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CHARACTERISATIONS OF PARTITION OF UNITIES GENERATED BY ENTIRE FUNCTIONS IN $\mathbb{C}^d$

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Abstract

Collections of functions forming a partition of unity play an important role in analysis. In this paper we characterise for any $N \in \mathbb{N}$ the entire functions P for which the partition of unity condition $\sum_{\mathbf{n}\in\mathbb{Z}^d} P(\mathbf{x}+\mathbf{n})\chi_{[0,N]^d}(\mathbf{x}+\mathbf{n}) = 1$ holds for all $\mathbf{x} \in \mathbb{R}^d$. The general characterisation leads to various easy ways of constructing such entire functions as well. We demonstrate the flexibility of the approach by showing that additional properties like continuity or differentiability of the functions $(P\chi_{[0,N]^d})(\cdot + \mathbf{n})$ can be controlled. In particular, this leads to easy ways of constructing entire functions P such that the functions in the partition of unity belong to the Feichtinger algebra.

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1. Introduction

In its most basic form a collection of functions $g_n : X \to \mathbb{C}$, indexed by a countable set I, forms a *partition of unity* if $\sum_{n\in I} g_n(x) = 1$ for all $x \in X$. Often, additional constraints restrict the class of partitions of unity being considered. For example, it might be assumed that the functions g_n are bounded or that there are only a finite number of nonzero contributions in the sum for each $x \in X$. The second type of constraint removes the question of how to interpret the convergence.

In this paper we consider the set $X = \mathbb{R}^d$ and assume that the functions g_n are translates of a fixed function g. Basically we are interested in the case where g is an entire function. Unfortunately this excludes the possibility that g is compactly supported, a property that is highly desirable in many applications. Motivated by this, we consider functions g that are formed by the product of an entire function $P : \mathbb{C}^d \to \mathbb{C}$ and a characteristic function $\chi_{[0,N]^d}$ for some $N \in \mathbb{N}$ and such that

$$\sum_{\mathbf{n}\in\mathbb{Z}^d} P(\mathbf{x}+\mathbf{n})\chi_{[0,N]^d}(\mathbf{x}+\mathbf{n}) = 1 \quad \text{for all } \mathbf{x} \in \mathbb{R}^d. \tag{1.1}$$

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The main new contribution of the paper is a characterisation of the entire functions P which satisfy (1.1) for a given and fixed $N \in \mathbb{N}$. In order to give a clear impression about the flow of the paper, we will state the result now; we will later formulate it in a more elegant notation that will be introduced in Section 3.

THEOREM 1.1. *Consider an entire function $P : \mathbb{C}^d \to \mathbb{C}$ and let $N \in \mathbb{N}$. Then the following are equivalent:*

(a) *P satisfies (1.1);*
(b) *P has the form $P(x_1, \dots, x_d) = N^{-d} + \sum_{j=1}^{d} P^j(x_1, \dots, x_d)$ for some functions $P^j(x_1, \dots, x_d) = \sum_{k \in \mathbb{Z} \setminus N\mathbb{Z}} r_k^j(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_d) e^{2\pi i k x_j / N}$ with entire coefficient functions $r_k^j : \mathbb{C}^{d-1} \to \mathbb{C}$.*

The hard part in Theorem 1.1 is the proof of (a) ⇒ (b). The characterisation leads to a number of ways of constructing appropriate entire functions P explicitly. The flexibility of the approach is demonstrated by showing that additional properties like continuity or differentiability of the functions $(P\chi_{[0,N]^d})(\cdot + \mathbf{n})$ can be controlled as well. In particular, this leads to easy ways of constructing entire functions P such that the functions in the partition of unity belong to the Feichtinger algebra.

The paper is a sequel to [1, 2]. In [1] we considered the case $d = 1$ and proved that an entire function P satisfying (1.1) must be N-periodic. In [2] the higher-dimensional case was analysed for entire functions having the additional property of being periodic, but the general case was not considered. In Section 2 we will provide an alternative characterisation of the entire functions P satisfying (1.1) in the special case $d = 1$; this turns out to be the key to the higher-dimensional case treated in Section 3.

Partition of unity conditions appear in many areas of mathematics. For example, the proof of Riesz' representation theorem in Rudin's book [8] is based on a construction of a partition of unity with certain additional properties. Partitions of unity also play key roles in the analysis of the Feichtinger algebra [3, 4] and, in the context of multiresolution analysis, it is well known that the integer-translates of the scaling function yield a partition of unity. Finally, the duality conditions for Gabor frames and wavelet frames (or even generalised shift-invariant systems) in $L^2(\mathbb{R})$ involve a partition of unity. In the special case of a Gabor system the partition of unity is formed by the integer-translates of a fixed function as considered in the current paper (see [6, 7]).

