A NOTE ON QUASI-FROBENIUS RINGS AND RING EPIMORPHISMS

H. H. Storrer*

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0. In this note, we characterize quasi-Frobenius rings by a weakened form of the usual condition, that every ideal is an annihilator ideal.

We then apply this result to pure rings in the sense of Cohn and to dominant rings, a concept arising in the study of ring epimorphisms. All rings considered have a unit element.

1. A ring $A$ is called quasi-Frobenius, if it is left and right Artinian and if the conditions

\[
\begin{align*}
(a^l) & \quad \ell(r(L)) = L \quad \text{and} \\
(a^r) & \quad r(\ell(R)) = R
\end{align*}
\]

are satisfied for all left ideals $L$ and right ideals $R$, $\ell(X)$ and $r(X)$ denoting the left and right annihilators of a subset $X$ of $A$.

Clearly, it suffices to assume the minimum condition only on one side. (The maximum condition would also do.)

On the other hand, Dieudonné has proved [3], that, for a left and right Artinian ring, it is sufficient to assume $(a^l)$ and $(a^r)$ for all minimal left and right ideals only. We denote these modified conditions by $(a'^l)$ and $(a'^r)$. Let us note the well-known fact, that $(a'^l)$ and $(a'^r)$ hold automatically for the non-nilpotent minimal ideals, since they are generated by an idempotent.

We first show, that for Dieudonné's result, one needs only the minimum condition on the left.

* N. R. C. of Canada Postdoctoral Fellow.


287
Recall, that a ring $A$ is left perfect, it has the minimum condition on principal right ideals or, equivalently, if the Jacobson radical $\text{Rad} A$ is T-nilpotent and $A/\text{Rad} A$ is completely reducible (i.e. semisimple Artinian) [1]. In particular, a left Artinian ring is both left and right perfect.

PROPOSITION 1. Let $A$ be a left Artinian ring such that the conditions $(a \, ')$ and $(a \, '')$ hold. Then $A$ is a quasi-Frobenius ring.

Proof. Consider the condition $(d \, r)$: The dual $M^* = \text{Hom}_A(M, A)$ (which is a right $A$-module in a canonical way) of any simple left $A$-module $M$ is simple or zero. $(d \, r)$ is defined similarly.

Dieudonné [3, 3.4] has shown, that a left and right Artinian ring satisfying $(d \, l)$ and $(d \, r)$ is quasi-Frobenius. He also proved [3, 3.4] that for $A$ a left Artinian $(d \, l)$ implies the existence of a composition series for the right $A$-module $A \cong (A^n)^r$. Thus $A$ is also right Artinian, and we only have to show, that the implications $(a \, '') \Rightarrow (d \, l)$ and $(a \, '') \Rightarrow (d \, r)$ hold for a left Artinian ring $A$.

This requires a slight modification of Dieudonné's arguments as follows: if $M$ is simple, then $M \cong A/I$, where $I$ is a maximal left ideal and $M^* \cong r(I)$. If $r(I)$ were not simple or zero, then it would properly contain a minimal right ideal $R$, since $A$ satisfies the minimum condition on principal right ideals. $0 \subset R \subset r(I)$ implies $A \supset (R) \supset (r(I)) = I$, and since $I$ is maximal, this yields $(R) = A$ or $(R) = I$, but both cases are impossible, by virtue of $(a \, '')$. Thus $(a \, '') \Rightarrow (d \, l)$ and similarly $(a \, '') \Rightarrow (d \, r)$.

q.e.d.

If $A$ is merely Noetherian, then $(a \, '')$ and $(a \, '')$ are of course not sufficient, even if there exist proper minimal ideals. In this case we have to assume, that $(a \, l)$ and $(a \, r)$ are satisfied for all principal left and right ideals. We will denote these new conditions by $(a \, l \, ')$ and $(a \, r \, ')$.

PROPOSITION 2. Let $A$ be a left Noetherian ring such that the conditions $(a \, l \, ')$ and $(a \, r \, ')$ are satisfied. Then $A$ is a quasi-Frobenius ring.

Proof. Since $A$ is left Noetherian, it satisfies the maximum condition for left annihilator ideals (i.e. left ideals $I$ such that $(r(I)) = I$) and hence the minimum condition for right annihilator
ideals. But then, by \((a \,')\), \(A\) satisfies the minimum condition on principal right ideals and is therefore left perfect. It is well known, that a left Noetherian left (or right) perfect ring is left Artinian, and we can thus apply Prop. 1. q.e.d.

We note, that for this proof one needed only \((a \,')\) and not \((a \,')\).

Since, in sections 2 and 3, we will consider only commutative rings, it may be useful to give a simple criterion for a commutative Artinian ring to be quasi-Frobenius.

The left socle \(\text{Soc}_l(A)\) is defined to be the sum of all minimal left ideals of \(A\). It is well known, that for an Artinian ring \(A\) one has \(\text{Soc}_l(A) = r(Rad A)\).

**PROPOSITION 3.** A commutative Artinian ring \(A\) is a quasi-Frobenius ring if and only if its socle \(S\) is a principal ideal.

**Proof.** \(A\) is a product of a finite number of local Artinian rings and it is enough to prove the proposition for the local case.

