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Simplicial Cohomology of Some Semigroup Algebras

F. Gourdeau, A. Pourabbas, and M. C. White

Abstract. In this paper, we investigate the higher simplicial cohomology groups of the convolution algebra $\ell^1(S)$ for various semigroups S. The classes of semigroups considered are semilattices, Clifford semigroups, regular Rees semigroups and the additive semigroups of integers greater than a for some integer a. Our results are of two types: in some cases, we show that some cohomology groups are 0, while in some other cases, we show that some cohomology groups are Banach spaces.

1 Introduction

In this paper, we investigate the higher simplicial cohomology groups of the convolution algebra $\ell^1(S)$ for various semigroups S. Our results are of two types: in some cases, we show that some cohomology groups are 0, while in some other cases, we show that some cohomology groups are Banach spaces.

There are several reasons why one might wish to show that a cohomology group of a Banach algebra is a Banach space. The first reason is that if one can show that the algebraic cohomology group is trivial, then this often leads to the conclusion that the space of coboundaries is dense in the space of cocycles. If one can additionally prove that the space of coboundaries is closed, then one has a proof that the cohomology is trivial. This is the method adopted in the proof that $\mathcal{H}^3(\ell^1(S), \ell^\infty(S)) = 0$ in this paper for the semilattice case. The second reason for wanting cohomology groups to be Banach spaces, is that in more advanced calculations [6, 7] one wishes to take projective tensor products of cohomology groups: this works well when the groups are Banach spaces. A third reason is that one can see showing the cohomology is a Banach space as a step to identifying the Banach space and hence the cohomology group. Finally, in the examples we have in this paper, as the first simplicial cohomology groups are trivial, the fact that the second cohomology groups are Banach spaces is equivalent to the first simplicial homology groups being trivial (see [6, Corollary 4.9]).

We now give an outline of the paper. In [4] second order cohomology groups of some semigroup Banach algebras were determined. For the semilattice S (that is, a commutative semigroup S in which $e^2 = e$ for each $e \in S$) Dales and Duncan [4] showed that $\mathcal{H}^2(\ell^1(S), \mathcal{X}) = 0$ for any commutative Banach $\ell^1(S)-$module $\mathcal{X}$. The cohomology of semilattices, which is addressed in Section 2, is of interest because

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many algebras contain subalgebras generated by idempotents. When these generate an amenable algebra, as is often the case, one can easily normalize with respect to this subalgebra. When the algebra they generate is not amenable, as here, one would still wish to normalize with respect to this subalgebra. We do not show that is possible, but we do make a first step, which is to show that the subalgebra has cohomology which is simpler than one might expect. We show that $\mathcal{H}^3(\ell^1(S), \mathcal{X})$, for any commutative Banach $\ell^1(S)$-module $\mathcal{X}$, is a Banach space. In particular we show that the third simplicial cohomology group of $\ell^1(S)$ vanishes.

For the semigroup $\mathbf{Z}_+$, we know [5] that all simplicial cohomology groups of $l^1(\mathbf{Z}_+)$ vanish for $n \geq 2$. If we consider the semigroup of integers $\mathbf{N}_a = \{n \in \mathbf{Z}_+ : n \geq a\}$ where $a > 0$, the situation becomes more complicated. This is the situation we consider in Section 3. In fact, all we are able to show in the general case is that $\mathcal{H}^2(\ell^1(\mathbf{N}_a), \ell^\infty(\mathbf{N}_a))$ is a Banach space. This is shown by considering approximately additive functions.

In Section 4 and Section 5, we respectively consider the Clifford semigroups and the regular Rees semigroups. The main results are that for those two classes of semigroup, the second simplicial cohomology group of $\ell^1(S)$ is a Banach space.

Before giving our notation, we explain the general idea for showing that a cohomology group is a Banach space. Let $\delta\colon \mathcal{C}^n(\mathcal{A}, \mathcal{X}) \to \mathcal{C}^{n+1}(\mathcal{A}, \mathcal{X})$ be the boundary map. Then $\mathcal{H}^n(\mathcal{A}, \mathcal{X})$ is a Banach space if and only if the range of δ is closed, which is the case if and only if δ is open onto its range, that is, there exists a constant K such that if $\psi = \delta(\phi)$ is such that $\|\psi\| < 1$, then there exists $\phi_1 \in \mathcal{C}^n(\mathcal{A}, \mathcal{X})$ such that $\|\phi_1\| < K$ and $\psi = \delta(\phi_1)$. This is in turn equivalent to the existence of $\phi_0 \in \ker \delta$ such that $\|\phi - \phi_0\| < K$ (where $\phi_0 = \phi - \phi_1$).