2. Partition of unity for entire functions $P : \mathbb{C} \to \mathbb{C}$

Our starting point is to derive a characterisation of the entire functions $P : \mathbb{C} \to \mathbb{C}$ satisfying (1.1). This will pave the way to the higher-dimensional case considered in Section 3. For reasons that will become clear in the proof of Theorem 2.2, we will first characterise the entire functions $R : \mathbb{C} \to \mathbb{C}$ satisfying the equation

$$\sum_{\ell=0}^{N-1} R(x + \ell) = 0 \quad \text{for all } x \in [0, 1]. \tag{2.1}$$

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LEMMA 2.1. *Let $N \in \mathbb{N}$ and consider an entire function $R : \mathbb{C} \to \mathbb{C}$. Then the following are equivalent:*

(a) *R satisfies (2.1);*
(b) *$R(x) = \sum_{k\in\mathbb{Z}\setminus N\mathbb{Z}} a_k e^{2\pi ikx/N}$ for $x \in \mathbb{R}$ and, for any $\alpha > 0$, there exists a constant C_α such that $|a_k| \le C_\alpha e^{-\alpha|k|}$ for all $k \in \mathbb{Z}$;*
(c) *$R(x) = G(e^{2\pi ix/N})$ for $x \in \mathbb{R}$ and a function $G(z) = \sum_{k\in\mathbb{Z}\setminus N\mathbb{Z}} a_k z^k$ which is holomorphic in $\mathbb{C} \setminus \{0\}$.*

PROOF. (a) $\Rightarrow$ (b). Applying Lemma 2.1 in [1] with $R(x) := P(x) - 1/N$, we see that (2.1) holds if and only if the restriction of R to $\mathbb{R}$ is N-periodic and the Fourier coefficients a_k in the expansion $R(x) = \sum_{k\in\mathbb{Z}} a_k e^{2\pi ikx/N}$, $x \in \mathbb{R}$, satisfy $a_k = 0$ for $k \in N\mathbb{Z}$; thus, R has the form in (b). For $k \le 0$ and any $\alpha > 0$, Cauchy's theorem implies that

$$\begin{aligned}|a_k| &= \left|\frac{1}{N}\int_0^N R(x)e^{-2\pi ikx/N}\,dx\right| = \frac{1}{N}\left|\int_0^N R(x+iN\alpha)e^{-2\pi ik(x+iN\alpha)/N}\,dx\right|\\ &\le \frac{1}{N}e^{2\pi\alpha k}\int_0^N |R(x+iN\alpha)|\,dx = C_\alpha e^{-2\pi\alpha|k|},\end{aligned}$$

where $C_\alpha := N^{-1}\int_0^N |R(x+iN\alpha)|\,dx$; a similar proof holds for $k > 0$.

(b) $\Rightarrow$ (c). Define $G(z) = \sum_{k\in\mathbb{Z}\setminus N\mathbb{Z}} a_k z^k$. From (b), G is holomorphic in $\mathbb{C} \setminus \{0\}$ and $R(x) = G(e^{2\pi ix/N})$.

(c) $\Rightarrow$ (a). Note that $\sum_{\ell=0}^{N-1} e^{2\pi ik\ell/N} = 0,\ k \notin N\mathbb{Z}$. Hence,

$$\sum_{\ell=0}^{N-1} R(z+\ell) = \sum_{\ell=0}^{N-1} G(e^{2\pi i(z+\ell)/N}) = \sum_{k\in\mathbb{Z}\setminus N\mathbb{Z}} a_k e^{2\pi ikz/N}\Big(\sum_{n=0}^{N-1} e^{2\pi ikn/N}\Big) = 0. \qquad \square$$

Via Lemma 2.1, we now arrive at a characterisation of the entire functions satisfying the partition of unity property.

THEOREM 2.2. *Let $N \in \mathbb{N}$ and consider an entire function $P : \mathbb{C} \to \mathbb{C}$. Then the following are equivalent:*

(a) *P satisfies (1.1);*
(b) *$P(x) = N^{-1} + \sum_{k\in\mathbb{Z}\setminus N\mathbb{Z}} a_k e^{2\pi ikx/N}$ for $x \in \mathbb{R}$ and, for any $\alpha > 0$, there exists a constant C_α such that $|a_k| \le C_\alpha e^{-\alpha|k|}$ for all $k \in \mathbb{Z}$;*
(c) *$P(x) = N^{-1} + G(e^{2\pi ix/N})$ for $x \in \mathbb{R}$ and a function $G(z) = \sum_{k\in\mathbb{Z}\setminus N\mathbb{Z}} a_k z^k$ which is holomorphic in $\mathbb{C} \setminus \{0\}$.*

PROOF. Note that the expression in (1.1) is 1-periodic; thus, (1.1) holds if and only if $\sum_{\ell=0}^{N-1} P(x+\ell) = 1$ for all $x \in [0, 1]$ or, since P is entire, $\sum_{\ell=0}^{N-1} P(x+\ell) = 1$ for all $x \in \mathbb{R}$. Writing $R(x) := P(x) - 1/N$, it is clear that P satisfies (1.1) if and only if R satisfies (2.1); thus, the result follows from Lemma 2.1. $\square$

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3. Partition of unity for entire functions $P : \mathbb{C}^d \to \mathbb{C}$

We will now consider the partition of unity property in dimension d. Given $N \in \mathbb{N}$ and any $j \in \{1, \ldots, d\}$, let us denote the sum over the $\mathbf{n} = (n_1, \ldots, n_j) \in \mathbb{Z}^j$ for which all coordinates are between 0 and $N-1$ by $\sum_{\mathbf{n}\in\mathbb{Z}_N^j} := \sum_{n_1=0}^{N-1} \cdots \sum_{n_j=0}^{N-1}$. Repeating the first steps in the proof of Theorem 2.2, we see that the partition of unity condition (1.1) is equivalent to

$$\sum_{\mathbf{n}\in\mathbb{Z}_N^d} P(\mathbf{x}+\mathbf{n}) = 1 \quad \text{for all } \mathbf{x} \in \mathbb{R}^d. \tag{3.1}$$

Again, we will first characterise the entire functions for which the sum on the left-hand side of (3.1) vanishes.

