Exceptional Sets of Slices for Functions From the Bergman Space in the Ball

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Abstract. Let $B_N$ be the unit ball in $\mathbb{C}^N$ and let $f$ be a function holomorphic and $L^2$-integrable in $B_N$. Denote by $E(B_N, f)$ the set of all slices of the form $\Pi = L \cap B_N$, where $L$ is a complex one-dimensional subspace of $\mathbb{C}^N$, for which $f \mid \Pi$ is not $L^2$-integrable (with respect to the Lebesgue measure on $L$). Call this set the exceptional set for $f$. We give a characterization of exceptional sets which are closed in the natural topology of slices.

1 Introduction

Let $B_N$ be the unit ball in $\mathbb{C}^N$. We have proved in [3] that there exists a function $f$ holomorphic in $B_N$ such that for every complex subspace $L$ of $\mathbb{C}^N$, $f \mid L \cap B_N \notin L^2(L \cap B_N)$ (where the space $L^2(L \cap B_N)$ is considered with respect to the Lebesgue measure in $L \cap B_N$). In this note we are interested in another problem: Let $E$ be a subset of the slices of the form $\Pi = L \cap B_N$, where $L$ is a complex one-dimensional subspace of $\mathbb{C}^N$. We are interested in determining those $E$ for which there exists a function $f$ holomorphic in $B_N$ and $L^2$-integrable with respect to the Lebesgue measure (we write $f \in L^2_H(B_N)$) such that for every one-dimensional complex subspace $L$ of $\mathbb{C}^N$, $f \mid L \cap B_N \notin L^2(L \cap B_N)$ (with respect to the Lebesgue measure in $L$) iff $L \cap B_N \in E$. Let $\tilde{E} = \bigcup \{ L \cap \partial B_N \mid L \cap B_N \in E \}$. Denote by $\nu$ the surface measure on $\partial B_N$. If a function $f$ with the above described properties exists then, by Fubini’s theorem, $\nu(\tilde{E}) = 0$.

We can identify $E$ with a subset $\hat{E}$ of the complex projective space $\mathbb{CP}^N$. Similarly to [2] one can prove that $\hat{E}$ must be a $G_\delta$-set in the natural topology of $\mathbb{CP}^N$: this is equivalent to say that $\tilde{E}$ is a $G_\delta$-subset of $\partial B_N$. Following [1] or [2] we will call the set $E$ the exceptional set of complex slices for $f$, and denote it by $E(B_N, f)$.

We will prove the following theorem:

**Theorem 1** Let $E$ be a subset of one-dimensional complex slices such that $\nu(\tilde{E}) = 0$, and $\hat{E}$ is closed in $\mathbb{CP}^N$ (this is equivalent to assume that $\tilde{E}$ is closed in $\partial B_N$). Then there exists a function $f \in L^2_H(B_N)$ such that $E(B_N, f) = E$.

A weaker result would be the following: Given a set $E$ of one-dimensional complex slices with $\nu(\tilde{E}) = 0$ and $\hat{E}$ closed in $\mathbb{CP}^N$, find a bounded domain of holomorphy $C$ with $0 \in C$ and a function $f \in L^2_H(C)$ such that for every one-dimensional complex...
subspace $L$ of $\mathbb{C}^N$, $f$ is not $L^2$-integrable on $L \cap C$ if and only if $L \cap B_N \in E$. (In this case, we will write $E = E(C, f)$).

We begin with such a weaker result, i.e., we prove the following:

**Theorem 2** Let $E$ be as in Theorem 1. Then there exists a strictly convex and balanced domain $C$ in $\mathbb{C}^N$ and a function $f \in L^2(\mathbb{C})$ such that $E(C, f) = E$.

(We recall that a domain $C \subset \mathbb{C}^N$ is called balanced if for every $z \in C$ and every $\lambda \in \mathbb{C}$ with $|\lambda| \leq 1$, $\lambda z \in C$).