We now recall some basic results and introduce our notation. Let $\mathcal{A}$ be a Banach algebra and let $\mathcal{A}'$ be a Banach $\mathcal{A}$-bimodule in the usual way. An n-cochain is a bounded n-linear map T from $\mathcal{A}$ to $\mathcal{A}'$, which we denote by $T \in \mathcal{C}^n(\mathcal{A}, \mathcal{A}')$. The map $\delta^n\colon \mathcal{C}^n(\mathcal{A}, \mathcal{A}') \to \mathcal{C}^{n+1}(\mathcal{A}, \mathcal{A}')$ is defined by

$$\begin{aligned}(\delta^n T)(a_1, \dots, a_{n+1})(a_0) = {}& T(a_2, a_3, \dots, a_{n+1})(a_0 a_1) \\ & - T(a_1 a_2, a_3, \dots, a_{n+1})(a_0) \\ & + T(a_1, a_2 a_3, a_4, \dots, a_{n+1})(a_0) + \cdots \\ & + (-1)^n T(a_1, \dots, a_{n-1}, a_n a_{n+1})(a_0) \\ & + (-1)^{n+1} T(a_1, \dots, a_n)(a_{n+1} a_0).\end{aligned}$$

The n-cochain T is an *n-cocycle* if $\delta^n T = 0$ and it is an *n-coboundary* if $T = \delta^{n-1} S$ for some $S \in \mathcal{C}^{n-1}(\mathcal{A}, \mathcal{A}')$. The linear space of all n-cocycles is denoted by $\mathcal{Z}^n(\mathcal{A}, \mathcal{A}')$, and the linear space of all n-coboundaries is denoted by $\mathcal{B}^n(\mathcal{A}, \mathcal{A}')$. We also recall that $\mathcal{B}^n(\mathcal{A}, \mathcal{A}')$ is included in $\mathcal{Z}^n(\mathcal{A}, \mathcal{A}')$ and that the n-th simplicial cohomology group $\mathcal{H}^n(\mathcal{A}, \mathcal{A}')$ is defined by the quotient

$$\mathcal{H}^n(\mathcal{A}, \mathcal{A}') = \frac{\mathcal{Z}^n(\mathcal{A}, \mathcal{A}')}{\mathcal{B}^n(\mathcal{A}, \mathcal{A}')}.$$

The n-cochain T is called *cyclic* if

$$T(a_1, a_2, \ldots, a_n)(a_0) = (-1)^n T(a_0, a_1, \ldots, a_{n-1})(a_n),$$

and we denote the linear space of all cyclic n-cochains by $\mathcal{CC}^n(\mathcal{A}, \mathcal{A}')$. It is well known (see [9]) that the cyclic cochains $\mathcal{CC}^n(\mathcal{A}, \mathcal{A}')$ form a subcomplex of $\mathcal{C}^n(\mathcal{A}, \mathcal{A}')$, that is $\delta^n \colon \mathcal{CC}^n(\mathcal{A}, \mathcal{A}') \to \mathcal{CC}^{n+1}(\mathcal{A}, \mathcal{A}')$, and so we have cyclic versions of the spaces defined above. We denote by $\mathcal{HC}^n(\mathcal{A})$ the cyclic cohomology group of order n.

Let $\mathcal{X}$ be a Banach $\ell^1(S)$-module. As usual, we identify the element of S with point masses in $\ell^1(S)$. There is an obvious one-to-one correspondence between bounded n-cochain $\phi \in \mathcal{C}^n(\ell^1(S), \mathcal{X})$ and the bounded function from $S \times \cdots \times S$ into $\mathcal{X}$. Thus we use the same notation for $\phi \in \mathcal{C}^n(\ell^1(S), \mathcal{X})$ and ϕ as a function on $S \times \cdots \times S$.

We shall use a much simplified version of the main results of [12]. Before stating it, we recall a definition. If S is a topological space and G is a (discrete) group, then we say that S is a G-set if the product gx is defined for all g in G and x in S in such a way that $g(hx) = (gh)x\,(g, h \in G, x \in S)$ and $x \mapsto gx$ is a homeomorphism of S onto S for every g in G.