We will need a classical result about pointwise convergence of Fourier series. Recall that a (possibly noncontinuous) function $f : [0, N] \to \mathbb{C}$ is piecewise continuously differentiable if there exist finitely many $x_0 = 0 < x_1 < \cdots < x_n = N$ such that:

(1) f is continuously differentiable on $]-x_{i-1}, x_i[$ for every $i \in \{1, \ldots, n\}$;
(2) the one-sided limits $f(x_{i-1}^+) := \lim_{x\to x_{i-1}^+} f(x)$ and $f(x_i^-) := \lim_{x\to x_i^-} f(x)$ exist for every $i \in \{1, \ldots, n\}$;
(3) the one-sided limits $\lim_{x\to x_{i-1}^+} f'(x)$ and $\lim_{x\to x_i^-} f'(x)$ exist for every $i \in \{1, \ldots, n\}$.

Define the function $\mathcal{P}_N f$ on $[0, N[$ by

$$\mathcal{P}_N f(x) := \begin{cases} f(x), & x \neq x_i, i = 0, 1, \ldots, n-1, \\ \frac{1}{2}(f(x_i^+) + f(x_i^-)), & x = x_i,\ i = 1, 2, \ldots, n-1, \\ \frac{1}{2}(f(x_0^+) + f(x_n^-)), & x = 0 \end{cases} \tag{3.2}$$

and extend it to be N-periodic in $\mathbb{R}$. Under the stated conditions, it is known that

$$\mathcal{P}_N f(x) = \sum_{k=-\infty}^{\infty} \left(\frac{1}{N} \int_0^N f(t) e^{-2\pi ikt/N}\, dt \right) e^{2\pi ikx/N}, \ x \in \mathbb{R}, \tag{3.3}$$

where the infinite sum should be interpreted as a limit of the symmetric partial sums, that is, $\sum_{k=-\infty}^{\infty} = \lim_{n\to\infty} \sum_{k=-n}^{n}$.

Lemma 3.1. *Let $N \in \mathbb{N}$. Assume that $g : [0, N] \to \mathbb{C}$ is piecewise continuously differentiable. Define $g^\sigma : [0, 1] \to \mathbb{C}$ by $g^\sigma(x) := N^{-1} \sum_{\ell=0}^{N-1} g(x+\ell)$. Then the following hold:*

(a) $\mathcal{P}_1 g^\sigma(x) = \sum_{k\in N\mathbb{Z}} (N^{-1} \int_0^N g(t) e^{-2\pi ikt/N}\, dt) e^{2\pi ikx/N}$, $x \in \mathbb{R}$;
(b) $\mathcal{P}_N g(x) - \mathcal{P}_1 g^\sigma(x) = \sum_{k\in\mathbb{Z}\setminus N\mathbb{Z}} (N^{-1} \int_0^N g(t) e^{-2\pi ikt/N}\, dt) e^{2\pi ikx/N}$, $x \in \mathbb{R}$.

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PROOF. (a) Applying (3.3) to the 1-periodic piecewise continuously differentiable function $\mathcal{P}_1 g^\sigma$,

$$\begin{aligned}\mathcal{P}_1 g^\sigma(x) &= \sum_{k=-\infty}^{\infty}\Big(\int_0^1\Big(\frac{1}{N}\sum_{\ell=0}^{N-1} g(t+\ell)\Big)e^{-2\pi ikt}\,dt\Big)e^{2\pi ikx}\\ &= \sum_{k=-\infty}^{\infty}\Big(\frac{1}{N}\int_0^N g(t)e^{-2\pi ikt}\,dt\Big)e^{2\pi ikx} = \sum_{k\in N\mathbb{Z}}\Big(\frac{1}{N}\int_0^N g(t)e^{-2\pi ikt/N}\,dt\Big)e^{2\pi ikx/N}.\end{aligned}$$

(b) This follows from (a) and (3.3). □

LEMMA 3.2. *Let $N \in \mathbb{N}$ and $j \geq 2$. Consider an entire function $R : \mathbb{C}^j \to \mathbb{C}$ for which*

$$\sum_{\mathbf{n}\in\mathbb{Z}_N^j} R(\mathbf{x}+\mathbf{n}) = 0 \quad \text{for all } \mathbf{x} \in \mathbb{R}^j. \tag{3.4}$$