The reason to prove first Theorem 2, which is weaker than Theorem 1 is because of the clarity of the construction. One of the main ingredients of the proof of Theorem 2 is the following result by Wojtaszczyk:

**Theorem 3** ([4], Theorem 1) There exists an integer $K = K(N)$ and a sequence $\{p_n\}$ of homogeneous polynomials in $\mathbb{C}^N$ of degree $n$ (for $n$ large enough, say $n \geq N_0$) such that

\begin{align*}
(1) & \quad |p_n(z)| \leq 2 \text{ for all } z \in \partial B_N; \\
(2) & \quad \text{for each } s \text{ large enough, say } s \geq S_0, \sum_{n=Ks}^{K(s+1)-1} |p_n(z)| \geq 0, 5 \text{ for all } z \in \partial B_N.
\end{align*}

In the proof of Theorem 2 we use this result exactly in the form stated in Theorem 3; in order to prove Theorem 1 we need first to show that the assertion of Theorem 3 holds also for strictly convex and balanced domains which are in some sense not too far from the unit ball; this requires further explanations, which might obscure the main proof.

In the sequel, we will denote by $B_N(z, r)$ the ball with center $z \in \mathbb{C}^N$ and of radius $r$, and $D(w, r)$ will denote the disc in the complex plane, centered at $w \in \mathbb{C}$, and of radius $r$. Also, we set $U$ to be the unit disc in $\mathbb{C}$.

If $D$ is a domain in $\mathbb{C}^N$, and $h \in L^2(D)$, we will denote by $\|h\|_D$ the $L^2$-norm of $h$ in $D$. The Lebesgue measure (of arbitrary dimension) in a subset of $\mathbb{C}^N$ or of a subspace of $\mathbb{C}^N$ will be denoted by $m$.

## 2 The Exceptional Sets of Complex Planes in $\mathbb{C}^N$

In this section we will prove Theorem 2. We will begin with the result which is rather obvious, and can be proved by standard methods:

**Lemma 4** Let $E$ be a closed subset of $\mathbb{C}P^N$. Then there exists a strictly convex domain $C \subset \mathbb{C}^N$ such that $C \subset B_N$, $\partial C \cap \partial B_N = E$, $\partial C \setminus \partial B_N \subset B_N$, and $C$ is balanced. Moreover, there exists a function $\sigma$, which is strictly convex and smooth in $\mathbb{C}^N$, is non-negatively homogeneous (i.e., $\sigma(\lambda z) = |\lambda|\sigma(z)$ for $z \in \mathbb{C}^N$ and $\lambda \in \mathbb{C}$), and which is a defining function for $C$ (i.e., $C = \{z \in \mathbb{C}^N \mid \sigma(z) < 1\}$ and $\text{grad } \sigma(w) \neq 0$ for $w \in \partial C$).

Let $\sigma$ be a defining function for $C$, with the properties listed in Lemma 4. Given $w \in \mathbb{C}^N$, $w \neq 0$, denote by $[w]$ the class of $w$ in $\mathbb{C}P^N$. For $[w] \in \mathbb{C}P^N$, set $\sigma([w]) = \sigma(w)$. The Lebesgue measure (of arbitrary dimension) in a subset of $\mathbb{C}P^N$ or of a subspace of $\mathbb{C}P^N$ will be denoted by $m$. The Exceptional Sets of Complex Planes in $\mathbb{C}^N$
Suppose also that $f \in \mathcal{O}(B_N)$ (the space of functions holomorphic in $B_N$) is such that for every $z \in \mathbb{C}^N$ with $\|z\| = 1$, for every $0 < r < 1$,
\[
\|f\|_{\{\lambda|z|<r\}}^2 = \int_{D(0,r)} |f(\lambda z)|^2 \, dm(\lambda) \leq \psi(r).
\]
Then there exists a constant $c > 0$, independent of $f$, such that
\[
\int_{\mathbb{C}^N} |f(z)|^2 \, dm(z) \leq c \int_{\mathbb{C}^N} \left( \int_{D(0,\lambda)} |f(\lambda w)|^2 \, dm(\lambda) \right) \, d\tilde{\nu}([w])
\]
\[
\leq c \int_{\mathbb{C}^N} \psi \left( \frac{1}{\tilde{\sigma}([w])} \right) \, d\tilde{\nu}([w])
\]
(4) \hspace{1cm} 