Theorem 1.1 ([12]) *Let G be a discrete group, and let S be a G-set. Then for every $\psi \in \mathcal{C}^1(\ell^1(G), \ell^\infty(S))$ there exists a $\bar{\psi} \in \mathcal{C}^1(\ell^1(G), \ell^\infty(S))$ such that $\delta\bar{\psi} = \delta\psi$ and $\|\bar{\psi}\| \le 2\,\|\delta\psi\|$, that means $\mathcal{H}^2(\ell^1(G), \ell^\infty(S))$ is a Banach space.*

2 Semilattice Algebra

The semigroup S is called a semilattice if S is a commutative semigroup such that $e^2 = e$ for every $e \in S$. The main result of this section is:

Theorem 2.1 *Let $\mathcal{A} = \ell^1(S)$, where S is a semilattice, and let $\mathcal{X}$ be a commutative $\mathcal{A}$-module. Then $\mathcal{H}^3(\mathcal{A}, \mathcal{X})$ is a Banach space.*

Before we give the proof, let us explain the idea behind the calculations in the proof, which may otherwise seem entirely *ad hoc*. As mentioned in the introduction, if one knows that the algebraic cohomology vanishes, this often implies that the coboundaries are dense in the space of cocycles. If only we can show that the coboundary map is open onto its range, then we will be able to show that the coboundary map has closed range. A method of showing that the map is open is to try the following strategy. Take a proof that $\mathcal{H}^n(A, A')$ is trivial, so that all cocycles are coboundaries. This will show that a coboundary map is surjective, so certainly open onto its range. Now try to rewrite this proof to show that if ϕ is an approximate n-cocycle, that is $\|\delta\phi\| < 1$, then it is approximately equal to a coboundary, *i.e.*, there exists a ψ so that $\|\phi - \delta\psi\| < K$ (for some K). Then we will have a small $\phi' = \phi - \delta\psi$, which has $\delta\phi' = \delta\phi$.

Now let us see how this works in the particular case of Theorem 2.1. We take the standard proof that derivations vanish on symmetrically acting idempotents.

$$D(e) = D(e^2) = eD(e) + D(e)e = 2eD(e).$$

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Hence $eD(e) = 2eD(e)$ and so $eD(e) = 0$ and so $D(e) = 0$.

Then if we are given a small 2-coboundary, $\delta\psi$, say $\|\delta\psi\| < 1$, we can think of this as saying that ψ is an approximate derivation. Then we have $\psi(e) = \psi(e^2) \approx 2e\psi(e)$, hence $e\psi(e) \approx 2e\psi(e)$, and so $e\psi(e) \approx \psi(e)$ and $\psi(e) \approx 0$. This shows that ψ is small on symmetrically acting idempotents. It remains only to make this idea rigorous.

We should note that the proof of Theorem 2.2 has exactly the same motivation, but it takes as its starting point the rigorous proof below and it is itself a rigorous proof, not an outline as given above.

Proof of Theorem 2.1 Let $\phi \in \mathcal{C}^2(\mathcal{A}, \mathcal{X})$. We define $\psi \in \mathcal{C}^1(\mathcal{A}, \mathcal{X})$ by

$$\psi(u) = (2u - 1)\phi(u, u).$$

For $\phi' \in \mathcal{C}^2(\mathcal{A}, \mathcal{X})$ given by $\phi'(u, v) = \phi(u, v) - \delta\psi(u, v)$, we have $\delta\phi' = \delta\phi \in \mathcal{B}^3(\mathcal{A}, \mathcal{X})$.

We claim that there exists a constant M such that $\|\phi'(u, v)\| \le M \|\delta\phi\|$ for every $u, v \in S$ which is equivalent to $\mathcal{H}^3(\mathcal{A}, \mathcal{X})$ being a Banach space.

Let us prove our claim. We have

$$\begin{aligned}\phi'(u, u) &= \phi(u, u) - 2u\psi(u) + \psi(u)\\ &= \phi(u, u) - 2u(2u - 1)\phi(u, u) + (2u - 1)\phi(u, u) = 0,\end{aligned}$$

as $\mathcal{X}$ is a commutative module. Using the 2-coboundary map, for every $a, b, c \in S$ we have

$$\|\delta\phi'(a, b, c)\| = \|a\phi'(b, c) - \phi'(ab, c) + \phi'(a, bc) - \phi'(a, b)c\| \le \|\delta\phi\|. \tag{2.1}$$

Let $u, v \in S$ be such that $uv = v$. Using (2.1) with u, u, v instead of a, b, c, respectively, we obtain (using $\phi'(u, u) = 0$)

$$\|u\phi'(u, v)\| \le \|\delta\phi\|. \tag{2.2}$$

Using (2.1) with u, v, v, along with $\phi'(v, v) = 0$ and commutativity of the module actions, we obtain

$$\|(1 - v)\phi'(u, v)\| = \|\phi'(u, v) - v\phi'(u, v)\| \le \|\delta\phi\|. \tag{2.3}$$

Combining (2.2) and (2.3) yields

$$\|v\phi'(u, v)\| \le \|u(1 - v)\phi'(u, v)\| + \|u\phi'(u, v)\| \le 2\|\delta\phi\|.$$

Thus for every $u, v \in S$ with $uv = v$ we have

$$\|\phi'(u, v)\| \le \|\phi'(u, v) - v\phi'(u, v)\| + \|v\phi'(u, v)\| \le 3\|\delta\phi\|. \tag{2.4}$$