If \(A\) is local with radical \(N\) and if it is quasi-Frobenius, then the annihilator of every minimal ideal is \(N\); thus there exists only one minimal ideal and \(S\) is clearly principal.

Conversely, let \(S\) be a principal ideal generated by \(s\). \(S\) is a finite direct sum of minimal ideals \(M_i (i = 1, \ldots, n)\) and \(s\) has a unique decomposition \(s = s_1 + \ldots + s_n \ (s_i \in M_i, s_i \neq 0)\). If \(n > 1\), then there must be an \(a \in A\) such that \(as = s_i\), i.e. \(as = s_1\) and \(as_2 = \ldots = as_n = 0\). Since \(S\) annihilates \(N\), \(a \notin N\), but then \(a\) has an inverse and we have a contradiction. Thus \(n = 1\) and \(A\) has a unique minimal ideal \(M = S\), which is an annihilator. Dieudonné's result implies then, that \(A\) is a quasi-Frobenius ring. q.e.d.

**Remarks.**

(i) In a special case, the above proof can be generalized to the non-commutative case: Let \(A\) be a left and right Artinian ring which is a direct product of completely primary rings. Then \(A\) is quasi-Frobenius if and only if \(\text{Soc}_l(A)\) is a principal left ideal and \(\text{Soc}_r(A)\) is a principal right ideal. (Of course, one has then \(\text{Soc}_l(A) = \text{Soc}_r(A)\).)
(ii) Statement (i) is not true for general Artinian rings. Nakayama [6] has proved, that a finite-dimensional algebra over a field is a Frobenius algebra if and only if $\text{Soc}(A)$ is a principal right ideal, and there exist quasi-Frobenius algebras which are not Frobenius.

(iii) Prop. 3 together with Nakayama's result implies that a commutative quasi-Frobenius algebra over a field is a Frobenius algebra. This fact has recently been noted by Wenger [8].

2. Let $A$ be a subring of the commutative ring $B$. The dominion $\text{Dom}(A,B)$ of $A$ in $B$ is the set of all $d \in B$ such that $f(d) = g(d)$ for all pairs of ring homomorphisms with domain $B$ and common range, coinciding on $A$. One can show, that it doesn't matter, if the common range of $f$ and $g$ runs through all rings or through the commutative rings only. The dominion has, in a more general setting, been defined by Isbell [4].

The embedding $A \subset B$ is an epimorphism in the category of commutative rings (or, equivalently, of all rings) if and only if $\text{Dom}(A,B) = B$.

If $\text{Dom}(A,B) = A$ for all $B$, then we call $A$ dominant, thereby modifying Isbell's terminology.

It is known, that a self-injective ring is dominant [7, Kor. 5.4] and that in a dominant ring every principal ideal is an annihilator [7, Kor. 4.4]. Since a quasi-Frobenius ring is self-injective, Prop. 2 implies immediately.

PROPOSITION 4. A commutative Noetherian ring is dominant if and only if it is a quasi-Frobenius ring.

We say, that $A$ is strongly dominant [7, Def. 3.2] if every homomorphic image of $A$ (including $A$) is dominant. Levy [5] has shown, that every homomorphic image of a commutative Noetherian ring $A$ is self-injective if and only if $A$ is an Artinian principal ideal ring. This yields

PROPOSITION 5. A commutative Noetherian ring is strongly dominant if and only if it is an Artinian principal ideal ring.

3. A submodule $N$ of a left $A$-module $M$ is called pure in $M$ if the sequence $0 \to P \otimes_A N \to P \otimes_A M$ is exact for every right $A$-module $P$. A ring $A$ is called left pure if the left $A$-module $A$ is a pure $A$-submodule of every ring $B$ containing $A$. These definitions are due to Cohn [2]. Left self-injective rings for example are left pure.
The notion of a commutative pure ring may be ambiguous, in that it may depend on the category of rings under consideration. In fact, we do not know, if there exists a commutative ring \( A \) which is a pure \( A \)-submodule of every commutative ring \( B \supseteq A \), but which is not a pure \( A \)-submodule of a certain noncommutative ring \( C \supseteq A \). (A similar problem arises for dominant rings).

However, in the case considered here, it happens, that no such ambiguity exists; in fact, we have

**Proposition 6.** A commutative Noetherian ring is pure if and only if it is a quasi-Frobenius ring.

**Proof.** By [7, Lemma 5.3], we have the implication pure \( \Rightarrow \) dominant and by Prop. 4 we have dominant \( \Rightarrow \) quasi-Frobenius.

Thus a commutative Noetherian pure ring is self-injective and therefore has a property, which could be called "absolutely pure": it is a pure submodule of every module containing it. This justifies the remark preceding Prop. 6. By [7, Satz 6.1], commutative semiprime pure rings are also "absolutely pure" and so are all pure commutative rings, which are algebras over a field, by a remark of Cohn [2].

Finally, let us remark, that using Theorem 5.5. of [2], one easily proves, that a (not necessarily commutative) Noetherian algebra over a field is left pure if and only if it is a quasi-Frobenius algebra.

**REFERENCES**


McGill University
Montreal

Cornell University
Ithaca, N.Y.