**Lemma 5** Suppose that $E$ is as in Theorems 1 or 2. Let $C$ be a strictly convex and balanced domain in $\mathbb{C}^N$, constructed with respect to $E$ according to Lemma 4. Then there exists $\psi$ satisfying (3), and such that
\[
\int_{\mathbb{C}^N} \psi \left( \frac{1}{\tilde{\sigma}([w])} \right) \, d\tilde{\nu}([w]) < +\infty.
\]
(5) \hspace{1cm} 

**Proof of Lemma 5** Since $\tilde{\nu}(\mathbb{C}^N) < +\infty$, $\tilde{\nu}(\tilde{E}) = \tilde{\nu}\left( \{ [w] \in \mathbb{C}^N \mid \tilde{\sigma}([w]) = 1 \} \right) = 0$, for every $[w] \in \mathbb{C}^N$, $\tilde{\sigma}([w]) \geq 1$, and $\tilde{\sigma}$ is continuous, there exists a sequence $\{t_n\}_{n=1}^{\infty}$, $0 < t_1 < t_2 < \cdots < 1$, with $\lim_{n \to \infty} t_n = 1$, and such that
\[
\tilde{\nu}\left( \left\{ [w] \in \mathbb{C}^N \mid \frac{1}{t_{n+1}} < \tilde{\sigma}([w]) \leq \frac{1}{t_n} \right\} \right) < \frac{1}{n^3}.
\]
Define the function $\chi$ by $\chi(t) = n + 1$ for $t \in [t_n, t_{n+1})$, $n = 1, 2, \ldots$, and $\chi(t) = 1$ for $t \in [0, t_1)$. Then
\[
\int_{\mathbb{C}^N} \chi \left( \frac{1}{\tilde{\sigma}([w])} \right) \, d\tilde{\nu}([w])
\]
\[
\leq \tilde{\nu}(\mathbb{C}^N) + \sum_{n=1}^{\infty} (n+1) \tilde{\nu}\left( \left\{ [w] \in \mathbb{C}^N \mid \frac{1}{t_{n+1}} < \tilde{\sigma}([w]) \leq \frac{1}{t_n} \right\} \right)
\]
\[
\leq \tilde{\nu}(\mathbb{C}^N) + \sum_{n=1}^{\infty} \frac{n+1}{n^3} < +\infty.
\]
Then it is sufficient to take $\psi$ satisfying (3) and such that $\psi \leq \chi$ on $[0, 1)$.

**Lemma 6** Given $\psi$ satisfying (3), there exists a function $f \in O(B_N)$ such that for every one-dimensional complex subspace $L$ of $\mathbb{C}^N$ and every $0 < r < 1$, $f|_{L \cap B_N} \notin L^2(L \cap B_N)$, and

$$\|f\|_{L^2(L \cap B_0, (0, r))} \leq \psi(r).$$

Suppose for a moment that Lemma 6 is proved. Let $E$, $C$, $\sigma$ and $\tilde{\sigma}$ be as before, and choose $\psi$ to $\tilde{\sigma}$ according to Lemma 5. Construct $f$ with respect to $\psi$ like in Lemma 6. Then by (4) and (5), $f \in L^2H(C)$. Moreover, by Lemmas 6 and 4, for every one-dimensional complex subspace $L$ of $\mathbb{C}^N$,

$$f|_{L \cap L \cap B_N} \notin L^2(L \cap C) \iff L \cap B_N \in E.$$

This gives the desired domain $C$ and the function $f$, and ends the proof of Theorem 2.

Therefore in order to prove Theorem 2, it remains to prove Lemma 6.