If we now consider any $u, v \in S$, we deduce from (2.4) that

$$\|\phi'(u, uv)\| \le 3\|\delta\phi\|.$$

Using (2.1) with u, u, v we now obtain

$$\|(1-u)\phi'(u,v)\| = \|\phi'(u,v) - u\phi'(u,v)\| \leq \|\delta\phi\| + \|\phi'(u,uv)\| \leq 4\,\|\delta\phi\|\,.$$

A similar argument to the one deployed above (starting before (2.2), applying (2.1) for $uv = u$) yields $\|(1-v)\phi'(u,v)\| \leq 4\,\|\delta\phi\|$.

Using (2.1) with u, v, uv gives $\|u\phi'(v,uv) + \phi'(u,uv) - \phi'(u,v)uv\| \leq \|\delta\phi\|$. Thus

$$\begin{aligned}\|uv\phi'(u,v)\| \leq \|uv\phi'(u,v) - u\phi'(v,uv) - \phi'(u,uv)\| \\ + \|u\phi'(v,uv)\| + \|\phi'(u,uv)\| \leq 7\,\|\delta\phi\|\,,\end{aligned}$$

and we deduce that

$$\|v\phi'(u,v)\| \leq \|v\phi'(u,v) - uv\phi'(u,v)\| + \|uv\phi'(u,v)\| \leq 11\,\|\delta\phi\|\,.$$

Therefore, $\|\phi'(u,v)\| \leq \|(1-v)\phi'(u,v)\| + \|v\phi'(u,v)\| \leq 15\,\|\delta\phi\|$, which proves our claim, and the proof is complete. ∎

Theorem 2.2 *Let S be a semilattice. Then $\mathcal{H}^3(\ell^1(S), \ell^\infty(S)) = 0$.*

Proof Let F be a finite subset of S and let $S_F = \{e_i : i \in J\}$ be the finite semigroup generated by F, where J is a finite index set. Then $\ell^1(S_F)$ is finite dimensional and it is the image of a finite dimensional group algebra given as follows.

For each $e_i \in S_F$ set $u_i = (2e_i - 1)$. Since $u_i^2 = 1$, each u_i is invertible and the set $\{u_i\}$ generates a group G_F, that is,

$$G_F = \Big\{\prod_{i\in I} u_i : I \subseteq J\Big\}.$$

This group is finite as $u_i^2 = 1$. The map $\ell^1(G_F) \to \ell^1(S_F)\colon u_i \longmapsto e_i = \frac{1}{2}(u_i + 1)$ is a continuous and surjective homomorphism. This shows that $\ell^1(S_F)$ is amenable, which implies that $\mathcal{H}^n(\ell^1(S_F), \ell^\infty(S_F)) = 0$ for all n. (See for instance [1, 44.6].)

Now pick $\phi \in \mathcal{Z}^3(\ell^1(S), \ell^\infty(S))$. This restricts to $\phi_F \in \mathcal{Z}^3(\ell^1(S_F), \ell^\infty(S_F))$, as $\ell^1(S_F)$ is amenable $\phi_F = \delta\psi_F$ say.

By Theorem 2.1 there exists ψ_F' such that $\phi_F = \delta\psi_F'$ and $\|\psi_F'\| \leq 15\,\|\phi_F\|$. Now define a function $\tilde\psi_F' \in \mathcal{C}^2(\ell^1(S), \ell^\infty(S))$ by extending ψ_F' to be zero for $\tilde\psi_F'(s_1,s_2)(s_3)$ if any $s_i \notin S_F$ for $i = 1, 2, 3$.

Then $\{\tilde\psi_F'\}$ is a bounded net in $\mathcal{C}^2(\ell^1(S), \ell^\infty(S)) \simeq \ell^\infty(S \times S \times S)$ which has a weak-$*$ compact ball. Now let $\tilde\psi'$ be a limit point of some subnet of the $\tilde\psi_F'$. Then

$$\tilde\psi'(s_1,s_2)(s_3) = \lim \tilde\psi_F'(s_1,s_2)(s_3).$$

Therefore, $\delta\tilde\psi'(s_1,s_2)(s_3) = \lim \delta\tilde\psi_F'(s_1,s_2)(s_3) = \lim \phi_F(s_1,s_2)(s_3) = \phi(s_1,s_2)(s_3)$. Thus $\delta\tilde\psi' = \phi$ and we have $\mathcal{Z}^3(\ell^1(S), \ell^\infty(S)) = \mathcal{B}^3(\ell^1(S), \ell^\infty(S))$, which means $\mathcal{H}^3(\ell^1(S), \ell^\infty(S)) = 0$. ∎

