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ON WEAK-FRAGMENTABILITY OF BANACH SPACES

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Abstract

Many characterizations of fragmentability of topological spaces have been investigated. In this paper we deal with some properties of weak-fragmentability of Banach spaces.

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1. Introduction

A topological space X is *fragmentable* if there exists a metric $d(.\,,.)$ on X such that for every $\varepsilon > 0$ and every nonempty set $A \subseteq X$ there exists a nonempty subset $B \subseteq A$ which is relatively open in A and d-diam $(B) = \sup\{d(x, y) : x, y \in B\} < \varepsilon$. In such a case we say that the metric d *fragments* X.

Theorem 1.1 [1, Theorem 5.1.10]. *Let $f : (X, \tau_1) \to (Y, \tau_2)$ be an injective continuous map. If (Y, τ_2) is fragmentable then (X, τ_1) is fragmentable.*

In [3] the following *topological game* was used to characterize the fragmentability of the space X. Two players $\mathcal{A}$ and $\mathcal{B}$ alternately select a subset of X. Player $\mathcal{A}$ starts the game by choosing a nonempty subset A_1 of X, then player $\mathcal{B}$ chooses a nonempty relatively open subset B_1 of A_1. Then $\mathcal{A}$ again selects an arbitrary nonempty subset $A_2 \subseteq B_1$ and $\mathcal{B}$ responds by choosing a nonempty relatively open subset B_2 of A_2. Continuing this alternate selection of sets, the two players generate a sequence of sets

$$A_1 \supseteq B_1 \supseteq A_2 \supseteq B_2 \supseteq \cdots$$

which we call a *play* and denote by $p = (A_i, B_i)_{i\geq 1}$. We say that player $\mathcal{B}$ is the *winner* whenever the set $\bigcap_{i\geq 1} A_i = \bigcap_{i\geq 1} B_i$ contains at most one point, otherwise player $\mathcal{A}$ is the winner. A *strategy* w for player $\mathcal{B}$ is a mapping which assigns to each partial play, $A_1 \supseteq B_1 \supseteq A_2 \supseteq B_2 \supseteq \cdots \supseteq A_k$, some nonempty set $B_k = w(A_1, B_1, \ldots, A_k)$

which is a relatively open subset of A_k. We call the play $p = (A_i, B_i)_{i\geq 1}$, a *w-play* if $B_i = w(A_1, B_1, \ldots, A_i)$ for every $i \geq 1$. The strategy w is a *winning strategy* for $\mathcal{B}$ if player $\mathcal{B}$ wins every w-play. We denote this game by G_f.

THEOREM 1.2 [3]. *The topological space X is fragmentable if, and only if, player $\mathcal{B}$ has a winning strategy for G_f.*

We can see in [1] and elsewhere many characterizations of fragmentable spaces. In this paper we describe some properties of these spaces.

Let X be a Banach space and Y be a closed subspace of X. A property of X implies the same property on X/Y, but weak-fragmentability of X does not imply weak-fragmentability of X/Y. For example (l_∞, weak) is fragmentable but it is proved in [7] that $(l_\infty/c_0, \text{weak})$ is not even a countable union of fragmentable spaces. Also a property of X^* implies the same property on X, but weak-fragmentability of X^* does not imply weak-fragmentability of X. In the next section we prove this claim. Also we investigate a transfer property of weak-fragmentability by an injective bounded linear map.

Let τ_1, τ_2 be two (not necessarily distinct) topologies on a set X. We say that (X, τ_1) is fragmentable by a metric d which *majorizes* the topology τ_2 if the topology generated by d is stronger than or equal to the topology τ_2.

THEOREM 1.3 [4]. *Let τ_1, τ_2 be two (not necessarily distinct) topologies on a set X. The space (X, τ_1) is fragmentable by a metric d which majorizes τ_2 if and only if there exists a strategy w for player $\mathcal{B}$ in the game G_f in (X, τ_1) such that, for every w-play $p = (A_i, B_i)_{i\geq 1}$, either $\bigcap_{i\geq 1} A_i = \bigcap_{i\geq 1} B_i = \emptyset$ or $\bigcap_{i\geq 1} B_i = \{x\}$ for some $x \in X$, and for every τ_2-open set U that contains x, there exists some integer $k > 0$ with $B_k \subseteq U$.*

Let (X, τ) be a topological space and suppose that there exists a strategy w for player $\mathcal{B}$ in the game G in (X, τ) such that, for every w-play $p = (A_i, B_i)_{i\geq 1}$, either $\bigcap_{i\geq 1} A_i = \bigcap_{i\geq 1} B_i = \emptyset$ or there exist $k > 0$ and $x \in X$ such that $B_i = \{x\}$ for $i \geq k$. By Theorem 1.3 we can say that (X, τ) is fragmentable by a metric d which generates the discrete topology.

In the next section we will prove that if X is a nontrivial normed linear space, then (X, weak) is not fragmentable by a metric which generates the discrete topology.

A topological space (X, τ) is *scattered* if for every nonempty closed subset A of X, there is a relatively open subset U of A which contains exactly one point. The proof of the following theorem is obvious.

