbb{Z}z$, where $z := (x_1 + \dots + x_n)/2$.

Let us set $l := n - m$. The Weyl group W of P is the direct product of the symmetric group S_m and the Weyl group of $\text{O}^+(2l + 1)$, the latter being a semidirect product of S_l by $(\mathbb{Z}/2\mathbb{Z})^l$. The action of W on \hat{T}' is given by the action of S_m by permutations of x_1, \dots, x_m , the action of S_l by permutation of x_{m+1}, \dots, x_n and the action of the i th copy of $\mathbb{Z}/2\mathbb{Z}$ for $i = 1, \dots, l$ by changing the sign of x_{m+i} . This action extends uniquely to an action of W on \hat{T} .

To comply with requirements of the previous section, we assume that $m < n$. Recall that we wrote A for $(\mathbb{Z}/2\mathbb{Z})^{\times l}$ in the previous section. Let us identify $S(\hat{T}')$ with the ring R from there by identifying x_{m+i} with y_i for $i = 1, \dots, l$. The action of A we had there is the restriction of the action of $W \supset A$ we have now. The element z from the previous section corresponds to the element z introduced here, and $S(\hat{T}) = R[z]$. The product \tilde{z} from the previous section becomes an element of $S^{2l}(\hat{T})$.

The parabolic subgroup P contains the Borel subgroup B of G defined as the stabiliser of the flag of the totally isotropic subspaces

$$\langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \dots \subset \langle e_1, \dots, e_n \rangle.$$

We are going to study the image of the composition $S(\hat{T})^W \hookrightarrow S(\hat{T}) \rightarrow \text{CH}(G/B)$. Note that the variety G/B is the variety of complete flags of totally isotropic subspaces in V and the image of $R = S(\hat{T}') \hookrightarrow S(\hat{T}) \rightarrow \text{CH}(G/B)$ is the subring $H \subset \text{CH}(G/B)$ generated by the first Chern classes of the tautological line bundles on G/B (compare [10]). Note that these Chern classes are the images of x_1, \dots, x_n (and will be denoted x_1, \dots, x_n at a later point).

Proposition 4.1. *The image of $S(\hat{T})^W$ in the ring $\text{CH}(G/B)$ is contained in the H -subalgebra generated by the image of $\tilde{z} \in S^{2l}(\hat{T})$.*

Proof. We note that $S(\hat{T})^W \subset S(\hat{T})^A$ and apply Proposition 3.3. The Chern classes of the tautological rank n vector bundle on G/B satisfy relations (3.4) (see [6, Formula 86.15]). Besides, $\text{CH}(G/B)$ is (torsion) free. Consequently, the ideal $I \subset R[z] = S(\hat{T})$ from Lemma 3.5 vanishes in $\text{CH}(G/B)$ and we are done by Lemma 3.5. \square

Let us recall the dimension formula for the variety X_m :

$$\dim X_m = m(m - 1)/2 + m(2n + 1 - 2m).$$

Theorem 4.2. *For $1 \leq m \leq n$, let q be a generic $(2n + 1)$ -dimensional quadratic form with trivial Clifford invariant and let X_m be its m th orthogonal Grassmannian. Then $i(X_m) = 2^m$ provided that $2^{n-m} > \dim X_m$.*

Proof. Let X be the variety of complete flags of totally isotropic subspaces for the form q . We are going to work with the varieties X, X_1, X_m, X_n and their base changes $\bar{X}, \bar{X}_1, \bar{X}_m, \bar{X}_n$ to the algebraic closure of the base field.

As shown in [18, Statement 2.15], the class $[pt] \in \text{CH}(\bar{X}_m)$ of a rational point pt on \bar{X}_m is equal to

$$[pt] = \xi_m(l_0)\xi_m(l_1) \cdots \xi_m(l_{m-1}),$$

where $l_i \in \text{CH}_i(\bar{X}_1)$ is the class of a projective linear i -dimensional subspace on \bar{X}_1 and ξ_m is the composition

$$\xi_m : \text{CH}(\bar{X}_1) \rightarrow \text{CH}(\bar{X}_{1,m}) \rightarrow \text{CH}(\bar{X}_m)$$

of the pullback followed by the push-forward with respect to the projections of the flag variety $X_{1,m} \subset X_1 \times X_m$. The image of $[pt]$ under the pullback $\text{CH}(\bar{X}_m) \rightarrow \text{CH}(\bar{X})$ with respect to the projection $X \rightarrow X_m$ can be computed via the following:

Lemma 4.3 ([18, Lemma 2.6]). *For any $i = 0, \dots, n - 1$, the image of $\xi_m(l_i)$ under the pullback $\text{CH}(\bar{X}_m) \rightarrow \text{CH}(\bar{X}_{m,m+1})$ is equal to $\xi_{m+1}(l_{i-1}) + \xi_{m+1}(l_i)x_{m+1}$. Here we view the ring $\text{CH}(\bar{X}_{m,m+1})$ as a $\text{CH}(\bar{X}_{m+1})$ -algebra via the pullback with respect to the projective bundle $X_{m,m+1} \rightarrow X_{m+1}, x_{m+1}$ is the first Chern class of the tautological line bundle on $X_{m,m+1}$, and $l_{-1} := 0$.*

It follows that the image of $[pt]$ under $\text{CH}(\bar{X}_m) \rightarrow \text{CH}(\bar{X}_{m,m+1})$ equals

$$(\xi_{m+1}(l_0)x_{m+1}) \cdot (\xi_{m+1}(l_0) + \xi_{m+1}(l_1)x_{m+1}) \cdots \cdots (\xi_{m+1}(l_{m-2}) + \xi_{m+1}(l_{m-1})x_{m+1}),$$

so that the coefficient at x_{m+1}^m is $\xi_{m+1}(l_0)\xi_{m+1}(l_1) \cdots \xi_{m+1}(l_{m-1})$. By iterating, the image of $[pt]$ under the pullback $\text{CH}(\bar{X}_m) \rightarrow \text{CH}(\bar{X})$ will be a polynomial in x_{m+1}, \dots, x_n with coefficients in $\text{CH}(\bar{X}_n)$ such that the coefficient at the monomial $x_{m+1}^m x_{m+2}^m \cdots x_n^m$ is

$$\xi_n(l_0)\xi_n(l_1) \cdots \xi_n(l_{m-1}) \in \text{CH}(\bar{X}_n).$$

By Proposition 2.1, the image of the composition

$$\text{CH}(X_m) \rightarrow \text{CH}(\bar{X}_m) \rightarrow \text{CH}(\bar{X}) \tag{4.4}$$

coincides with the image of $\text{CH}(BP) \rightarrow \text{CH}(G/P) \rightarrow \text{CH}(G/B) = \text{CH}(\bar{X})$. By Lemma 2.2, the latter image is contained in the image of $S(\hat{T})^W \rightarrow \text{CH}(G/B)$. By Proposition 4.1, since $\bar{z} \in S^{2^{n-m}}(\hat{T})$ and $2^{n-m} > \dim X_m$, every element a in the image of formula (4.4) is a polynomial (with integer coefficients) in the first Chern classes x_1, \dots, x_n of the tautological line bundles on X . By Lemma 4.5, the element a can be written uniquely as a polynomial in x_1, \dots, x_n with coefficients in $\text{CH}(\bar{X}_n)$, where each x_i appears only in degrees $< i$. By Lemma 4.6, the coefficients are polynomials in the Chern classes of the tautological rank n vector bundle T on X_n .

We conclude: if r is such that $i(X_m) = 2^r$, the element

$$2^r \xi_n(l_0) \xi_n(l_1) \cdots \xi_n(l_{m-1}) \in \text{CH}(\bar{X}_n)$$

is a polynomial in the Chern classes of T .

Recall ([6, Theorem 86.12 and Formula 86.5], originally proved in [17]) that the group $\text{CH}(\bar{X}_n)$ is free with a basis given by all 2^n products of distinct $\xi_n(l_0), \dots, \xi_n(l_{n-1})$. The elements $2\xi_n(l_0), \dots, 2\xi_n(l_{n-1})$ are, up to a sign, the Chern classes of T [6, Proposition 86.13]. Due to the already mentioned fact that the Chern classes of T satisfy formula (3.4), the additive group of the subring in $\text{CH}(\bar{X}_n)$ generated by these Chern classes is free with the basis given by the 2^n products of distinct $2\xi_n(l_0), \dots, 2\xi_n(l_{n-1})$. (For the generalisation of this fact to all X_m see [10, Theorem 2.1].) Therefore $r = m$. \square