Then there exist a function $R_p : \mathbb{R}^j \to \mathbb{C}$ of the form

$$R_p(\mathbf{y}, x) = \sum_{k\in\mathbb{Z}\setminus N\mathbb{Z}} r_k(\mathbf{y})e^{2\pi ikx/N} \quad \text{for all } (\mathbf{y}, x) \in \mathbb{R}^{j-1}\times\mathbb{R},$$

with an entire function $r_k : \mathbb{C}^{j-1} \to \mathbb{C}$, and a function $R_h : \mathbb{R}^j \to \mathbb{C}$ such that

$$R(\mathbf{y}, x) = R_p(\mathbf{y}, x) + R_h(\mathbf{y}, x) \quad \text{for all } (\mathbf{y}, x) \in \mathbb{R}^{j-1}\times\mathbb{R}$$

and

$$\sum_{\mathbf{n}\in\mathbb{Z}_N^{j-1}} R_h(\mathbf{y}+\mathbf{n}, x) = 0 \quad \text{for all } (\mathbf{y}, x) \in \mathbb{R}^{j-1}\times\mathbb{R}. \tag{3.5}$$

Moreover, fixing the jth coordinate $x \in \mathbb{R}$, the functions $R_p(\cdot, x), R_h(\cdot, x) : \mathbb{C}^{j-1} \to \mathbb{C}$ are entire.

PROOF. Consider the entire function

$$Q : \mathbb{R}^j \to \mathbb{C},\ \ Q(\mathbf{y}, x) := \sum_{\mathbf{n}\in\mathbb{Z}_N^{j-1}} R(\mathbf{y}+\mathbf{n}, x) \quad \text{for all } (\mathbf{y}, x) \in \mathbb{R}^{j-1}\times\mathbb{R}.$$

Fix $\mathbf{y} \in \mathbb{R}^{j-1}$. Then (3.4) takes the form $\sum_{n=0}^{N-1} Q(\mathbf{y}, x+n) = 0$ for all $x \in \mathbb{R}$. Applying Lemma 2.1 to $Q(\mathbf{y}, \cdot)$ in the place of R,

$$Q(\mathbf{y}, x) = \sum_{k\in\mathbb{Z}\setminus N\mathbb{Z}}\Big(\frac{1}{N}\int_0^N Q(\mathbf{y}, t)e^{-2\pi ikt/N}\,dt\Big)e^{2\pi ikx/N} \quad \text{for all } x \in \mathbb{R}.$$

Define $R_{\mathbf{y}}(x) := R(\mathbf{y}, x)$ and $R_p(\mathbf{y}, x) := \mathcal{P}_N R_{\mathbf{y}}(x) - \mathcal{P}_1 R_{\mathbf{y}}^\sigma(x)$. By Lemma 3.1(b), we have the representation $R_p(\mathbf{y}, x) = \sum_{k\in\mathbb{Z}\setminus N\mathbb{Z}} r_k(\mathbf{y})e^{2\pi ikx/N}$ for all $x \in \mathbb{R}$, where

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$r_k(\mathbf{y}) := N^{-1} \int_0^N R(\mathbf{y}, t)e^{-2\pi ikt/N}\,dt$. Note that $r_k : \mathbb{C}^{j-1} \to \mathbb{C}$ is an entire function and, for $x \in \mathbb{R}$,

$$\begin{aligned}
Q(\mathbf{y}, x) &= \sum_{k\in\mathbb{Z}\setminus N\mathbb{Z}} \bigg(\frac{1}{N}\int_0^N \sum_{\mathbf{n}\in\mathbb{Z}_N^{j-1}} R(\mathbf{y}+\mathbf{n}, t)e^{-2\pi ikt/N}\,dt\bigg)e^{2\pi ikx_j/N} \\
&= \sum_{k\in\mathbb{Z}\setminus N\mathbb{Z}} \bigg(\frac{1}{N}\int_0^N \sum_{\mathbf{n}\in\mathbb{Z}_N^{j-1}} R(\mathbf{y}+\mathbf{n}, t)e^{-2\pi ikt/N}\,dt\bigg)e^{2\pi ikx_j/N} \\
&= \sum_{\mathbf{n}\in\mathbb{Z}_N^{j-1}} \sum_{k\in\mathbb{Z}\setminus N\mathbb{Z}} r_k(\mathbf{y}+\mathbf{n})e^{2\pi ikx_j/N} = \sum_{\mathbf{n}\in\mathbb{Z}_N^{j-1}} R_p(\mathbf{y}+\mathbf{n}, x).
\end{aligned}$$

Thus, the function R_p gives a particular solution of the inhomogeneous equation $\sum_{\mathbf{n}\in\mathbb{Z}_N^{j-1}} R(\mathbf{y}+\mathbf{n}, x) = Q(\mathbf{y}, x)$. The general solution $R(\mathbf{x})$ of this equation has the form

$$R(\mathbf{y}, x) = R_p(\mathbf{y}, x) + R_h(\mathbf{y}, x),$$

where R_h satisfies the homogeneous equation (3.5). We now fix the jth coordinate x. Note that $R_p(\cdot, x) : \mathbb{C}^{j-1} \to \mathbb{C}$ is an entire function; hence also $R_h(\cdot, x) : \mathbb{C}^{j-1} \to \mathbb{C}$ is an entire function. This completes the proof. □