**Proof of Lemma 6** We prove first an auxiliary lemma:

**Lemma 7** Let $\psi$ be a function satisfying (3). Then there exists a function $h$ holomorphic in the unit disc $U$ in $\mathbb{C}$ such that for every $0 < r < 1$,

$$\|h\|_{D(0, r)}^2 = \int_{D(0, r)} |h(w)|^2 \, dm(w) \leq \psi(r)$$

and

$$\int_{U} |h(w)|^2 \, dm(w) = +\infty.$$

**Proof of Lemma 7** Shrinking $\psi$ if necessary we may assume that $\psi$ is continuous. If

$$h(w) = \sum_{n=0}^{\infty} a_n w^n$$

is holomorphic in $U$, then

$$\int_{D(0, r)} |h(w)|^2 \, dm(w) = \pi \sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1} r^{2(n+1)},$$

and

$$\int_{U} |h(w)|^2 \, dm(w) = \pi \sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1}.$$

Therefore it is sufficient to choose non-negative numbers $\{a_n\}_{n=0}^{\infty}$ such that

$$\pi \sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1} = +\infty.$$
and for every $r$ with $0 < r < 1$,

$$
\pi \sum_{n=0}^{\infty} \frac{a_n^2}{n+1} r^{2(n+1)} \leq \psi(r).
$$

(Note that if the numbers $\{a_n\}$ satisfy (8), then the series $\sum_{n=0}^{\infty} a_n w^n$ is convergent uniformly on compact subsets of $U$, so it defines a holomorphic function in $U$).

Denote further

$$
b_n = \frac{\pi a_n^2}{n+1},
$$

$n = 0, 1, 2, \ldots$ If we can choose $\{b_n\}_{n=0}^{\infty}$ such that $b_n \geq 0, n = 0, 1, \ldots$,

$$
\sum_{n=0}^{\infty} b_n = +\infty,
$$

and for every $r$ with $0 < r < 1$,

$$
\sum_{n=0}^{\infty} b_n r^{2(n+1)} \leq \psi(r),
$$

and then we compute $a_n$ by means of $b_n$ according to (9), we get the desired coefficients $\{a_n\}_{n=0}^{\infty}$.

We claim that we can choose $b_n$ satisfying (10) and (11), and it is sufficient to allow $b_n$ to assume only the values 0 or 1 for convenient $n$. We do this inductively. Choose a positive integer $k_1$ so large that

$$
r^{2(k_1+1)} < \psi(r) \quad \text{for } 0 \leq r < 1
$$

(this is possible because of the assumptions on $\psi$). Set $b_{k_1} = 1$. The function $\psi_1(r) =: \psi(r) - r^{2(k_1+1)}, 0 \leq r < 1$, is positive, continuous, and $\lim_{r \to 1^-} \psi_1(r) = +\infty$. There exists $k_2$ so large that $k_2 > k_1$, and

$$
r^{2(k_2+1)} < \psi_1(r) \quad \text{for } 0 \leq r < 1.
$$

Set $b_{k_2} = 1$. Similarly, the function $\psi_2(r) =: \psi_1(r) - r^{2(k_2+1)}$ is positive, continuous, and $\lim_{r \to 1^-} \psi_2(r) = +\infty$. Then there exists $k_3$ so large that $k_3 > k_2$, and

$$
r^{2(k_3+1)} < \psi_2(r) \quad \text{for } 0 \leq r < 1.
$$

We set $b_{k_3} = 1$, $\psi_3(r) =: \psi_2(r) - r^{2(k_3+1)}$, and choose the integer $k_4$, and so on. In this way we have defined $b_k = 1$ for $k = k_i, i = 1, 2, \ldots$ For other values of $k$ we set $b_k = 0$.