We recall two classical facts used in the foregoing proof:

Lemma 4.5. *Let X be a smooth variety with a rank n vector bundle \mathcal{E} and let Y be the variety of complete flags in \mathcal{E} . Let x_1, \dots, x_n be the first Chern classes of the tautological line bundles on Y . The $\text{CH}(X)$ -module $\text{CH}(Y)$ is free with a basis given by the monomials $x_1^{a_1} \dots x_n^{a_n}$ satisfying the condition $a_i < i$ for all i .*

Proof. Viewing $Y \rightarrow X$ as a chain of projective bundles, the statement follows from the projective bundle theorem [6, Theorem 57.14]. \square

Lemma 4.6. *The ring of polynomials $\mathbb{Z}[x_1, \dots, x_n]$ in variables x_1, \dots, x_n , considered as a module over the subring of symmetric polynomials, is free with a basis given by the monomials $x_1^{a_1} \dots x_n^{a_n}$ satisfying the condition $a_i < i$ for all i .*

Proof. Apply Lemma 4.5, taking for X the Grassmannian of n -dimensional subspaces in a vector space of large (better infinite) dimension and taking for \mathcal{E} the tautological vector bundle. \square

Part II

Even dimension

5. Introduction

Let us repeat the introduction of Part 1, replacing the quadratic forms of odd dimension with the even-dimensional ones and making the other necessary changes. There are many similarities between the even- and odd-dimensional cases; we apologise for repetitions. On the other hand, the picture here is in some places somewhat messier or more complicated.

Let F be any field and let V be an F -vector space of finite even dimension $2n$ for some integer $n \geq 1$. Let $q: V \rightarrow F$ be a nondegenerate quadratic form. For any $m = 1, \dots, n$, the m th orthogonal Grassmannian X_m of q is defined as the variety of m -dimensional totally isotropic subspaces in V . Thus, X_m is a closed subvariety inside the usual m -Grassmannian of V . The two extremes here are studied the most in the literature: the projective quadric X_1 and the highest orthogonal Grassmannian X_n .

The variety X_m has a rational point if and only if the Witt index $i_W(q)$ of the quadratic form q is at least m . Since q becomes hyperbolic over some multiquadratic field extension, the index $i(X_m)$ of the variety X_m is a power of 2. Namely, $i(X_m)$ is the maximal 2-power dividing the degree of every finite field extension L/F satisfying $i(q_L) \geq m$.

The question considered in this part is as follows: Given n and m , what is the maximal value of $i(X_m)$ when F , V and q vary (more precisely, we let F vary over all field extensions of a fixed field) and the discriminant and the Clifford invariant of q (now given by the Brauer class of the total Clifford algebra $C(q)$) are trivial? Since this maximal value is realised at the generic quadratic forms with trivial

discriminant and Clifford invariant (defined later), what we want is just to determine $i(X_m)$ in the case of generic q . Note that if we drop the condition on triviality of the invariants, the answer to the modified question becomes 2^m for any m . Clearly, this is an upper bound for the original question.

In the case of trivial invariants, the isomorphism class of the quadratic form q is given by a $\text{Spin}(2n)$ -torsor E over F . And any $\text{Spin}(2n)$ -torsor over F yields an isomorphism class of a $(2n)$ -dimensional quadratic form with trivial discriminant and Clifford invariant. (The ‘generic q ’ earlier means that it is given by a *generic torsor* defined as the generic fibre of the quotient morphism $\text{GL}(N) \rightarrow \text{GL}(N)/\text{Spin}(2n)$ for an embedding $\text{Spin}(2n) \hookrightarrow \text{GL}(N)$ for some N .)

If $m \neq n$, one has $X_m \simeq E/P$ for an appropriate parabolic subgroup $P \subset \text{Spin}(2n)$. The variety X_n consists of two connected components each isomorphic to E/P . For an arbitrary proper parabolic subgroup $P \subset \text{Spin}(2n)$, the twisted flag variety E/P is either the variety of flags of totally isotropic subspaces in V of some dimensions $1 \leq m_1 < \dots < m_r \leq n$ (with some $r \geq 1$) or (if and only if $m_r = n$) one of its two isomorphic components. The index of this variety coincides with $i(X_{m_r})$ and therefore does not require additional investigation.