THEOREM 1.4. *If (X, τ) is scattered then (X, τ) is fragmentable by a metric which generates the discrete topology.*

In general, the converse of this theorem is not true. For example, $\mathbb{Q}$ is not scattered but we will prove that this space is fragmentable by a metric which generates the discrete topology. However, we will show in the next section that the converse of Theorem 1.4 is true in some particular cases.

Let X be a Banach space. We say that (X, weak) is *σ-fragmentable* if, given $\epsilon > 0$, there exists a countable family of sets $(X_i)_{i\geq 1}$ such that $X = \bigcup_{i\geq 1} X_i$ and for every $i \geq 1$ and nonempty $A \subseteq X_i$, there is a relatively nonempty open subset $B \subseteq A$ such that norm-diam $(B) < \epsilon$.

Theorem 1.5 [4]. *For a Banach space X the following assertions are equivalent:*

(a) (X, weak) *is fragmented by a metric d which majorizes the weak topology.*
(b) *There exists a strategy w for player $\mathcal{B}$ such that, for every w-play $p = (A_i, B_i)_{i\geq 1}$, either $\bigcap_{i\geq 1} B_i = \emptyset$ or $\lim_{i\to\infty}(\text{norm-diam}\,(B_i)) = 0$.*
(c) (X, weak) *is σ-fragmentable.*

Let X be a Banach space. We say that the $\|.\|$ on X is *Kadec* if the norm and weak topologies coincide on the unit sphere S_X.

Theorem 1.6 [2]. *Let X be a Banach space. If X admits an equivalent Kadec norm then (X, weak) is σ-fragmentable.*

2. Results

Theorem 2.1. *Let (X, τ_1) and (Y, τ_2) be two topological spaces and $f : (X, \tau_1) \to (Y, \tau_2)$ be a continuous injective map. If (Y, τ_2) is fragmentable by a metric which generates the discrete topology then (X, τ_1) is fragmentable by a metric which generates the discrete topology. In particular, subspaces of Y are fragmentable by a metric which generates the discrete topology.*

Proof. Let d be a fragmenting metric on (Y, τ_2) that generates the discrete topology on Y. Define ρ on $X \times X$ by, $\rho(x, y) := d(f(x), f(y))$. Since f is injective, ρ is a metric on X. Since d generates the discrete topology, for $x_0 \in X$ there exists $r > 0$ such that $\{y \in Y : d(f(x_0), y) < r\} := \{f(x_0)\}$, then $\{x \in X : \rho(x_0, x) < r\} := \{x_0\}$ which implies that ρ generates the discrete topology. Let A be a nonempty subset of X and let $\varepsilon > 0$. Then $f(A)$ is a nonempty subset of Y. Let W be an open subset of (Y, τ_2) such that $\emptyset \neq f(A) \cap W$ and d-diam $(f(A) \cap W) < \varepsilon$. Then $f^{-1}(W)$ is an open subset of X, $A \cap f^{-1}(W) \neq \emptyset$ and ρ-diam $(A \cap f^{-1}(W)) < \varepsilon$ since $f(A \cap f^{-1}(W)) \subseteq f(A) \cap W$. □

Theorem 2.2. *Let X be a countable set and let τ be a T_1 topology on X. Then (X, τ) is fragmented by a metric that generates the discrete topology.*

Proof. If X is finite then the result is trivially true, so we shall suppose that X is infinite. Let $X = \{x_n\}_{n=1}^{\infty}$. By Theorem 1.3 it is enough to show that there exists a strategy w for player $\mathcal{B}$ in the game G_f such that, for every w-play $p = (A_i, B_i)_{i\geq 1}$, either $\bigcap_{i\geq 1} A_i = \bigcap_{i\geq 1} B_i = \emptyset$ or there exist $x \in X$ and $k > 0$ such that $B_n = \{x\}$ for every $n \geq k$. Suppose that $\mathcal{A}$ chooses a nonempty set A_1 as their first move. If A_1 is finite set, let $B_1 := \{x_i : i = \min\{k \in \mathbb{N} : x_k \in A_1\}\}$. If A_1 is an infinite set, let $B_1 := A_1 \setminus \{x_1\}$. In either case B_1 is nonempty open relatively open subset of A_1. Define $w(A_1) = B_1$. Let player $\mathcal{A}$ choose a nonempty set $A_2 \subseteq B_1$. If A_2 is finite let $B_2 := \{x_i : i = \min\{k \in \mathbb{N} : x_k \in A_2\}\}$. If A_2 is infinite, let $B_2 := A_2 \setminus \{x_1, x_2\}$. In either case B_2 is nonempty