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Another method which can be used to prove Theorem 2.2. is based on the following general idea. Constructing a bounded linear operator $t^n\colon \mathcal{C}^{n+1}(\mathcal{A},\mathcal{A}') \to \mathcal{C}^n(\mathcal{A},\mathcal{A}')$ such that $\delta^n t^n + t^{n+1}\delta^{n+1}$ is the identity map on $\mathcal{C}^{n+1}(\mathcal{A},\mathcal{A}')$ shows the vanishing of the cohomology group $\mathcal{H}^{n+1}(\mathcal{A},\mathcal{A}')$. Indeed, if $\phi \in \mathcal{C}^{n+1}(\mathcal{A},\mathcal{A}')$ is such that $\delta^{n+1}\phi = 0$, then $\delta^n(t^n(\phi)) = \phi$, which means that $\mathcal{H}^{n+1}(\mathcal{A},\mathcal{A}') = 0$. The family of maps t^n is called the contracting homotopy if t^n exists for each n. Here, we are not able to build a contracting homotopy but succeed in finding t^1 and t^2 which give the result. As we mentioned earlier, the method of constructing these maps is based on the ideas used in the proof of Theorem 2.1.

An alternate proof for Theorem 2.2 Let $\mathcal{A} = \ell^1(S)$, where S is a semilattice, and let $T \in \mathcal{C}^3(\mathcal{A},\mathcal{A}')$. We define

$$\begin{aligned} t^2(T)(u,v) =& 2uvT(u,u,uv) + uvT(v,v,uv) - uvT(uv,v,v) \\ &+ uT(v,uv,uv) + uT(u,v,v) - uT(uv,uv,v) \\ &+ 2T(u,uv,uv) - T(u,v,uv) - T(u,u,v). \end{aligned}$$

We claim that $\delta^1 t^1 + t^2\delta^2 = id$, where $t^1\colon \mathcal{C}^2(\mathcal{A},\mathcal{A}') \to \mathcal{C}^1(\mathcal{A},\mathcal{A}')$ is defined by $t^1(\phi)(e) = (2e-1)\phi(e,e)$. To prove our claim for $\phi \in \mathcal{C}^2(\mathcal{A},\mathcal{A}')$ we have

$$\begin{aligned} t^2(\delta^2)(\phi)(u,v) =& 2uv\delta^2\phi(u,u,uv) + uv\delta^2\phi(v,v,uv) - uv\delta^2\phi(uv,v,v) \\ &+ u\delta^2\phi(v,uv,uv) + u\delta^2\phi(u,v,v) - u\delta^2\phi(uv,uv,v) \\ &+ 2\delta^2\phi(u,uv,uv) - \delta^2\phi(u,v,uv) - \delta^2\phi(u,u,v). \end{aligned}$$

Using the definition of boundary map δ^2 we obtain the value of all terms on the right-hand side of the above as follows

$$\begin{aligned} t^2(\delta^2\phi)(u,v) &= \phi(u,v) - \big[u(2v-1)\phi(v,v) - (2uv-1)\phi(uv,uv) \\ &\qquad + v(2u-1)\phi(u,u)\big] \\ &= (id - \delta^1 t^1)(\phi)(u,v), \end{aligned}$$

which proves our claim, and the proof is complete. ■

3 Approximately Additive Functions and the Semigroup $\mathbf{N}_a$

Definition 3.1 A real-valued function f defined on a subset X of a semigroup S is called 1-*additive* if $|f(x) + f(y) - f(x+y)| < 1$ when $x, y, x+y \in X$, and *additive* if $|f(x) + f(y) - f(x+y)| = 0$ when $x, y, x+y \in X$.

The following proposition will enable us to deduce that the boundary map

$$\delta\colon \mathcal{C}^1(\ell^1(\mathbf{N}_a), \ell^\infty(\mathbf{N}_a)) \to \mathcal{C}^2(\ell^1(\mathbf{N}_a), \ell^\infty(\mathbf{N}_a))$$

is open onto its range, and hence that $\mathcal{H}^2(\ell^1(\mathbf{N}_a), \ell^\infty(\mathbf{N}_a))$ is a Banach space.

Proposition 3.2 *Let f be a real-valued 1-additive function on $[s,t] = \{n \in \mathbf{N} : s \leq n \leq t\}$. Then there exists a universal constant K and an additive function g on $[s,t]$ such that $\|f - g\|_\infty < K$ where $\|f\|_\infty = \max_{x\in[s,t]} |f(x)|$.*

Proof We can assume that $f(t) = 0$ by subtracting the linear function $g(x) = \frac{x}{t} f(t)$ which is additive. The proof will proceed through four cases.