In order to simplify the description of the d-dimensional problem, we will use the following notation. For $\mathbf{x} \in \mathbb{C}^d$ and $j \in \{1, \ldots, d\}$, we write $\mathbf{x}_j := (x_1, \ldots, x_j)$, $\widetilde{\mathbf{x}}_j := (x_{j+1}, \ldots, x_d)$ and $\hat{\mathbf{x}}_j = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_d)$. Then $\hat{\mathbf{x}}_1 = \widetilde{\mathbf{x}}_1$, $\hat{\mathbf{x}}_d = \mathbf{x}_{d-1}$ and $\mathbf{x} = (x_1, \ldots, x_d) = (\mathbf{x}_j, \widetilde{\mathbf{x}}_j) = (\mathbf{x}_{j-1}, x_j, \widetilde{\mathbf{x}}_j)$. We first characterise the solutions to the homogeneous equation associated with the partition of unity property.

Theorem 3.3. *Consider an entire function $R : \mathbb{C}^d \to \mathbb{C}$ and let $N \in \mathbb{N}$. Then the following are equivalent:*

(a) $\sum_{\mathbf{n}\in\mathbb{Z}_N^d} R(\mathbf{x}+\mathbf{n}) = 0$ *for all* $\mathbf{x} \in \mathbb{R}^d$;

(b) *R has the form $R(\mathbf{x}) = \sum_{j=1}^d R_p^j(\mathbf{x})$, where $R_p^j(\mathbf{x}) = \sum_{k\in\mathbb{Z}\setminus N\mathbb{Z}} r_k^j(\hat{\mathbf{x}}_j)e^{2\pi ikx_j/N}$ and the functions $r_k^j : \mathbb{C}^{d-1} \to \mathbb{C}$ are entire.*

Proof. (a) ⇒ (b). We use induction for $j = 2, \ldots, d$ in reverse order. Assume that $R_h^j(\cdot, \widetilde{\mathbf{x}}_j) : \mathbb{C}^j \to \mathbb{C}$, $j = 2, \ldots, d$, is an entire function such that $\sum_{\mathbf{n}\in\mathbb{Z}_N^j} R_h^j(\mathbf{x}_j+\mathbf{n}, \widetilde{\mathbf{x}}_j) = 0$ for $\mathbf{x}_j \in \mathbb{R}^j$, with $R_h^d(\mathbf{x}) := R(\mathbf{x})$, $\mathbf{x} \in \mathbb{C}^d$. Applying Lemma 3.2 to $R_h^j(\cdot, \widetilde{\mathbf{x}}_j)$ in place of $R(\cdot)$ and $\widetilde{\mathbf{x}}_j$ fixed,

$$R_h^j(\mathbf{x}_{j-1}, x_j, \widetilde{\mathbf{x}}_j) = R_p^j(\mathbf{x}_{j-1}, x_j, \widetilde{\mathbf{x}}_j) + R_h^{j-1}(\mathbf{x}_{j-1}, x_j, \widetilde{\mathbf{x}}_j), \quad (\mathbf{x}_{j-1}, x_j) \in \mathbb{R}^{j-1}\times\mathbb{R},$$

where R_p^j has the form $R_p^j(\mathbf{x}_{j-1}, x_j, \widetilde{\mathbf{x}}_j) = \sum_{k\in\mathbb{Z}\setminus N\mathbb{Z}} r_k^j(\hat{\mathbf{x}}_j)e^{2\pi ikx_j/N}$ and R_h^{j-1} satisfies $\sum_{\mathbf{n}\in\mathbb{Z}_N^{j-1}} R_h^{j-1}(\mathbf{x}_{j-1}+\mathbf{n}, x_j, \widetilde{\mathbf{x}}_j) = 0$. Moreover, fixing $(x_j, \widetilde{\mathbf{x}}_j) \in \mathbb{R}\times\mathbb{R}^{d-j}$, the function $R_h^{j-1}(\cdot, x_j, \widetilde{\mathbf{x}}_j) : \mathbb{C}^{j-1} \to \mathbb{C}$ is entire. Thus, $R(\mathbf{x}) = \sum_{j=2}^d R_p^j(\mathbf{x}_{j-1}, x_j, \widetilde{\mathbf{x}}_j) + R_h^1(x_1, \widetilde{\mathbf{x}}_1)$, where R_h^1 satisfies $\sum_{n=0}^{N-1} R_h^1(x_1+n, \widetilde{\mathbf{x}}_1) = 0$. Let $R_p^1 := R_h^1$. Fixing $\widetilde{\mathbf{x}}_1 \in \mathbb{R}^{d-1}$, by Lemma 2.1, R_p^1 has the form $R_p^1(x_1, \widetilde{\mathbf{x}}_1) = \sum_{k\in\mathbb{Z}\setminus N\mathbb{Z}} r_k^1(\hat{\mathbf{x}}_1)e^{2\pi ikx_1/N}$. Hence (b) holds.