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relatively open subset of A_2. Define $w(A_1, B_1, A_2) := B_2$. If we follow this process inductively, then in the nth stage we have $B_n := \{x_i : i = \min\{k \in \mathbb{N} : x_k \in A_n\}\}$ if A_n is finite and $B_n := A_n \setminus \{x_1, x_2, \ldots, x_n\}$ if A_n is infinite. In either case B_n is a nonempty open relatively open subset of A_n and we define $w(A_1, B_1, A_2, B_2, \ldots, A_n) = B_n$; this completes the definition of w. In the w-play $p = (A_i, B_i)_{i\geq 1}$, if there exists $m \in \mathbb{N}$ such that A_m is finite then there exists $x \in X$ such that $B_m := \{x\}$ and then $B_n := \{x\}$ for every $n \geq m$, otherwise $\bigcap_{i\geq 1} B_i \subseteq X \setminus \{x_n\}_{n=1}^{\infty}$ then $\bigcap_{i\geq 1} A_i = \bigcap_{i\geq 1} B_i = \emptyset$. □

Theorem 2.3. *Let (X, τ) be a hereditarily Baire topological space, i.e. every nonempty closed subset of X is a Baire space with respect to the relative topology defined on it. If (X, τ) is fragmented by a metric ρ that generates the discrete topology then (X, τ) is scattered.*

Proof. Let Y be a nonempty closed subset of X. We show that Y has an isolated point. Without loss of generality we may assume that $Y = X$. Fix $\varepsilon > 0$ and consider the following open subset of X:

$$O_\varepsilon := \bigcup\{U \in \tau : \rho\text{-diam}\,(U) < \varepsilon\}.$$

Let W be a nonempty open subset of X. Since ρ fragments X there exists a nonempty relatively open (and hence open, since W is open) subset U of W such that ρ-diam $(U) < \varepsilon$. Then

$$\emptyset \neq U \subseteq O_\varepsilon \cap W.$$

Therefore, O_ε is dense in (X, τ). Let $G = \bigcap_{n\in\mathbb{N}} O_{1/n}$. Since (X, τ) is a Baire space, $G \neq \emptyset$. Let $x_0 \in G$. Since ρ generates the discrete topology there exists $r > 0$ such that $\{x \in X : \rho(x, x_0) < r\} := \{x_0\}$. There exists $m \in \mathbb{N}$ such that $1/m < r$. Since $x_0 \in O_{1/m}$ there exists $U \in \tau$ such that $x_0 \in U$ and ρ-diam $(U) < 1/m$. If $x \in U$ then $\rho(x, x_0) < 1/m < r$, which implies $x = x_0$. Therefore x_0 is an isolated point of X. □

The proof is very similar to the proof of Proposition 2.2 in [5].

Corollary 2.4. *If X is a nontrivial normed linear space then (X, weak) is not fragmented by a metric which generates the discrete topology.*

Proof. Let $x_0 \in X$; then the map $f : \mathbb{R} \to (X, \text{weak})$, defined by $f(r) = rx_0$ for $r \in \mathbb{R}$, is continuous and injective. Therefore by Theorem 2.1, it is enough to show that $\mathbb{R}$ is not fragmented by a metric which generates the discrete topology. We know that $\mathbb{R}$ is hereditarily a Baire space, but is not scattered; then by Theorem 2.3, $\mathbb{R}$ is not fragmented by a metric which generates the discrete topology. □

Theorem 2.5. *Let X, Y be Banach spaces and $T : X \to Y$ be an injective bounded linear map:*

(a) *If (Y, weak) is fragmentable then (X, weak) is fragmentable.*

(b) *If (Y, weak) is fragmented by a metric which majorizes the weak topology, and T is also an isomorphism, then (X, weak) is fragmented by a metric which majorizes the weak topology.*

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PROOF. (a) Since T is linear and continuous, $T : (X, \text{weak}) \to (Y, \text{weak})$ is continuous. Therefore by Theorem 1.1 if (Y, weak) is fragmentable then (X, weak) is fragmentable.

(b) Since T is an isomorphism, $T : (X, \text{weak}) \to (Y, \text{weak})$ is a homeomorphism. Therefore if (Y, weak) is fragmented by metric d which majorizes the weak topology, the metric

$$\rho : X \times X \to [0, \infty)$$
$$\rho(x, y) := d(T(x), T(y))$$

fragments X and majorizes the weak topology. □

THEOREM 2.6. *There exists a Banach space X in which (X^*, weak) is fragmented by a metric which majorizes the weak topology but (X, weak) is not even a countable union of fragmentable spaces.*

PROOF. Let $\beta\mathbb{N}$ be Stone–Cech compactification of $\mathbb{N}$. Then $C(\beta\mathbb{N})$ is isometrically isomorphic to l_∞, so l_∞^* is also isometrically isomorphic to $C(\beta\mathbb{N})^*$. If K is compact then there exists an isometric isomorphism from $C(K)^*$ to some $L_1(X, \mu)$ where μ is a finite measure [6]. Since the common norm on L_1 is Kadec, by Theorems 1.5 and 1.6, (L_1, weak) is fragmented by a metric which majorizes the weak topology. By Theorem 2.5, $(l_\infty^*, \text{weak})$ is fragmented by a metric which majorizes the weak topology. Let $X = l_\infty/c_0$; then there exists an isometric isomorphism from X^* to $c_0^\perp$. As $c_0^\perp$ is a subspace of l_∞^*, (X^*, weak) is fragmented by a metric which majorizes the weak topology, but (X, weak) is not even a countable union of fragmentable spaces. □

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