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MONOLITHIC MODULES OVER NOETHERIAN RINGS

PAULA A. A. B. CARVALHO*

Departamento de Matemática, Faculdade de Ciências, Universidade do Porto, Rua do Campo Alegre 687, 4169-007 Porto, Portugal
e-mail: pbcarval@fc.up.pt

and IAN M. MUSSON

University of Wisconsin-Milwaukee, PO Box 413, Milwaukee, WI 53201 USA
e-mail: musson@csd.uwm.edu

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Abstract. We study finiteness conditions on essential extensions of simple modules over the quantum plane, the quantised Weyl algebra and Noetherian down-up algebras. The results achieved improve the ones obtained by Carvalho et al. (Carvalho et al., Injective modules over down-up algebras, *Glasgow Math. J.* **52A** (2010), 53–59) for down-up algebras.

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1. Introduction. In this paper we consider the following property of a Noetherian ring A:

($\diamond$) Injective hulls of simple left A-modules are locally Artinian.

Property ($\diamond$) has an interesting history. Indeed, it was shown by Jategaonkar [**12**] and Roseblade [**20**] that if G is a polycylic-by-finite group, then the group ring RG has property ($\diamond$) whenever R is the ring of integers, or is a field that is algebraic over a finite field; see also [**18**, Section 12.2]. This result is a key step in the positive solution of a problem of Hall [**9**]. Hall questioned whether every finitely generated abelian-by-(polycylic-by-finite) group is residually finite. In [**20**] a module M is called *monolithic* if it has a unique minimal submodule. Note that A has property ($\diamond$) if and only if every finitely generated monolithic A-module is Artinian. We have revived the older and shorter terminology in the title of this paper. Jategaonkar showed in [**11**] that a fully bounded Noetherian ring R satisfies property ($\diamond$), and used this fact to show that Jacobson's conjecture holds for R. In particular, Noetherian rings satisfying a polynomial identity (PI rings) have property ($\diamond$).

Returning to the group ring situation, suppose G is a polycylic-by-finite group, K is a field, $A = KG$ and E is the injective hull of a finite-dimensional A-module. It was shown by Brown [**3**] that if K has characteristic zero, then E is locally finite dimensional, and this fact and some Hopf algebra theory was used by Donkin [**8**] to show that E is in fact Artinian. Note that injective comodules over co-algebras are always locally finite dimensional. Similar results were obtained, when K has positive

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characteristic, by the second author [15] using methods that more closely follow the argument used for commutative rings in [21].

The first examples of Noetherian rings for which property ($\diamond$) does not hold were given by the second author for group algebras and enveloping algebras, see [16, 17] and [6, Example 7.15]. On the other hand Dahlberg [7] showed that injective hulls of simple modules over $U(\mathfrak{sl}(2))$ are locally Artinian.

Interest in property ($\diamond$) was renewed recently by a question of P. F. Smith to the first author, see [5]. He asked whether Noetherian down-up algebras have property ($\diamond$). Given a field K and α, β, γ arbitrary elements of K, the associative algebra $A = A(\alpha, \beta, \gamma)$ over K with generators d and u and defining relations

$$(R1) \qquad d^2u = \alpha dud + \beta ud^2 + \gamma d$$

$$(R2) \qquad du^2 = \alpha udu + \beta u^2d + \gamma u$$

is called a down-up algebra. Down-up algebras were introduced by Benkart and Roby [1]. In [13] it is shown that $A(\alpha, \beta, \gamma)$ is Noetherian if and only if $\beta \neq 0$. Some examples of down-up algebras with property ($\diamond$) were given in [5]. In this paper, we study Noetherian down-up algebras having property ($\diamond$), and in particular we exhibit the first examples that do not have this property. These examples are constructed using the fact that when $\gamma = 0$, (resp. $\gamma = 1$) the quantum plane, (resp. the quantised Weyl algebra) is an image of A.