Note that the condition (11) is satisfied by the construction. Moreover, since infinitely many $b_k$’s are equal to 1, the condition (10) is also satisfied.
Consider the constant $K = K(N)$ from the assertion of Theorem 3. Note that we can assume that the numbers $k_1, k_2, \ldots$ in the proof of Lemma 7 can be chosen so that for all $l = 1, 2, \ldots$,

$$k_{l+1} - k_l > K = K(N),$$

and for each $l$ there exists a positive integer $s_l$ such that

$$k_l = K(N)s_l.$$  

We need also further modification of the function $h$ obtained in Lemma 7. For every $l = 1, 2, \ldots$, consider the number $k_l$, where $\{k_l\}_{l=1}^{\infty}$ are chosen according to the proof of Lemma 7, and satisfy (12) and (13). Then, because of (9), (12), and the choice of the numbers $b_k$, we have $a_{k_1} > 0$ and $a_{k_1} = \cdots = a_{k+N-1} = 0$. Define $c_{k_1}$, $c_{k_1+1}$, $\ldots$, $c_{k_1+K(N)-1}$ by

$$\frac{c_{k+1}}{k+1} = \frac{c_{k+2}}{k+2} = \cdots = \frac{c_{k+K(N)-1}}{k+K(N)} = \frac{1}{K(N)} \frac{a_k^2}{k+1}, \quad l = 1, 2, \ldots.$$

This gives the numbers $c_n$ for some values of $n$. For other $n$, set $c_n = 0$. Note that because of (12), the definition of $c_n$ is correct. Set

$$g(w) = \sum_{n=0}^{\infty} c_n w^n.$$

Then $g$ is holomorphic in $U$, and by (14),

$$\sum_{n=0}^{\infty} \frac{c_n}{n+1} = \sum_{n=0}^{\infty} \frac{a_n^2}{n+1} = +\infty.$$

Moreover, for every $r$ with $0 < r < 1$, and every $l = 1, 2, \ldots$, we have by (14)

$$\frac{\pi a_k^2}{k+1} r^{2(k+1)} = \left( \frac{\pi c_k^2}{k+1} + \frac{\pi^2 c_{k+K(N)-1}^2}{k+K(N)} \right) r^{2(k+1)}$$

$$\geq \frac{\pi c_k^2}{k+1} r^{2(k+1)} + \frac{\pi c_{k+1}^2}{k+2} r^{2(k+2)} + \cdots + \frac{\pi c_{k+K(N)-1}^2}{k+K(N)} r^{2(k+K(N))},$$

and so, for every $0 < r < 1$,

$$\sum_{n=0}^{\infty} \frac{\pi^2 n^2}{n+1} r^{2(n+1)} \leq \sum_{n=0}^{\infty} \frac{\pi a_n^2}{n+1} r^{2(n+1)} \leq \psi(r).$$

Hence the function $g(w)$ also satisfies the assertions of Lemma 7, but the coefficients $c_n$ satisfy further properties, which we need later.
Define now for \( z \in B_N \),

\[
F(z) = \sum_{n=N_0}^{\infty} c_n p_n(z),
\]

where \( \{p_n\} \) are polynomials from Theorem 3. Since \( |p_n(z)| \leq 2 \) for \( z \in \partial B_N \), and \( c_n \) grow to infinity at most like \( Cn \) for some \( C > 0 \), it is not difficult to show that the series on the right-hand side of (16) converges in all of \( B_N \) to a function holomorphic in \( B_N \).

Now fix \( z \in \partial B_N \). Consider the function

\[
F_z: U \ni w \to F(wz).
\]

(We recall that \( U \) denotes the unit disc in \( \mathbb{C} \)). Then for \( 0 < r < 1 \) we have by (1) and (15)

\[
\|F_z\|_{D(0,r)} = \int_{D(0,r)} |F(wz)|^2 \, dm(w)
\]

\[
= \sum_{n=N_0}^{\infty} c_n^2 \int_{D(0,r)} |p_n(wz)|^2 \, dm(w) = \sum_{n=N_0}^{\infty} c_n^2 \int_{D(0,r)} |p_n(z)|^2 |w|^{2n} \, dm(w)
\]

\[
= \pi \sum_{n=N_0}^{\infty} \frac{c_n^2}{n+1} |p_n(z)|^2 n^{2(n+1)} \leq 4\pi \sum_{n=N_0}^{\infty} \frac{c_n^2}{n+1} n^{2(n+1)} \leq 4\psi(r).
\]