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(b) ⇒ (a). Under the assumption in (b), a similar argument as the one used in the proof of Lemma 2.1 (c) ⇒ (a) yields

$$\sum_{\mathbf{n}\in\mathbb{Z}_N^d} R(\mathbf{x}+\mathbf{n}) = \sum_{j=1}^{d}\sum_{n_j=0}^{N-1}\sum_{\mathbf{n}_{d-1}\in\mathbb{Z}_N^{d-1}}\sum_{k\in\mathbb{Z}\setminus N\mathbb{Z}} r_k^j(\hat{\mathbf{x}}_j+\mathbf{n}_{d-1})e^{2\pi ik(x_j+n_j)/N}$$
$$= \sum_{j=1}^{d}\sum_{\mathbf{n}_{d-1}\in\mathbb{Z}_N^{d-1}}\sum_{k\in\mathbb{Z}\setminus N\mathbb{Z}} r_k^j(\hat{\mathbf{x}}_j+\mathbf{n}_{d-1})\Big(\sum_{n_j=0}^{N-1} e^{2\pi ikn_j/N}\Big)e^{2\pi ikx_j/N} = 0,$$

as desired. □

In Theorem 3.3(b), the functions R_p^j may not be entire. As demonstrated by the following example, they might not even be continuous.

EXAMPLE 3.4. Let $N = d = 2$. Consider the entire function $R(x_1, x_2) := x_1 e^{\pi ix_2} + x_2 e^{\pi ix_1}$. By Proposition 3.3, R satisfies

$$R(x_1, x_2) + R(x_1, x_2 + 1) + R(x_1 + 1, x_2) + R(x_1 + 1, x_2 + 1) = 0, \quad (x_1, x_2) \in \mathbb{R}.$$

As in the proof of Lemma 3.2, we write $R(x_1, x_2) = R_p(x_1, x_2) + R_h(x_1, x_2)$. Fix x_1. Define R_p by

$$R_p(x_1, x_2) := \sum_{k\in\mathbb{Z}\setminus 2\mathbb{Z}}\Big(\frac{1}{2}\int_0^2 R(x_1, t)e^{-\pi ikt}\,dt\Big)e^{\pi ikx_2} = \sum_{k\in\mathbb{Z}\setminus 2\mathbb{Z}}\Big(x_1\delta_{k,1} - \frac{e^{\pi ix_1}}{\pi ik}\Big)e^{\pi ikx_2}.$$

From (3.2),

$$\mathcal{P}_2 R(x_1, x_2) = \begin{cases} x_1 e^{\pi ix_2} + x_2 e^{\pi ix_1}, & x_2 \in [0, 2] \setminus \{0, 2\},\\ x_1 + e^{\pi ix_1}, & x_2 \in \{0, 2\},\end{cases}$$
$$\mathcal{P}_1 R^{\sigma}(x_1, x_2) = \begin{cases} (x_2 + \frac{1}{2})e^{\pi ix_1}, & x_2 \in [0, 2] \setminus \{0, 1, 2\},\\ e^{\pi ix_1}, & x_2 \in \{0, 1, 2\}.\end{cases}$$

By Lemma 3.1(b) and (3.6),

$$R_p(x_1, x_2) = \begin{cases} x_1 e^{\pi ix_2} - \frac{1}{2}e^{\pi ix_1}, & x_2 \in [0, 1] \setminus \{0, 1\},\\ x_1 e^{\pi ix_2} + \frac{1}{2}e^{\pi ix_1}, & x_2 \in [1, 2] \setminus \{1, 2\},\\ x_1, & x_2 \in \{0, 2\},\\ -x_1, & x_2 = 1.\end{cases}$$

Thus, $R_p(x_1, \cdot)$ is not continuous, for example, in $x_2 = 0$.

Applying Proposition 3.3 with $R(\mathbf{x}) := P(\mathbf{x}) - 1/N^d$, we immediately obtain the desired characterisation of the partition of unity property.

THEOREM 3.5. *Consider an entire function $P : \mathbb{C}^d \to \mathbb{C}$ and let $N \in \mathbb{N}$. Then the following are equivalent:*

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(a) *P satisfies (1.1);*
(b) *P has the form* $P(\mathbf{x}) = N^{-d} + \sum_{j=1}^{d} P^j(\mathbf{x})$ *for some functions*

$$P^j(\mathbf{x}) = \sum_{k\in\mathbb{Z}\setminus N\mathbb{Z}} r_k^j(\hat{\mathbf{x}}_j)e^{2\pi i k x_j/N}$$

with entire coefficient functions $r_k^j : \mathbb{C}^{d-1} \to \mathbb{C}$.

As a consequence of Theorem 3.5, we immediately obtain the following easy way of constructing entire functions having the partition of unity property.

Corollary 3.6. *Consider a finite set* $\mathcal{K} \subset \mathbb{Z} \setminus N\mathbb{Z}$ *and a collection of entire functions* r_k^j, $k \in \mathcal{K}$, $j \in \{1, 2, \ldots, d\}$. *Define* $P : \mathbb{C}^d \to \mathbb{C}$ *by*

$$P(\mathbf{x}) = \frac{1}{N^d} + \sum_{j=1}^{d}\sum_{k\in\mathcal{K}} r_k^j(\hat{\mathbf{x}}_j)e^{2\pi i x_j k/N}, \ \mathbf{x} \in \mathbb{C}^d. \tag{3.6}$$

Then P is an entire function and (1.1) holds.