An interesting class of down-up algebras arises in the following way. For $\eta \neq 0$, let A_η be the algebra with generators h, e, f and relations

$$he - eh = e,$$

$$hf - fh = -f,$$

$$ef - \eta fe = h.$$

Then A_η is isomorphic to a down-up algebra $A(1 + \eta, -\eta, 1)$ and conversely any down-up algebra $A(\alpha, \beta, \gamma)$ with $\beta \neq 0 \neq \gamma$ and $\alpha + \beta = 1$ has the above form. Note that $A_1 \simeq U(\mathfrak{sl}(2))$ and $A_{-1} \simeq U(\mathfrak{osp}(1, 2))$. When η is not a root of unity, we have been unable to determine whether property ($\diamond$) holds. However, we resolve the issue in all other cases. Our main result is as follows.

THEOREM 1.1. *Suppose that $A = A(\alpha, \beta, \gamma)$ is a Noetherian down-up algebra, and assume that if $\alpha + \beta = 1$ and $\gamma \neq 0$, then β is a root of unity. Then any finitely generated monolithic A-module is Artinian if and only if the roots of $X^2 - \alpha X - \beta$ are roots of unity.*

We remark that a characterisation of property ($\diamond$) for Noetherian rings remains rather elusive. Even a comparison of the examples for the quantum plane and quantised Weyl algebra does not seem easy to make, see Section 4 for further remarks. Thus, it seems worthwhile to study examples of rings with low GK-dimension, and down-up algebras provide an interesting test-case for property ($\diamond$). Much current research in non-commutative algebraic geometry also centers on low-dimensional algebras, and in particular down-up algebras have been studied as non-commutative threefolds by Kulkarni in [14].

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2. Preliminaries. If $r \in K$ and x and y are elements of a K-algebra, then we set $[x, y]_r = xy - ryx$. Throughout this paper we will assume that the equation $0 = X^2 - \alpha X - \beta$ has roots $r, s \in K$. Suppose $q \in K$ is non-zero and consider the algebra $B(q) = K[a, b]$ generated by a, b subject to the relation $ab = qba$. In addition, let $C(q) = K[a, b]$ denote the algebra generated by a, b subject to the relation $ab - qba = 1$. The algebras $B(q)$, $C(q)$ are known as the *coordinate algebra of the quantum plane* and the *quantised Weyl algebra*, respectively.

LEMMA 2.1.

(a) *The algebra $B(r)$ is a homomorphic image of $A = A(\alpha, \beta, 0)$.*
(b) *If $s \neq 1$ the algebra $C(r)$ is a homomorphic image of $A = A(\alpha, \beta, 1)$.*

Proof. If $\gamma = 0$, relations (R1) and (R2) can be written in the form

$$[d, [d, u]_r]_s = [[d, u]_r, u]_s = 0.$$

Thus, both relations follow from the relation $[d, u]_r = 0$, so there is a map from $A = A(\alpha, \beta, 0)$ onto $B(r)$ sending d to a and u to b.

On the other hand, if $\gamma \neq 0$, we can assume $\gamma = 1$. If $s \neq 1$, let $t \in K$ be such that $t(1 - s) = 1$. Relations $(R1)$ and $(R2)$ can now be written in the form

$$[d, [d, u]_r - t]_s = [[d, u]_r - t, u]_s = 0.$$

Since $[ta, b]_r - t = 0$ in $C(r)$, there is an homomorphism from A onto $C(r)$ sending d to ta and u to b. $\square$

The above Lemma will be used, together with the results of the next two subsections, to produce examples of down-up algebras that do not satisfy property $(\diamond)$. However, note that if exactly one of the roots of the equation $X^2 - \alpha X - \beta = 0$ is equal to 1, then the Lemma tells us only that the first Weyl algebra is a homomorphic image of $A = A(\alpha, \beta, 1)$. In this case the Lemma is of no use in constructing counterexamples.

3. The coordinate ring of the quantum plane. If q is an element of K, which is not a root of unity, we show that $B = B(q)$ does not satisfy property $(\diamond)$. Consider the left ideals $I = B(ab - 1)(a - 1) \subset J = B(a - 1)$, and set $M = B/I$, $V = J/I$ and $W = B/J$. Then there is an exact sequence

$$0 \longrightarrow V \longrightarrow M \longrightarrow W \longrightarrow 0.$$

THEOREM 3.1.

