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ANALOGUES OF THE BRAID GROUP WHOSE GRAPHS ARE STARS

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1. Introduction

Let G be a group having presentation

$$\langle \{x_i\colon i=1,\ldots,n+1\} \,|\, r_{ij},\ 1 \leqq i < j \leqq n+1 \rangle \tag{1}$$

where

$$r_{ij} = x_i x_j x_i x_j^{-1} x_i^{-1} x_j^{-1} := (x_i, x_j)$$

or

$$x_i x_j x_i^{-1} x_j^{-1} := [x_i, x_j],$$

a braid relator and a commutator respectively, and define the graph of such a group as having vertices labelled $x_1, \ldots, x_{n+1}$ and such that x_i and x_j are joined if and only if $r_{ij} = (x_i, x_j)$.

With this definition Artin's braid group, B_n, [**2**] has graph which is an interval and in [**1**] Al-bar and Johnson examine those groups whose graphs are regular of degree two.

In this paper we investigate those groups, henceforth called G_n, whose graph has exactly one vertex of degree n and all others have degree one, in other words the bipartite graphs $K_{1,n}$. (For graph theoretic terminology and notation see Wilson [**6**]).

2. The derived group of G_n

We shall always assume that x_1 is a vertex of degree one and that the vertex of degree n is x_2. Note that any necessary re-labelling of vertices corresponds to elementary Nielsen transformations and therefore preserves the group (up to isomorphism).

Using an argument similar to that of Gorin and Lin [**3**] it can be seen that $G_n/G'_n \cong Z$, and we may choose $\{x_1^i : i \in Z\}$ as a Schreier transversal for G'_n. The Reidemeister–Schreier rewriting process now yields generators

$$X_{j,i} := x_1^i x_j x_1^{-(i+1)}, \qquad j = 2, \ldots, n+1,$$

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and relations

$$X_{2,i}X_{2,i+2}=X_{2,i+1},$$

$$X_{j,i}=X_{j,i+1}, \qquad j=3,\dots,n+1$$

$$X_{2,i}X_{j,i+1}X_{2,i+2}=X_{j,i}X_{2,i+1}X_{j,i+2}, \qquad j=3,\dots,n+1$$

$$X_{j,i}X_{k,i+1}=X_{k,i}X_{j,i+1}, \qquad 3\leqq j<k\leqq n+1$$

for all $i\in Z$.

It has been shown (in [5]) that this presentation can be reduced to a finite one, and this may be further modified (by Tietze transformations) to the following:

$$\langle u,v,\{a_i,\alpha_i: i=3,\dots,n+1\}\,|\,[a_i,a_j],[\alpha_i,\alpha_j],$$

$$ua_iu^{-1}=\alpha_i, \qquad \text{(i)}$$

$$u\alpha_iu^{-1}=\alpha_i^2a_i^{-1}\alpha_i, \qquad \text{(ii)}$$

$$va_iv^{-1}=a_i^{-1}\alpha_i, \qquad \text{(iii)}$$

$$v\alpha_iv^{-1}=(a_i^{-1}\alpha_i)^3a_i^{-2}\alpha_i, \qquad \text{(iv)}$$

$$\text{for } i,j\in\{3,\dots,n+3\}\rangle \qquad (2)$$

where $u=x_2x_1^{-1}$, $v=x_1x_2x_1^{-2}$, $a_i=x_ix_1^{-1}$, $\alpha_i=x_2x_ix_1^{-1}x_2^{-1}$. Note here, that when $n=2$ we have B_4 and the above presentation is that found by Gorin and Lin [3].

Write $T_n:=\langle\{a_i,\alpha_i: i=3,\dots,n+1\}\rangle$, then the presentation (2) together with the relations $u^{-1}a_iu=a_i\alpha_i^{-1}a_i^2$, $u^{-1}\alpha_iu=a_i$, $v^{-1}a_iv=a_i\alpha_i^{-1}a_i^3$, $v^{-1}\alpha_iv=a_i\alpha_i^{-1}a_i^4$ for all $i=3,\dots,n+1$, shows that $T_n\lhd G_n'$. G_n'/T_n is $\langle u,v\,|\,\rangle$, a free group of rank two and we determine a presentation for T_n as follows.

Define

$$R_n=\langle\{b_i,\beta_i: i=3,\dots,n+1\}\,|\,[b_i,b_j],[\beta_i,\beta_j] \qquad \text{for } 3\leqq i<j\leqq n+1\rangle,$$

a free product of two copies of $Z^{\times(n-1)}$ and let $U=\langle\mu,\nu\,|\,\rangle$ be free of rank two.

Consider the split extension E of R_n by U having presentation

$$\langle\mu,\nu,\{b_i,\beta_i: i=3,\dots,n+1\}\,|\,[b_i,b_j],[\beta_i,\beta_j],$$

$$\mu b_i\mu^{-1}=\beta_i,$$

$$\mu\beta_i\mu^{-1}=\beta_i^2b_i^{-1}\beta_i,$$

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$$vb_iv^{-1}=b_i^{-1}\beta_i,$$

$$v\beta_iv^{-1}=(b_i^{-1}\beta_i)^3b_i^{-2}\beta_i,$$

$$\text{for } i, j\in\{3,\ldots,n+1\}\rangle.$$

This gives us two extensions

$$\begin{array}{ccccccccc} 1 & \to & T_n & \to & G_n' & \to & G_n'/T_n & \to & 1 \\ \uparrow & & & & \uparrow & & \uparrow & & \uparrow \\ 1 & \to & R_n & \to & E & \to & U & \to & 1 \end{array}$$

where all of the vertical arrows indicate isomorphisms and the square commutes. A modified version of the five lemma now implies that R_n is isomorphic to T_n so that G_n' is the split extension of a free product of two copies of $Z^{\times(n-1)}$ by a free group of rank two.