The case of the highest orthogonal Grassmannian X_n has been addressed [15].¹ (Note that $i(X_n) = i(X_{n-1}) = i(X_{n-2}) = i(X_{n-3})$ if defined.) The answer and proof there appear to be quite complicated. For generic q , the integer $i(X_n)$ is equal to 2^t , where t is approximately $n - 2 \log_2(n)$.

For the other extreme, the projective quadric X_1 , the answer is much simpler: it is just $2 = 2^m$ if we look only at the forms q of dimension at least 12. In other terms, the evident upper bound $i(X_m) \leq 2^m$ turns out to be sharp here. Note that this answer is equivalent to the following assertion: there are anisotropic forms q with trivial discriminant and Clifford invariant in every dimension $\dim q \geq 12$. (We do not know a simple proof for this statement. Several different proofs are presented in Section 8.)

In this part we show for arbitrary $m < n$ that the evident upper bound 2^m is sharp, provided that $2^{n-m-1} > \dim X_m$ (see Theorem 7.2). This implies that an overwhelming majority of the natural numbers n have the following property: the formula $i(X_m) = 2^m$ holds for $m < t$ and fails for $m > t$. More precisely, the proportion of such $n < N$ tends to 1 when $N \rightarrow \infty$.

Theorem 7.2 confirms [9, Conjecture 3.3] for $n \geq 17$. For large n , the theorem is stronger than the statement of the conjecture.

6. Computation of invariants

In the notation of Section 3, let A' be the subgroup of A consisting of the elements with the trivial sum of components.

The orbit of the element z under the action of A' consists of 2^{l-1} elements. We write $\check{z} \in R[z]^{A'}$ for the product of the elements in the orbit of z . As a warm-up, note that the ring $R^{A'}$ is generated by x_1, \dots, x_m , the squares y_1^2, \dots, y_l^2 and the product $y_1 \dots y_l$.

Proposition 6.1. *The $R^{A'}$ -algebra $R[z]^{A'}$ is generated by the elements*

$$\check{z}, f_1, \dots, f_{l-2}.$$

Proof. Just repeat the proof of Proposition 3.3 (including Lemma 3.2), replacing A by A' , \bar{z} by \check{z} and l by $l - 1$. □

7. Main result

In this section, G is the split spin group $\text{Spin}(2n)$ for some $n > 1$ over an arbitrary field F . As in [12, §8.4], we construct G out of a hyperbolic quadratic form q defined on a $(2n)$ -dimensional vector space V with a basis given by vectors $e_i, f_i, i = 1, \dots, n$, where e_i, f_i are pairwise orthogonal hyperbolic pairs.

¹Due to the so-called *exceptional* (in the sense of [3]) isomorphism between X_n and the highest orthogonal Grassmannian of any nondegenerate 1-codimensional subform of q , our question on the highest orthogonal Grassmannian need not be considered for quadratic forms of even dimension.

Let us fix some $m \in \{1, \dots, n - 1\}$, consider the m -dimensional totally isotropic subspace generated by e_1, \dots, e_m and let $P \subset G$ be its stabiliser. Then P is a parabolic subgroup in G and the variety G/P is the m th orthogonal Grassmannian X_m of q .

We take for $T \subset P$ the split maximal torus, mapped under the isogeny $G \rightarrow G/\mu_2 = O^+(2n)$ (with the special orthogonal group) to the split maximal torus $T' := \mathbb{G}_m^n \hookrightarrow O^+(2n)$ given by $t(e_i) = t_i e_i$ and $t(f_i) = t_i^{-1} f_i$, where $t = (t_1, \dots, t_n) \in \mathbb{G}_m^n(F)$. We have an exact sequence

$$1 \rightarrow \mu_2 \rightarrow T \rightarrow T' \rightarrow 1.$$

Writing x_1, \dots, x_n for the standard basis of $\mathbb{Z}^n = \hat{T}'$, we therefore have $\hat{T} = \hat{T}' + \mathbb{Z}z$, where $z := (x_1 + \dots + x_n)/2$.