(a) *The module M is a non-Artinian essential extension of the simple submodule V.*
(b) *The submodules of W are linearly ordered by inclusion, and are pairwise non-isomorphic.*

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Proof. Step 1. *V is simple.* Clearly V is generated by the element $v_0 = (a-1) + I$. For $n \geq 0$, set

$$v_n = b^n v_0, \quad v_{-n} = a^n v_0.$$

Then using $abv_0 = v_0$, we obtain for all $n \geq 0$,

$$av_{n+1} = q^n v_n, \quad bv_{-n-1} = q^{-n-1} v_{-n}. \tag{1}$$

Furthermore, for all integers n,

$$abv_n = q^n v_n. \tag{2}$$

It is easy to see that V is spanned by the set $X = \{v_n | n \in \mathbb{Z}\}$, and it follows from equation (2) that the set X is linearly independent. Equation (2) also implies that any submodule of V is spanned by a subset of X. Then simplicity of V follows from equation (1).

Step 2. *Proof of (b).* Clearly W is generated by the element $w_0 = 1 + J$ and spanned over K by the set $Y = \{w_n | n \geq 0\}$, where $w_n = b^n w_0$. Furthermore, for all $n \geq 0$,

$$aw_n = q^n w_n. \tag{3}$$

As in the proof of Step 1, Y is linearly independent. Equation (3) also implies that any submodule of W is spanned by a subset of Y. Now for all $n \geq 0$ set

$$W_n = span\{w_m | m \geq n\} = Bw_n.$$

Consideration of the action of b now shows that a complete list of non-zero submodules of W is

$$W = W_0 \supset W_1 \supset W_2 \ldots.$$

In order to complete the proof of (b) we observe that a acts as multiplication by q^n on the unique simple quotient of W_n.

Step 3. *There is no element $v \in V$ such that* $(a - q^m)v = v_m$. If $v \in V$ is non-zero, we can write v as a linear combination of basis elements, $v = \sum_{i=c}^{d} \lambda_i v_i$, where λ_c and λ_d are non-zero elements. Then we set $|v| = d - c$. From equation (1), it follows that $|(a - q^m)v| = d - c + 1$. Clearly, this gives the assertion.

Step 4. *Proof of (a).* Set $m_n = b^n + I$ for $n \geq 0$. Then m_n maps onto w_n under the natural map $M \longrightarrow W$. Thus, the set $\{v_n, m_p | n, p \in \mathbb{Z}, p \geq 0\}$ is a basis for M. Since $am_0 = m_0 + v_0$, it follows that

$$\begin{aligned} am_n &= q^n b^n am_0 \\ &= q^n(m_n + v_n). \end{aligned}$$

Suppose that $m = \sum_{i \in I} \lambda_i m_i + v$ is a non-zero element of M. We assume that $v \in V$, I is non-empty and λ_i is a non-zero scalar for all $i \in I$. Then we show by induction on $|I|$ that $Bm \cap V$ is non-zero. Suppose that $n \in I$, and without loss $\lambda_n = 1$. If $|I| = 1$, then $Bm \cap V$ contains

$$(a - q^n)(m_n + v) = q^n v_n + (a - q^n)v,$$

and by Step 3, this is non-zero. Similarly, if $|I| > 1$, then Bm contains $(a - q^n)m$ and we have $(a - q^n)m = \sum_{j\in J} \mu_j m_j + v'$ with $J = I\backslash\{n\}$, $v' \in V$, and $\mu_j \neq 0$ for $j \in J$. Thus, the result follows by induction. □

4. The quantised Weyl algebra. Throughout this section assume that q is an element of K, which is not a root of unity. We show that the quantised Weyl algebra $C = C(q)$ does not have property (◇). We begin with some comments that may serve to motivate our construction. Observe that in Theorem 3.1, the submodules of $W = Bw_0$ have the form $Bn^k w_0$ for some normal element n of B. An analogous statement holds for the Example from **[6]** mentioned in the Introduction. Now the element $n = ab - ba \in C$ is normal, and we can in fact repeat this strategy. However, note that n has degree two with respect to a natural filtration on C, whereas in the earlier examples the normal element had degree one. For this reason, we have not attempted to give a more unified treatment of our results.