Corollary 3.7. *For a collection of entire functions* r_k^j, $k \in \mathbb{Z} \setminus N\mathbb{Z}$, $j \in \{1, \ldots, d\}$ *such that for each j,* $\sum_{k\in\mathbb{Z}\setminus N\mathbb{Z}} r_k^j(\hat{\mathbf{x}}_j)$ *converges uniformly on compact subsets* $\mathcal{A} \subset \mathbb{C}^{d-1}$,

$$P(\mathbf{x}) = \frac{1}{N^d} + \sum_{j=1}^{d}\sum_{k\in\mathbb{Z}\setminus N\mathbb{Z}} r_k^j(\hat{\mathbf{x}}_j)e^{2\pi i x_j k/N}, \ \mathbf{x} \in \mathbb{C}^d,$$

is an entire function and (1.1) holds.

As an illustration of Corollary 3.7, consider for $k \in \mathbb{Z}$ and $j \in \{1, \ldots, d\}$ some monomials $\mathbf{x}^{\alpha(j,k)} = x_1^{\alpha_1}\cdots x_{d-1}^{\alpha_{d-1}}$, where $|\alpha(j,k)| := \alpha_1 + \cdots + \alpha_{d-1} = |k|$. Letting $r_k^j(\mathbf{x}) := \mathbf{x}^{\alpha(j,k)}/|k|!$, the function

$$P(\mathbf{x}) = \frac{1}{N^d} + \sum_{j=1}^{d}\sum_{k\in\mathbb{Z}\setminus N\mathbb{Z}} r_k^j(\hat{\mathbf{x}}_j)e^{2\pi i x_j k/N}, \quad \mathbf{x} \in \mathbb{C}^d,$$

is an entire function and (1.1) holds.

We will now demonstrate the flexibility of the approach by constructing an entire function which satisfies the partition of unity condition and vanishes on the boundary of $[0, N]^d$. We first observe that $P\chi_{[0,N]^d} \in C(\mathbb{R}^d)$ if and only if for $j_0 \in \{1, \ldots, d\}$ and $\alpha \in \{0, N\}$, $P(x_1, \ldots, x_{j_0-1}, \alpha, x_{j_0+1}, \ldots, x_d) = 0$. Thus, it is sufficient to search for a function P of the form (3.6) for some entire functions $r_k^j : \mathbb{C}^{d-1} \to \mathbb{C}$, $k \in \mathcal{K}$, and such that for $j_0 \in \{1, \ldots, d\}$ and $\alpha \in \{0, N\}$,

$$\begin{aligned} 0 = \frac{1}{N^d} &+ \sum_{j=1}^{j_0-1}\sum_{k\in\mathcal{K}} r_k^j(\mathbf{x}_{j-1}, (x_{j+1}, \ldots, x_{j_0-1}, \alpha, x_{j_0+1}, \ldots, x_d))e^{2\pi i x_j k/N} \\ &+ \sum_{j=j_0+1}^{d}\sum_{k\in\mathcal{K}} r_k^j((x_1, \ldots, x_{j_0-1}, \alpha, x_{j_0+1}, \ldots, x_{j-1}), \widetilde{\mathbf{x}}_j)e^{2\pi i x_j k/N} + \sum_{k\in\mathcal{K}} r_k^{j_0}(\hat{\mathbf{x}}_{j_0}). \end{aligned} \tag{3.7}$$

Let us consider a concrete example in $\mathbb{R}^2$.

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EXAMPLE 3.8. Let $d = 2$, $N = 2$ and $\mathcal{K} = \{1, 3\}$. Then (3.7) means that for $\alpha \in \{0, 2\}$,

$$\frac{1}{4} + \sum_{k\in\mathcal{K}} r_k^2(\alpha)e^{\pi i x_2 k} + \sum_{k\in\mathcal{K}} r_k^1(x_2) = 0, \quad \frac{1}{4} + \sum_{k\in\mathcal{K}} r_k^1(\alpha)e^{\pi i x_1 k} + \sum_{k\in\mathcal{K}} r_k^2(x_1) = 0. \tag{3.8}$$

Consider entire functions $r_k^1,\ r_k^2 : \mathbb{C} \to \mathbb{C}$, $k = 1, 3$, defined by

$$r_k^1(x) = \begin{cases} -\frac{1}{4}, & k = 1, \\ 0, & k = 3, \end{cases} \qquad r_k^2(x) = \begin{cases} -x(x-2) - \frac{1}{4} + \frac{1}{4}e^{\pi i x}, & k = 1, \\ x(x-2), & k = 3. \end{cases}$$

Then we have, for $\alpha \in \{0, 2\}$,

$$\frac{1}{4} + \sum_{k\in\mathcal{K}} r_k^2(\alpha)e^{\pi i x_2 k} + \sum_{k\in\mathcal{K}} r_k^1(x_2) = \frac{1}{4} + r_1^1(x_2) = 0,$$
$$\frac{1}{4} + \sum_{k\in\mathcal{K}} r_k^1(\alpha)e^{\pi i x_1 k} + \sum_{k\in\mathcal{K}} r_k^2(x_1) = \frac{1}{4} - \frac{1}{4}e^{\pi i x_1} + \sum_{k\in\mathcal{K}} r_k^2(x_1) = 0.$$

Thus, (3.8) holds. Define P by

$$P(x_1, x_2) = \tfrac{1}{4} - \tfrac{1}{4}e^{\pi i x_1} + (-x_1(x_1-2) - \tfrac{1}{4} + \tfrac{1}{4}e^{\pi i x_1})e^{\pi i x_2} + x_1(x_1-2)e^{3\pi i x_2}.$$

Then $P\chi_{[0,2]^2} \in C(\mathbb{R}^2)$ and $\sum_{\mathbf{n}\in\mathbb{Z}_2^2} P(\mathbf{x} + \mathbf{n}) = 1$ for all $\mathbf{x} \in \mathbb{R}^2$.