Let us set $l := n - m$. The Weyl group W of P is the direct product of the symmetric group S_m and the Weyl group of $O^+(2l)$, the latter being a semidirect product of S_l by $A' \subset (\mathbb{Z}/2\mathbb{Z})^l$ for A' introduced in Section 6. The action of W on \hat{T}' is given by the action of S_m by permutations of x_1, \dots, x_m , the action of S_l by permutation of x_{m+1}, \dots, x_n and the action of A' obtained by restriction of the action of $(\mathbb{Z}/2\mathbb{Z})^l$, where the i th copy of $\mathbb{Z}/2\mathbb{Z}$ acts by changing the sign of x_{m+i} . This action extends uniquely to an action of W on \hat{T} .

Let us identify $S(\hat{T}')$ with the ring R from the previous section by identifying x_{m+i} with y_i for $i = 1, \dots, l$. The action of A' we had there is the restriction of the action of $W \supset A'$ we have now. The element z from the previous section corresponds to the element z introduced here, and $S(\hat{T}) = R[z]$. The product \check{z} from the previous section becomes an element of $S^{2^{l-1}}(\hat{T})$.

The parabolic subgroup P contains the Borel subgroup B of G defined as the stabiliser of the flag of the totally isotropic subspaces

$$\langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \dots \subset \langle e_1, \dots, e_n \rangle.$$

We are going to study the image of the composition $S(\hat{T})^W \hookrightarrow S(\hat{T}) \rightarrow \text{CH}(G/B)$. Note that G/B is the variety of flags of totally isotropic subspaces in V of dimensions $1, \dots, n - 1$, which is a component of the variety of flags of totally isotropic subspaces in V of dimensions $1, \dots, n - 1, n$. The image of $S(\hat{T}') \hookrightarrow S(\hat{T}) \rightarrow \text{CH}(G/B)$ is the subring $H \subset \text{CH}(G/B)$ generated by the Chern classes of the tautological line bundles on the latter variety of flags.

Proposition 7.1. *The image of $S(\hat{T})^W$ in the ring $\text{CH}(G/B)$ is contained in the H -subalgebra generated by the image of $\check{z} \in S(\hat{T})$.*

Proof. We note that $S(\hat{T})^W \subset S(\hat{T})^{A'}$ and apply Proposition 6.1. The Chern classes of the tautological rank n vector bundle on G/B satisfy formula (3.4) (see [6, Formula 86.15]). In addition, $\text{CH}(G/B)$ is (torsion) free. Consequently, the ideal $I \subset R[z] = S(\hat{T})$ from Lemma 3.5 vanishes in $\text{CH}(G/B)$ and we are done, by Lemma 3.5. □

The dimension formula for the variety X_m is

$$\dim X_m = m(m - 1)/2 + 2m(n - m).$$

Theorem 7.2. *For $1 \leq m < n$, let q be a generic $(2n)$ -dimensional quadratic form with trivial discriminant and Clifford invariant and let X_m be its m th orthogonal Grassmannian. Then $i(X_m) = 2^m$, provided that $2^{n-m-1} > \dim X_m$.*

Example 7.3. For $m = 1$, the condition $2^{n-m-1} > \dim X_m$ is satisfied if and only if $\dim q \geq 12$.

Proof (Proof of Theorem 7.2). Let X be a component of the variety of flags of totally isotropic subspaces of dimensions $1, \dots, n$ for the form q . We are going to work with the varieties X, X_1, X_m, X_n and their base changes $\bar{X}, \bar{X}_1, \bar{X}_m, \bar{X}_n$ to the algebraic closure of the base field. We replace X_n by one of its components – the one onto which X projects.

As shown in [18, Statement 2.15], the class $[pt] \in \text{CH}(\bar{X}_m)$ of a rational point pt on \bar{X}_m is equal to

$$[pt] = \xi_m(l_0)\xi_m(l_1) \cdots \xi_m(l_{m-1}),$$

where $l_i \in \text{CH}_i(\bar{X}_1)$ is the class of a projective linear i -dimensional subspace on \bar{X}_1 , and ξ_m is the composition

$$\xi_m: \text{CH}(\bar{X}_1) \rightarrow \text{CH}(\bar{X}_{1,m}) \rightarrow \text{CH}(\bar{X}_m)$$

of the pullback followed by the push-forward with respect to the projections of the flag variety $X_{1,m} \subset X_1 \times X_m$.