It is reasonable to look for a C-module W such that $W = K[n]$ as a $K[n]$-module with (n^i) a submodule of W for each i. Note that $\bar{C} = C/Cn \simeq K[a^{\pm 1}]$, and that if such a module W exists, then each factor $(n^i)/(n^{i+1})$ is a one-dimensional $\bar{C}$-module. Based on these considerations, it is not hard to determine the possibilities for W, and with a little experimentation, arrive at the required non-Artinian monolithic module.

Consider the K-vector space M with basis $\{v_i, w_i : i, j \in \mathbb{N}\}$, and let $V = span_K\{v_i : i \in \mathbb{N}\}$, $W = M/V$. Define linear operators a and b on V by

$$av_0 = 0, \tag{4}$$

$$av_n = \frac{q^n - 1}{q - 1} v_{n-1}, \tag{5}$$

$$bv_n = v_{n+1}. \tag{6}$$

Next, extend the action of a and b to M by setting

$$aw_n = q^n(w_n + w_{n+1}) \tag{7}$$

and

$$bw_n = \frac{q^{-n}}{1 - q} w_n + (-1)^n v_0. \tag{8}$$

We then have

$$(ab - ba)w_n = -\frac{1}{q} w_{n+1}, \tag{9}$$

$$(ab - qba)w_n = w_n. \tag{10}$$

It is now easy to see that M is a C-module, and V is a submodule of M.

LEMMA 4.1. *The C-module V is simple.*

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Proof. Since any element of V is of the form $v = a_0v_0 + a_1v_1 + \cdots + a_nv_n$ for some $a_i \in K$, by equation (5) we deduce that $v_0 \in Cv$ for any non-zero $v \in V$. Hence, V is simple and also $V = Cv_0$. □

THEOREM 4.2.
(a) *The module M is a non-Artinian essential extension of the simple submodule V.*
(b) *The submodules of W are linearly ordered by inclusion, and are pairwise non-isomorphic.*

Proof. First we prove (b). By equation (8) any submodule of W is spanned by a subset of $\{w_n : n \in \mathbb{N}_0\}$. For any $n \in \mathbb{N}$ set $W_n = \mathit{span}\{w_m : m \geq n\}$. Consideration on the actions of a and b shows that the complete list of non-zero submodules of W is

$$W = W_0 \supset W_1 \supset W_2 \supset \ldots.$$

Since b acts as a multiplication by $\dfrac{q^{-n}}{1-q}$ on the unique simple quotient of W_n, the proof of (b) is complete.

Next we prove (a). By Lemma 4.1, V is simple and by (b) M is not Artinian. The rest of the proof consists of the following three steps.

(i) Given $n \in \mathbb{N}$, by (8)

$$\left(b - \frac{q^{-n}}{1-q}\right) w_n = (-1)^n v_0 \in V \cap Cw_n, \tag{11}$$

so $Cw_n \cap V \neq 0$.

(ii) For any $n \in \mathbb{N}$ and $v \in V$, $C(w_n + v) \cap V \neq 0$. Indeed

$$\left(b - \frac{q^{-n}}{1-q}\right)(w_n + v) = (-1)^n v_0 + \left(b - \frac{q^{-n}}{1-q}\right) v. \tag{12}$$

So we must show that we can not have $v \in V\backslash\{0\}$ such that

$$\left(b - \frac{q^{-n}}{1-q}\right) v = (-1)^{n+1} v_0.$$

This follows that if $v = \lambda_0 v_0 + \cdots + \lambda_m v_m$, for some $\lambda_0, \ldots, \lambda_m \in K$ with $\lambda_m \neq 0$, then the coefficient of v_{m+1} in $(b - \frac{q^{-n}}{1-q})v$ is non-zero.

(iii) Let $m \in M\backslash V$. We show that $Cm \cap V \neq 0$. This will complete the proof. Without loss of generality we can write $m = w_n + \lambda_{n-1}w_{n-1} + \cdots + \lambda_0 w_0 + v$ for some $v \in V$ and $\lambda_0, \ldots, \lambda_{n-1} \in K$. Then $\left(b - \frac{q^{-n}}{1-q}\right)m$ is a linear combination of $w_{n-1}, \ldots, w_0$, and the v_i with $i \in \mathbb{N}$. Either we are in case (i) or (ii) or if not, then we apply $\left(b - \frac{q^{-k}}{1-q}\right)$ for a suitable k and repeat the process. □

5. A positive result. Let $A = A(\alpha, \beta, \gamma)$ be a down-up algebra and set $f(X) = X^2 - \alpha X - \beta$. Suppose that $f(X) = (X - r)^2$, where r is a primitive nth root of unity. Thus, $\alpha = 2r$ and $\beta = -r^2$. The goal of this section is to prove the following.