Higher-order regularity of $P\chi_{[0,N]^d}$ can be controlled in a similar fashion, just involving a larger number of equations. For example, $P\chi_{[0,N]^d} \in C^1(\mathbb{R}^d)$ if and only if for $j_0, \ell \in \{1, \dots, d\}$ and $\alpha \in \{0, N\}$,

$$P(x_1, \dots, x_{j_0-1}, \alpha, x_{j_0+1}, \dots, x_d) = 0 = \frac{\partial P}{\partial x_\ell}(x_1, \dots, x_{j_0-1}, \alpha, x_{j_0+1}, \dots, x_d).$$

In particular, these criteria can be used to ensure that the functions $P\chi_{[0,N]^d}(\cdot + n)$ for $n \in \mathbb{Z}$ in a partition of unity belong to certain function spaces arising in the literature. For example, it is known that a function $f \in L^1(\mathbb{R}^d)$ belongs to the Feichtinger algebra [3] if also the Fourier transform $\widehat{f} \in L^1(\mathbb{R}^d)$. This is satisfied if $f \in C_c^1(\mathbb{R}^d)$. Let us illustrate this with a concrete construction in $\mathbb{R}^2$. For a recent and detailed description of the Feichtinger algebra and its many applications, we refer to the paper [5].

EXAMPLE 3.9. Let $d = 2$, $N = 2$ and $\mathcal{K} = \{-3, -1, 1, 3\}$. Let

$$r_1^1(x_2) = r_{-1}^1(x_2) = \tfrac{1}{16}(e^{\pi i x_2} + e^{-\pi i x_2} - 2),$$
$$r_3^1(x_2) = r_{-3}^1(x_2) = 0,$$
$$r_1^2(x_1) = -\tfrac{1}{8}(3ix_1^2(x_1-2)^2 + 1), \quad r_{-1}^2(x_1) = \tfrac{1}{8}(3ix_1^2(x_1-2)^2 - 1),$$
$$r_3^2(x_1) = -r_{-3}^2(x_1) = \frac{i}{8}x_1^2(x_1-2)^2$$

and define P by

$$P(x_1, x_2) = \frac{1}{4} + \sum_{k\in\mathcal{K}} r_k^1(x_2)e^{\pi i x_1 k} + \sum_{\ell\in\mathcal{K}} r_\ell^2(x_1)e^{\pi i x_2 \ell}.$$

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A direct calculation shows that

$$P(x_1, x_2) = \sin^2(\pi x_1/2)\sin^2(\pi x_2/2) + x_1^2(x_1 - 2)^2 \sin^3(\pi x_2).$$

Then $P\chi_{[0,2]^2} \in C_c^1(\mathbb{R}^2)$ and $\sum_{\mathbf{n}\in\mathbb{Z}_2^2} P(\mathbf{x}+\mathbf{n}) = 1$ for all $\mathbf{x} \in \mathbb{R}^2$. In particular, the function $P\chi_{[0,2]^2}$ and its translates belong to the Feichtinger algebra. □

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References

[1] O. Christensen, H. O. Kim and R. Y. Kim, 'On entire functions restricted to intervals, partition of unities, and dual Gabor frames', *Appl. Comput. Harmon. Anal.* **38** (2015), 72–86.

[2] O. Christensen, H. O. Kim and R. Y. Kim, 'On partition of unities generated by entire functions and Gabor frames in $L^2(\mathbb{R}^d)$ and $\ell^2(\mathbb{Z}^d)$', *J. Fourier Anal. Appl.* **22** (2016), 1121–1140.

[3] H. G. Feichtinger, 'On a new Segal algebra', *Monatsh. Math.* **92** (1981), 269–289.

[4] H. G. Feichtinger, 'Atomic characterizations of modulation spaces through Gabor-type representations', *Rocky Mountain J. Math.* **19** (1989), 113–125.

[5] M. S. Jakobsen, 'On a (no longer new) Segal algebra', Preprint, 2016.

[6] A. J. E. M. Janssen, 'The duality condition for Weyl–Heisenberg frames', in: *Gabor Analysis: Theory and Applications* (eds. H. G. Feichtinger and T. Strohmer) (Birkhäuser, Boston, 1998), 33–84.

[7] A. Ron and Z. Shen, 'Frames and stable bases for shift-invariant subspaces of $L^2(\mathbb{R}^d)$', *Canad. J. Math.* **47**(5) (1995), 1051–1094.

[8] W. Rudin, *Real and Complex Analysis* (McGraw-Hill, New York, 1986).

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