The image of $[pt]$ under the pullback $\text{CH}(\bar{X}_m) \rightarrow \text{CH}(\bar{X})$ can be computed via [18, Lemma 2.6], a statement similar to Lemma 4.3. What we get is a polynomial in x_{m+1}, \dots, x_n (the first Chern classes of the corresponding tautological line bundles on X) with coefficients in $\text{CH}(\bar{X}_n)$ such that the coefficient at $x_{m+1}^m x_{m+2}^m \cdots x_n^m$ is

$$\xi_n(l_0)\xi_n(l_1) \cdots \xi_n(l_{m-1}) \in \text{CH}(\bar{X}_n).$$

By Proposition 2.1, Lemma 2.2 and Proposition 7.1, since $\tilde{z} \in S^{2^{n-m-1}}(\hat{T})$ and $2^{n-m-1} > \dim X_m$, every element a in the image of the composition

$$\text{CH}(X_m) \rightarrow \text{CH}(\bar{X}_m) \rightarrow \text{CH}(\bar{X})$$

is an integral polynomial in the first Chern classes x_1, \dots, x_n of the tautological line bundles on X . By Lemma 4.5, the element a can be written uniquely as a polynomial in x_1, \dots, x_n with coefficients in $\text{CH}(\bar{X}_n)$, where each x_i appears only in degrees $< i$. By Lemma 4.6, the coefficients are polynomials in the Chern classes of T .

We conclude: if r is such that $i(X_m) = 2^r$, the element

$$2^r \xi_n(l_0)\xi_n(l_1) \cdots \xi_n(l_{m-1}) \in \text{CH}(\bar{X}_n)$$

is a polynomial in the Chern classes of T .

Recall ([6, Theorem 86.12 and Formula 86.5], originally proved in [17]) that the group $\text{CH}(\bar{X}_n)$ is free with a basis given by all 2^{n-1} products of distinct $\xi_n(l_0), \dots, \xi_n(l_{n-2})$. The elements $2\xi_n(l_0), \dots, 2\xi_n(l_{n-2})$ are, up to a sign, the first $n - 1$ Chern classes of T [6, Proposition 86.13], the n th Chern class being 0 (see [6, Proposition 86.17] or [10, Theorem 2.1]). The additive group of the subring in $\text{CH}(\bar{X}_n)$ generated by these Chern classes is free with the basis given by the 2^{n-1} products of distinct $2\xi_n(l_0), \dots, 2\xi_n(l_{n-2})$. (For the generalisation of this fact to all X_m , see [10, Theorem 2.1].) Therefore $r = m$. □

8. Appendix: Quadrics

We list several different proofs of the fact Φ that $i(X_1) = 2$ for a generic quadratic form q of sufficiently large even dimension with trivial discriminant and Clifford invariant. Note that by taking a 1-codimensional subform in q , the statement Φ implies the similar statement on the quadratic forms of odd dimension.

Steenrod operations

Φ is a consequence of [6, Proposition 82.7] (which is due to A. Vishik), whose proof makes use of Steenrod operations on the modulo 2 Chow groups of algebraic varieties. In particular, it became available in characteristic 2 only after the recent [13].

More precisely, Steenrod operations are used in the proof of [6, Corollary 80.8] on the possible size of binary correspondences.

Essential dimension

In characteristic 0, Φ follows from [2, Theorem 4.2], which roughly states that a generic quadratic form with trivial discriminant, Clifford invariant and sufficiently large dimension contains no proper even-dimensional subforms of trivial discriminant. A nondegenerate quadratic form is anisotropic if and only if it contains no 2-dimensional subforms of trivial discriminant (i.e., hyperbolic planes).

Totaro's torsion index

Φ can be deduced from the computation of the index of the highest orthogonal Grassmannian made in [15]. Indeed, if for some $n \geq 6$, a generic $2n$ -dimensional quadratic form q with trivial discriminant and Clifford invariant is isotropic, then $l_0 \in \text{CH}_0(\bar{X}_1)$ is in the image of $\text{CH}(X_1) \rightarrow \text{CH}(\bar{X}_1)$ so that $\xi_n(l_0) \in \text{CH}(\bar{X}_n)$ is in the image of $\text{CH}(X_n) \rightarrow \text{CH}(\bar{X}_n)$. The latter image modulo 2 is known to be the subring generated by $\xi_n(l_{n-2})$. By [6, Formula (86.15)], it follows that $n - 1$ is a 2-power.