Case 1: $t < 2s$. In this case, any function is additive as there are no constraints, and we let $g = f$.

Case 2: $t = 2s$. As $f(2s) = 0$, we have $1 > |f(s) + f(s) - f(2s)| = |2f(s)|$ and thus $|f(s)| < 1/2$. Letting $g(s) = g(2s) = 0$ and $g(x) = f(x)$ for $s < x < 2s$, we have g additive and $\|f - g\|_\infty < 1/2$.

Case 3: $2s < t < 3s$. Here $t = 2s + u$ with $0 < u < s$. Let $s_1 = \lfloor t/2 \rfloor$, $I_1 = [s, s_1]$, $I_2 = [s_1 + 1, t - s]$, $I_3 = [t - s + 1, 2s - 1]$ and $I_4 = [2s, t]$. (Note that $I_3 = \varnothing$ if $t = 3s - 1$; all intervals are otherwise non-empty.)

The first step is to show that we can assume $f(s) = 0$, $f(2s) < 1$ and f is zero on I_3. To do so, let us consider the functions g_1 and g_2 defined by

$$g_1(x) = \begin{cases} f(x) & \text{if } x \in I_3, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$g_2(x) = \begin{cases} (1 - 2j/u) f(s) & \text{if } x = s + j \text{ for } 0 \leq j \leq u, \\ (2 - 2j/u) f(s) & \text{if } x = 2s + j \text{ for } 0 \leq j \leq u, \\ 0 & \text{otherwise.} \end{cases}$$

Note that for each x, at most one of these two functions is non-zero.

These two functions are additive. For g_1, this follows from the observation that for all $x \in I_3$, we have $s + s > x$ and $s + x > t$, and thus there are no constraints involving non-zero values of g_1. For g_2, it is an easy check. We can therefore subtract these two functions from f and the resulting function will still be 1-additive, and will vanish on I_3 and at s.

Therefore, without loss of generality, we can assume that f is 1-additive, that $f(s) = 0$ (from which we immediately deduce $f(2s) < 1$) and that f is zero on I_3. We show that such a function cannot take values of modulus greater or equal to 5.

Let $M = \|f\|_\infty$ and let x_0 be the smallest value such that $|f(x_0)| = M$. Without loss of generality, assume $f(x_0) = M$.

If $x_0 \in I_1$, then $|f(2x_0) - 2f(x_0)| < 1$ and therefore $2M - M < 1$. Thus $M < 1$ and we are done. Note that for any $x \in I_1$, we must have $|f(2x) - 2f(x)| < 1$ and therefore, as $|f(2x)| \leq M$,

$$(*) \qquad |f(x)| < \frac{M}{2} + \frac{1}{2}.$$

If $x_0 \in I_2$, then $t - x_0 \in I_1$, and thus from $|f(x_0) + f(t - x_0) - f(t)| < 1$ we get $M = f(x_0) < |f(t - x_0)| + 1 < \frac{M}{2} + \frac{3}{2}$ by $(*)$. This gives $M < 3$ and we are done.

Note that for $x \in I_2$, we have $t-x \in I_1$ and $|f(x)+f(t-x)| < 1$. As $f(t-x) < \frac{M}{2}+\frac{1}{2}$ (from $(*)$), we obtain

$$(**) \qquad |f(x)| < \frac{M}{2} + \frac{3}{2}.$$

The only non-trivial case remaining is $x_0 \in I_4$ as f vanishes on I_3. Then $x_0-s \in I_1 \cup I_2 \cup I_3$, and from the estimates already obtained in $(*)$ and $(**)$, we have $|f(x_0-s)| < \frac{M}{2}+\frac{3}{2}$. Using this estimate, we deduce from $|f(x_0-s)+f(s)-f(x_0)| < 1$ that $M = f(x_0) < |f(x_0-s)|+1 < \frac{M}{2}+\frac{5}{2}$, and therefore $M < 5$.

Case 4: $t \geq 3s$. Let $M = \|f\|_\infty$, let x_0 be the smallest value such that $|f(x_0)| = M$, assuming without loss of generality that $f(x_0) = M$, let I_1 and I_2 be defined as in Case 3, and let $I_3 = [t-s+1, t-1]$.

If x_0 is in I_1 or I_2, we argue as we did in Case 3 and we get $M < 1$ or $M < 3$.