Moreover, similarly as above, and by the choice of coefficients \( c_n \), in particular by (13) and (14), we conclude that there exist positive integers \( L_0 \) and \( M_0 \), depending only on \( N_0 \) and \( S_0 \) from Theorem 3, and a number \( c > 0 \) which depends only on \( K = K(N) \) from Theorem 3 (in particular, \( L_0 \), \( M_0 \) and \( c \) do not depend on \( z \in \partial B_N \)) such that the following estimate holds:

\[
\int_U |F(wz)|^2 \, dm_2(w) = \pi \sum_{n=N_0}^{\infty} \frac{c_n^2}{n+1} |p_n(z)|^2
\]

\[
\geq \pi \sum_{n=N_0}^{\infty} \left( \frac{c_n^2}{k_l+1} |p_{k_l}(z)|^2 \right) \left( \frac{c_n^2}{k_l+1} |p_{k_l}(z)|^2 \right) \left( \frac{c_n^2}{k_l+1} |p_{k_l}(z)|^2 \right)
\]

\[
= \pi \sum_{n=N_0}^{\infty} \frac{a_{k_l}^2}{K(N)(k_l+1)} (|p_{k_l}(z)|^2 + \cdots + |p_{k_l+N(N)-1}(z)|^2)
\]

\[
= \frac{\pi}{K(N)} \sum_{n=N_0}^{\infty} \frac{a_{k_l}^2}{K(N)(k_l+1)} \left( \sum_{n=k_l+N(N)-1}^{K(N)(n+1)-1} |p_n(z)|^2 \right)
\]

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\[
\geq \frac{c\pi}{K(N)} \sum_{l=L_0}^{\infty} \frac{a_l^{2K(N)n_0}}{K(N)l+1} \left( \sum_{n=K(N)n_0}^{K(N)(n+1)-1} |p_n(z)|^2 \right) \geq \frac{1}{4} \frac{c\pi}{K(N)} \sum_{l=L_0}^{\infty} \frac{a_l^{2K(N)n_0}}{K(N)l+1} = +\infty
\]

(the last inequality follows from (2)). In virtue of (18) and (19), it is sufficient to set \( f = \frac{1}{4} F \) in order to obtain the function \( f \) satisfying the assertion of Lemma 6. This ends the proof of Lemma 6.

We give now the outline of the proof of Theorem 1. Take any strictly convex and balanced domain \( C \) in \( \mathbb{C}^N \) such that \( B_N \subset C \), \( \partial B_N \cap \partial C = E = \partial B_N \setminus \partial C \subset C \). As in Lemma 1 there exists a strictly convex, smooth and non-negatively homogeneous defining function \( \sigma \) for \( C \). Since \( C \) is balanced, the homogeneous polynomials of different orders are mutually orthogonal in \( C \) with respect to the standard Lebesgue measure in \( \mathbb{C}^N \). Looking at the proof of Theorem 2 we see that the main ingredient of the proof of the present theorem would be the following generalization of Wojtaszczyk’s result:

**Lemma 8** Suppose that \( C \) is not far away from \( B_N \) (in the sense that the strictly convex, smooth, and non-negatively homogeneous defining function \( \sigma \) for \( C \) does not differ too much from the defining function for \( B_N \), together with derivatives up to order three, in the uniform norm on some open set \( W \supset \partial B_N \cup \partial C \)). Then there exists an integer \( K = K(N) \) and a sequence \( \{ p_n \} \) of homogeneous polynomials in \( \mathbb{C}^N \) of degree \( n \) (for \( n \) large enough, say \( n \geq N_0 \)) such that

\[
|p_n(z)| \leq 2 \quad \text{for all } z \in \partial C;
\]

\[
\sum_{n=K_s}^{K(s+1)-1} |p_n(z)| \geq 0,5 \quad \text{for all } z \in \partial C.
\]

**Note** We do not know whether the assertion of Lemma 8 is true for all strictly convex and balanced domains in \( \mathbb{C}^N \).