3. The main theorem

The structure of G_n may be revealed by showing that T_n is, in fact, normal in G_n and not just in G_n'. This may be done by direct calculation for

$$x_1a_ix_1^{-1}=a_i, \qquad x_1\alpha_ix_1^{-1}=a_i^{-1}\alpha_i,$$

$$x_2a_ix_2^{-1}=\alpha_i, \qquad x_2\alpha_ix_2^{-1}=\alpha_ia_i^{-1}\alpha_i,$$

$$x_ia_ix_i^{-1}=a_i, \qquad x_i\alpha_ix_i^{-1}=\alpha_ia_i^{-1},$$

$$x_ja_ix_j^{-1}=a_i, \qquad x_j\alpha_ix_j^{-1}=a_ja_i^{-1}\alpha_ia_j^{-1},$$

and

$$x_1^{-1}a_ix_1=a_i, \qquad x_1^{-1}\alpha_ix_1=a_i\alpha_i,$$

$$x_2^{-1}a_ix_2=a_i^{-1}\alpha_ia_i^{-1}, \qquad x_2^{-1}\alpha_ix_2=a_i,$$

$$x_i^{-1}a_ix_i=a_i, \qquad x_i^{-1}\alpha_ix_i=\alpha_ia_i,$$

$$x_j^{-1}a_ix_j=a_i, \qquad x_i^{-1}\alpha_ix_j=a_ia_j^{-1}\alpha_ia_j,$$

for $i, j\in\{3,\ldots,n+1\}$ with $i\neq j$.

Thus T_n is normal in G_n and, by putting each a_i and α_i equal to the identity in G_n, that is putting $x_1=x_i$ for each $3\leqq i\leqq n+1$, we obtain that $G_n/T_n\cong B_3$. In fact we prove the stronger result below.

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Theorem. (*a*) *Let*

$$H_0 = G'_n, H_1 = [H_0, H_0], \ldots, H_{k+1} = [H_k, H_0], \ldots$$

be the lower central series of the group G'_n. *Then* $T_n = \bigcap_{k=0}^{\infty} H_k$. *In particular,* T_n *is a fully invariant subgroup of both* G'_n *and* G_n.

(*b*) *The kernel of every homomorphism* $\phi: G_n \to B_3$ *contains* T_n.

(*c*) *The kernel of every epimorphism* $\phi: G_n \to B_3$ *coincides with* T_n.

Proof. (*a*) Since $H_0 = G'_n$, $T_n \subseteq H_0$ and we proceed by induction.

Assume that for some $k \geqq 0$, $T_n \subseteq H_k$. By the relations (i)

$$\alpha_i = u a_i u^{-1} = a_i a_i^{-1} u a_i u^{-1} = aq, \text{ say,}$$

where

$$q = a_i^{-1} u a_i u^{-1} \in [T_n, H_0] \subseteq [H_k, H_0] = H_{k+1}, \quad \text{for } i = 3, \ldots, n+1.$$

Thus $a_i^{-1}\alpha_i = q \in H_{k+1}$, for $i = 3, \ldots, n+1$. Now from (iii) we have

$$a_i = v^{-1}(a_i^{-1}\alpha_i)v \in H_{k+1}, \text{ and therefore}$$

$$\alpha_i = a_i q \in H_{k+1}, \text{ for } i = 3, \ldots, n+1.$$

Thus $T_n \subseteq H_{k+1}$, and, by induction, we have $T_n \subseteq \bigcap_{k=0}^{\infty} H_k$.

For the reverse inclusion, note that the image of the subgroup $\bigcap_{k=0}^{\infty} H_k \subseteq G'_n$ under the natural projection $\tau: G'_n \to G'_n/T_n$ is contained in the intersection of the lower central series of G'_n/T_n. However G'_n/T_n is free of rank two and a theorem of Magnus (see [4] p. 38) implies that this intersection is trivial. Thus $\bigcap_{k=0}^{\infty} H_k \subseteq \ker \tau = T_n$.

(*b*) Let $\phi: G_n \to B_3$ be an arbitrary homomorphism. Since $T_n \subseteq G'_n$ we have $\phi(T_n) \subseteq \phi(G'_n) \subseteq B'_3$. From part (*a*) we may deduce that $\phi(T_n)$ is contained in the intersection of the lower central series of B'_3. But B'_3 is free, and again, by Magnus, this intersection is trivial and $\phi(T) = \{e\}$.

(*c*) Let $S = \ker \phi$. Then the homomorphism $G_n/G'_n \to B_3/B'_3$ induced by ϕ has kernel SG'_n/G'_n. Now finitely generated free groups have the Hopf property (see [4] p. 59) and since G_n/G'_n and B_3/B'_3 are both infinite cyclic SG_n/G'_n is trivial so that $S \subseteq G'_n$.

Furthermore, the restriction of ϕ to G'_n is onto B'_3 which yields $G'_n/S \cong B'_3$, but $G'_n/T_n \cong B'_3$, both free groups of rank two and by part (*b*) G'_n/S is (isomorphic to) a factor group of G'_n/T_n. Using the Hopf property again gives us that $S \subseteq T_n$ as required.

It should be noted here that the proof of this theorem is a straightforward generalization of that which may be found in Gorin and Lin [3].

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