Suppose now that $x_0 \in I_3$. We have $|f(s)+f(x_0-s)-f(x_0)| < 1$. As $x_0-s \in I_1 \cup I_2$, we have $|f(x_0-s)| < \frac{M}{2}+\frac{3}{2}$ as in Case 3. Note that we also have some control on $|f(s)|$ as $2s \in I_1 \cup I_2$: we have $|f(2s)| < \frac{M}{2}+\frac{3}{2}$ and, as $|2f(s)-f(2s)| < 1$, we get $|f(s)| < \frac{M}{4}+\frac{5}{4}$. Thus we get $M < 1+\frac{M}{2}+\frac{3}{2}+\frac{M}{4}+\frac{5}{4}$ and we obtain $M < 15$. Hence we have proved the proposition with the constant $K = 15$. ∎

Remark 3.3 The proof of the previous proposition is long. If one tries to simplify the proof by extending the function, then the problem is that we cannot extend the definition of a 1-additive function on $[s,t]$ to a 1-additive function on $[s,\infty[$. An easy example is provided by the 1-additive function f on $[3,5]$ defined by $f(3) = f(5) = 10$ and $f(4) = 0$, which cannot be extended to take a value on 8, for instance. Also, we cannot in general subtract an additive function g in such a way that $(f-g)(s) = (f-g)(t) = 0$. This would give an easier argument. Finally, note that the proposition may well hold with a smaller constant, but this is not something of concern for us.

Theorem 3.4 *With the notation as above, $\mathcal{H}^2(\ell^1(\mathbf{N}_a), \ell^\infty(\mathbf{N}_a))$ is a Banach space.*

Proof Let $\phi \in \mathcal{C}^1(\ell^1(\mathbf{N}_a), \ell^\infty(\mathbf{N}_a))$ be such that $\|\delta\phi\| < 1$. Using the one-to-one correspondence between $\mathcal{C}^n(\ell^1(\mathbf{N}_a), \ell^\infty(\mathbf{N}_a))$ and bounded functions from the n-fold product $\mathbf{N}_a \times \cdots \times \mathbf{N}_a$ into $\ell^\infty(\mathbf{N}_a)$, we write

$$|\delta\phi(x,y)(z)| < 1 \qquad \forall x,y,z \in \mathbf{N}_a,$$

which is $|\phi(y)(x+z) - \phi(x+y)(z) + \phi(x)(y+z)| < 1$.

For each $N \geq 3a$, let $f_N\colon [a, N-a] \to \mathbf{R}$ be given by $f_N(x) = \phi(x)(N-x)$. Then f_N is 1-additive as, for $x, y, x+y \in [a, N-a]$, we have

$$|f_N(x) + f_N(y) - f_N(x+y)| = |\delta\phi(x,y)(N-(x+y))| < 1.$$

Therefore, it follows from Proposition 3.2 that for each $N \geq 3a$, there exists $g_N\colon [a, N-a] \to \mathbf{R}$ additive such that $\|f_N - g_N\|_\infty < K$ for a fixed constant K.

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Let $\psi \in \mathcal{C}^1(\ell^1(\mathbf{N}_a), \ell^\infty(\mathbf{N}_a))$ be induced by

$$\psi(x)(y) = \begin{cases} \phi(x)(y) & \text{if } x + y < 3a, \\ g_N(x) & \text{otherwise, where } N = x + y. \end{cases}$$

Then $\delta(\phi - \psi) = \delta(\phi)$ and $\|\phi - \psi\| < K$. The map δ is therefore open onto its range, which proves the theorem. ∎

4 Clifford Semigroup Algebra

In this section, we show that $\mathcal{H}^2(\ell^1(S), \ell^\infty(S))$ is a Banach space, where S is a Clifford semigroup. We recall that S is a Clifford semigroup if it is an inverse semigroup with each idempotent central, or equivalently, if it is a strong semilattice of groups. So we can write our Clifford semigroup as $S = \cup\{G_e : e \in E\}$ where E is the semilattice of idempotents and each G_e is a group with identity element e, and for every $e, e' \in E$, we have $G_e G_{e'} \subseteq G_{ee'}$ [3, §4.2].

Remark 4.1 Let $\phi \in \mathcal{C}^2(\ell^1(S), \ell^\infty(S))$ be a 2-cocycle. Then by [4, Theorem 2.5] there exists a $\psi \in \mathcal{C}^1(\ell^1(S), \ell^\infty(S))$ such that $\phi = \delta\psi$ on E. So if we define $\phi' = \phi - \delta\psi$, then $\phi'(e_1, e_2) = 0$ for every $e_1, e_2 \in E$. Thus without loss of generality, by replacing ϕ by $\phi - \delta\psi$ if necessary, we can assume that for any 2-cocycle ϕ we have $\phi(e_1, e_2) = 0$, where $e_1, e_2 \in E$.