**Sketch of the proof of Lemma 8** Consider the proof of [4], Proposition 1. Let \( \{ \zeta_1, \ldots, \zeta_s \} \) be a \( d/\sqrt{N} \)-separated subset of the unit sphere \( S \) (for definition, see [4]). Set

\[
p(z) := \sum_{j=1}^{s} \frac{1}{\| \zeta_j \|^2} (z, \zeta_j)^k,
\]

where \( \| \| \) and \( (, ) \) denote the usual Euclidean norm and scalar product in \( \mathbb{C}^N \). Fix \( j_0 \) with \( 1 \leq j_0 \leq s \). For \( z \in \partial C \), let \( \alpha \) denote the angle between \( z \) and \( \zeta_{j_0} \) (treated as the vectors in \( \mathbb{C}^N = \mathbb{R}^{2N} \)). Then, for \( z \in \partial C \) near \( \zeta_{j_0} \), we have

\[
\| z - \zeta_{j_0} \| \approx \alpha,
\]
and
\[ \frac{1}{\|\zeta_j\|^k} \langle z, \zeta_j \rangle^k = \left( \frac{\|z\|}{\|\zeta_j\|} \right)^k \langle \frac{z}{\|z\|}, \zeta_j \rangle^k. \]

Moreover,
\[ \left| \left\langle \frac{z}{\|z\|}, \zeta_j \right\rangle \right| = \cos \alpha \leq 1 - \frac{\alpha^2}{4} \]
for \( \alpha \) small, and if \( \partial C \) is sufficiently near to \( \partial B_N \), we have
\[ \frac{\|z\|}{\|\zeta_j\|} \approx 1 + c\alpha^2 \]
for some \( c > 0 \) independent of \( \zeta_j \) and \( z \), and this number \( c \) can be chosen arbitrarily close to zero. (This is the estimate to which we use the fact that \( \partial C \) is near to \( \partial B_N \).)

Hence
\[ \left( 1 - \frac{\alpha^2}{4} \right)^k \left( 1 + c\alpha^2 \right)^k \leq \left( 1 - \frac{1}{8}\alpha^2 \right)^k \]
for \( \alpha \) small (i.e., for \( z \) near \( \zeta_j \)). Moreover, assume that \( C \subset B(0, \xi) \). Then, for other values of \( j \), and \( z \in \partial C \) still near to \( \zeta_j \), the following estimate holds for \( N \leq k \leq 2N \):
\[ \frac{1}{\|\zeta_j\|^k} \langle z, \zeta_j \rangle^k = \left( \frac{\|z\|}{\|\zeta_j\|} \right)^k \left| \left\langle \frac{z}{\|z\|}, \zeta_j \right\rangle \right|^k \leq \left( \frac{e^2}{2} \right)^k e^{-\frac{\xi}{2}}. \]

This estimate is similar to [4], formula (5). Then, like in the proof of the estimates following [4], formula (5), we have by (22) and (23),
\[ |p(z)| \leq 1 + \sum_{k=1}^{\infty} \left( \frac{e^2}{2} \right)^k e^{-\frac{\xi}{2}} \frac{\xi^{k-2}}{2^{k-2}} 2^{N-1} (k+2) 2^{N-2}. \]

The last sum can be chosen to be \( \leq 0, 1 \) if \( d > 0, 5 \) was chosen sufficiently large, this would give the convenient modification of [4], Proposition 1. The rest of the proof of Lemma 8 follows the proof of [4], Theorem 1.

Having proved Lemma 8, we can repeat the proof of Theorem 2, beginning with the formula (12), in order to end the proof of Theorem 1.

References


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