THEOREM 5.1. *A finitely generated essential extension of a simple A-module is Artinian.*

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Suppose first that $char(K) = p$, and let $Z' = [d^{np}, u^{np}, (du - rud + \frac{\gamma}{r-1})^n]$. Using [**10**, Theorem 4.4] and [**22**, Lemma 2.2], it is easy to see that A is finitely generated over the central subalgebra Z'. Therefore, A is PI and property $(\diamond)$ holds. For the rest of this section we assume that $char(K) = 0$.

We denote the Krull dimension of ring B by $\mathrm{K.dim}\, B$. If $r = \gamma = 1$, then A is isomorphic to the enveloping algebra of the Lie algebra $\mathfrak{sl}(2)$, and Theorem 5.1 holds by [**7**]. The proof depends on the fact that $\mathrm{K.dim}\, A = 2$, and does not immediately adapt to our situation. A key step in our proof is the fact that a certain localisation of A has the Krull dimension 2; see Proposition 5.5.

We establish some preliminaries. Let $x = ud$, $y = du$, $R = K[x, y]$ the commutative polynomial ring and σ the automorphism of R defined by the rules $\sigma(x) = y$ and $\sigma(y) = \alpha y + \beta x + \gamma$. By [**5**, Corollary 3.2] we may assume that $r \neq 1$. Hence, case 3 of [**4**, Section 1.4] holds and we set

$$w_1 = (2\beta + \alpha)ud + (\alpha - 2)du + 2\gamma;$$

$$w_2 = 2du - 2ud$$

so that $\sigma(w_1) = rw_1$ and $\sigma(w_2) = rw_2 + w_1$. Set $w = w_1/2(r-1) = -rud + du + \varepsilon$, where $\varepsilon = \gamma/(r-1)$.

LEMMA 5.2. *$\overline{A} = A/Aw$ is a PI algebra.*

Proof. Denote the images of u and d in $\overline{A}$ by $\overline{u}$ and $\overline{d}$, respectively. Then $\overline{A}$ is generated by $\overline{u}$ and $\overline{d}$ and we have that

$$-r\overline{u}\overline{d} + \overline{d}\overline{u} + \varepsilon = 0.$$

It follows that $\overline{A}$ is isomorphic to the quantised Weyl algebra if $\gamma \neq 0$ and to the coordinate ring of a quantum plane if $\gamma = 0$. Since r is a primitive nth root of unity for $n > 1$, it is well known that these algebras are PI. □

Recall that given a ring D, an automorphism σ of D and a central element $a \in D$, the generalised Weyl algebra $D(\sigma, a)$ is the ring extension of D generated by X^+ and X^-, subject to the relations: $X^+b = \sigma(b)X^+$ and $bX^- = X^-\sigma(b)$ for all $b \in D$, $X^-X^+ = a$, $X^+X^- = \sigma(a)$. The Noetherian down-up algebras can the presented as generalised Weyl algebras. In fact, set as before, $x = ud$, $y = du$ and $R = K[x, y]$ the commutative polynomial ring and define the automorphism σ of R such that $\sigma(x) = y$ and $\sigma(y) = \alpha y + \beta x + \gamma$. The Noetherian down-up algebra is isomorphic to the generalised Weyl algebra $R(\sigma, x)$ under the isomorphism taking X^+ to d and X^- to u, see [**13**].

We need the following result of Bavula and van Oystaeyen [**2**, Theorem 1.2].

THEOREM 5.3. *Let D be a commutative Noetherian ring with* $\mathrm{K.dim}\, D = m$ *and let $T = D(\sigma, a)$ be the generalised Weyl algebra. Then* $\mathrm{K.dim}\, T = m$ *unless there is a height m maximal ideal P of D such that one of the following holds:*

(a) *$\sigma^n(P) = P$, for some $n > 0$;*

(b) *$a \in \sigma^n(P)$ for infinitely many n.*

If there is an ideal P as above such that (a) or (b) holds, then $\mathrm{K.dim}\, T = m + 1$.