On the other hand, since q is isotropic, the index $i(X_n)$, computed in [15], cannot be higher than the index $i(X'_{n-1})$ of the highest orthogonal Grassmannian X'_{n-1} of a generic $2(n-1)$ -dimensional quadratic form q' with trivial discriminant and Clifford invariant. It follows by [15, Theorem 0.1] that $n - 1$ is not a 2-power.

Our main result

Φ is a particular case of Theorem 7.2 (see Example 7.3).

Acknowledgements. The first author thanks the Max Planck Institute for Mathematics in Bonn for financial support and hospitality. The work of the second author was supported by a Discovery Grant from the National Science and Engineering Research Council of Canada and was accomplished during his stay at the Université vu Lëtzebuerg. The third author was supported by National Science Foundation grant DMS #1801530. We thank Burt Totaro and Alexander Vishik for valuable comments.

Conflict of Interest: None.

References

- [1] A. Borel, *Linear Algebraic Groups*, second edn, Graduate Texts in Mathematics, **126** (Springer-Verlag, New York, 1991).
- [2] V. Chernousov and A. Merkurjev, 'Essential dimension of spinor and Clifford groups', *Algebra Number Theory* **8**(2) (2014), 457–472.
- [3] M. Demazure, 'Automorphismes et déformations des variétés de Borel', *Invent. Math.* **39**(2) (1977), 179–186.
- [4] D. Edidin and W. Graham, 'Characteristic classes in the Chow ring', *J. Algebraic Geom.* **6**(3) (1997), 431–443.
- [5] D. Edidin and W. Graham, 'Equivariant intersection theory', *Invent. Math.* **131**(3) (1998), 595–634.
- [6] R. Elman, N. Karpenko and A. Merkurjev, *The Algebraic and Geometric Theory of Quadratic Forms*, American Mathematical Society Colloquium Publications, **56** (American Mathematical Society, Providence, RI, 2008).
- [7] N. A. Karpenko, 'A bound for canonical dimension of the (semi)spinor groups', *Duke Math. J.* **133**(2) (2006), 391–404.
- [8] N. A. Karpenko, 'Unitary Grassmannians', *J. Pure Appl. Algebra* **216**(12) (2012), 2586–2600.
- [9] N. A. Karpenko, 'Around 16-dimensional quadratic forms in I_q^3 ', *Math. Z.* **285**(1-2) (2017), 433–444.
- [10] N. A. Karpenko, 'On generic quadratic forms', *Pacific J. Math.* **297**(2) (2018), 367–380.
- [11] N. A. Karpenko, 'On generically split generic flag varieties', *Bull. Lond. Math. Soc.* **50** (2018), 496–508.
- [12] N. A. Karpenko and A. S. Merkurjev, 'Canonical p -dimension of algebraic groups', *Adv. Math.* **205**(2) (2006), 410–433.
- [13] E. Primožic, 'Motivic Steenrod operations in characteristic p ', *Forum Math. Sigma* **8** (2020), e52.
- [14] B. Totaro, 'The Chow ring of a classifying space', In *Algebraic K-theory (Seattle, WA, 1997)*, Proceedings of Symposia in Applied Mathematics, **67** (American Mathematical Society, Providence, RI, 1999), 249–281.
- [15] B. Totaro, 'The torsion index of the spin groups', *Duke Math. J.* **129**(2) (2005), 249–290.
- [16] A. Vishik, 'Motives of quadrics with applications to the theory of quadratic forms', in *Geometric Methods in the Algebraic Theory of Quadratic Forms*, Lecture Notes in Mathematics, **1835** (Springer, Berlin, 2004), 25–101.
- [17] A. Vishik, 'On the Chow groups of quadratic Grassmannians', *Doc. Math.* **10** (2005), 111–130.
- [18] A. Vishik, 'Fields of u -invariant $2^r + 1$ ', in *Algebra, Arithmetic, and Geometry: In Honor of Yu. I. Manin. Vol. II*, Progress in Mathematics, **270** (Birkhäuser Boston Inc., Boston, MA, 2009), 661–685.