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Given $\lambda_0, \lambda_1 \in K$ and $n \in \mathbb{Z}$, there is a unique $\lambda_n \in K$ such that

$$\lambda_n = \alpha\lambda_{n-1} + \beta\lambda_{n-2} + \gamma.$$

For all $n \in \mathbb{Z}$ we have, see [**4**, Lemma 2.3],

$$\sigma^{-n}(x - \lambda_0, y - \lambda_1) = (x - \lambda_n, y - \lambda_{n+1}),$$

where $(x - \lambda_0, y - \lambda_1)$ denotes the ideal generated by $x - \lambda_0$ and $y - \lambda_1$.

LEMMA 5.4. *If M is a maximal ideal of R such that $x \in \sigma^n(M)$ for infinitely many n, then $\sigma^n(M) = M$ for some $n > 0$.*

Proof. We can assume that $x \in M$, that is $M = (x - \lambda_0, y - \lambda_1)$ with $\lambda_0 = 0$. The solution to the recursive relation is then given by

$$\lambda_n = c_1(r^n - 1) + c_2nr^n$$

for some fixed $c_1, c_2 \in K$. If $\lambda_n = 0$, then $nc_2 = c_1(1 - r^{-n})$, but the right side of this equation can take only finitely values. Hence, $c_2 = 0$ and the sequence $\{\lambda_n\}$ is periodic. Clearly, this gives the result. □

Since w is a normal element of A, the set $\{w^n | n \geq 0\}$ satisfies the Ore condition. We denote by A_w, R_w the localisations of A and R with respect to this set.

PROPOSITION 5.5. K.dim $A_w = 2$.

Proof. Note that $A_w = R_w(\sigma, x)$ is a generalised Weyl algebra, so by Lemma 5.4 and Theorem 5.3, we need to show that for any maximal ideal P of R_w and $n > 0$, we have $\sigma^n(P) \neq P$. We show that equivalently if M is a maximal ideal of R such that $\sigma^n(M) = M$, then $w \in M$. Indeed, if $M = (w_1 - a_1, w_2 - a_2)$, then from [**4**, Lemma 2.2(ii)] we have $a_1 = 0$ and the result follows. □

Proof of Theorem 5.1. Let V be a simple A-module and M a finitely generated essential extension of V. There are two cases.

If $wV = 0$, then by the same argument used in [**5**, Section 1.5], it is enough to show that $N = \text{ann}_M(Aw)$ is Artinian. However, N is a module over the PI algebra A/Aw, and PI algebras have property (◇) as noted in the Introduction.

If $wV \neq 0$, then since w^n is central, there exists $\lambda \in K, \lambda \neq 0$ such that $(w^n - \lambda)V = 0$. By [**19**, Theorem 3.15] $P = (w^n - \lambda)A$ is prime. By a similar argument as before we can assume $PM = 0$. Let $g, h \in K[w]$ be such that

$$1 = gw + h(w^n - \lambda).$$

This implies that $M = wM$ and $ann_M(w) = 0$, otherwise $wV = 0$. So M is an A_w-module, which is annihilated by P_w. Since K.dim $A_w = 2$ and P_w is a non-zero prime ideal, A_w/P_w is a prime of the Krull dimension one and the result follows from [**16**, Proposition 5.5]. □

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6. Down-up algebras.

Proof of Theorem 1.1. If the roots of $X^2 - \alpha X - \beta = 0$ are both equal to one or distinct roots of unity, it follows from [**5**, Corollary 3.2] that any finitely generated monolithic A-module is Artinian. By Theorem 5.1, the same holds if both roots of the quadratic equation are equal roots of unity.

Suppose that the roots of $X^2 - \alpha X - \beta = 0$ are not both roots of unity. Note that if 1 is a root of this equation, then the other root is $-\beta$. By Lemma 2.1, either the coordinate algebra of the quantum plane $B(q)$ or the quantised Weyl algebra $C(q)$ (with q not a root of 1) is a homomorphic image of A depending on $\gamma = 0$ or $\gamma \neq 0$, respectively. Hence, by Theorems 3.1 and 4.2 it follows that A does not satisfy condition $(\diamond)$. □

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