If $\phi \in \mathcal{C}^2(\ell^1(S), \ell^\infty(S))$ is a 2-cocycle, then for every $e \in E$ and $h \in S$ such that $eh = h$, by the 2-cocycle equation $\delta\phi(e, e, h) = 0$, we have $e\phi(h, e) = 0$ and similarly $\phi(h, e)e = 0$

Lemma 4.2 *Let ϕ be a 2-cocycle. Then there exists $\psi \in \mathcal{C}^1(\ell^1(S), \ell^\infty(S))$ such that $(\phi - \delta\psi)(g, e) = 0$ for every $g \in S$, and $e \in E$ with $eg = g$.*

Proof If $eg = g$, then $g \in G_{e'}$ for some $e' \in E$ such that $ee' = e'$, (for instance, taking $e' = g^{-1}g$). Using the 2-cocycle equation $\delta\phi(e', g, e) = 0$, we obtain

$$e'\phi(g, e) - \phi(g, e) + \phi(e', g) - \phi(e', g)e = 0.$$

Since $eg = g$ and $ee' = e'$ we have

$$e'\phi(g, e) = e'e\phi(g, e) = 0,$$

thus $\phi(g, e) = (1 - e)\phi(e', g)$ whenever $eg = g$. Set $\psi(g) = -\phi(e, g)$ for every $g \in G_e$. Then for every $g \in G_{e'}$

$$\begin{aligned}(\phi - \delta\psi)(g, e) &= \phi(g, e) - g\psi(e) + \psi(g) - \psi(g)e \\ &= \phi(g, e) - \phi(e', g) + \phi(e', g)e \\ &= \phi(g, e) - (1 - e)\phi(e', g) = 0,\end{aligned}$$

whenever $eg = g$. ∎

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Remark 4.3 By the previous Lemma, replacing ϕ by $\phi - \delta\psi$ if necessary, we can assume that $\phi(g, e) = 0$ whenever $ge = g$. Applying the 2-cocycle equation

$$\delta\phi(e, g, h)(k) = 0$$

for $e \in E$, $g, h, k \in S$ with $ge = e$, we obtain (using $\phi(g, e) = 0$)

$$\phi(g, h)(k) = \phi(g, h)(ek).$$

Similarly $\phi(g, h)(k) = \phi(g, h)(ek)$ whenever $he = e$.

Lemma 4.4 *Let ϕ be a 2-cocycle and let ψ be defined by $\psi(g)(h) = \phi(g, e')(h)$, for $g \in G_{e_1}$ and $h \in G_{e_2}$, where $e' = e_1e_2$. Then $(\phi - \delta\psi)(g, e)(h) = 0$ for every $g, h \in S$ and $e \in E$.*

Proof For every $g, h \in S$ and $e \in E$, we have

$$\begin{aligned}\delta\psi(g, e)(h) &= \psi(e)(hg) - \psi(ge)(h) + \psi(g)(eh)\\ &= \phi(e, e'e)(hg) - \phi(ge, e'e)(h) + \phi(g, e'e)(eh)\\ &= \phi(e, e'e)(hg) - \phi(ge, e'e)(h) + \phi(g, e'e)(h)\end{aligned}$$

and, since $\delta\phi(g, e, e'e)(h) = 0$, we get

$$\begin{aligned}&= \phi(g, e)(e'eh) = \phi(g, e)(e_1eh)\\ &= \phi(g, e)(eh) = \phi(g, e)(h),\end{aligned}$$

where we have used Remark 4.3 several times. ■

Following Lemma 4.4, we can now assume without loss of generality that for any 2-cocycle ϕ, we have $\phi(g, e)(h) = 0$ for every $g, h \in S$ and $e \in E$.

Lemma 4.5 *For every 2-cocycle ϕ and for every $g \in G_{e_1}$, $h \in G_{e_2}$, $k \in G_{e_3}$ and $e = e_1e_2e_3$, we have $\phi(ge, he)(ke) = \phi(g, h)(k)$.*

Proof By the 2-cocycle equation $\delta\phi(g, e, he)(ke) = 0$, we have

$$g\phi(e, he)(ke) - \phi(ge, he)(ke) + \phi(g, he)(ke) - \phi(g, e)(hke) = 0.$$

By Lemma 4.4, the first and the last terms of the above equation are zero, and therefore $\phi(ge, he)(ke) = \phi(g, he)(ke)$. The 2-cocycle equation $\delta\phi(g, he, e)(ke) = 0$ gives $\phi(ge, he)(ke) = \phi(g, h)(ke)$. Finally by Remark 4.3, since $ge_1 = g$ and $he_2 = h$, we have $\phi(g, h)(ke) = \phi(g, h)(ke_1e_2) = \phi(g, h)(k)$. ■

Theorem 4.6 *Let S be a Clifford semigroup. Then $\mathcal{H}^2(\ell^1(S), \ell^\infty(S))$ is a Banach space.*

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Proof Let $\psi \in \mathcal{C}^1(\ell^1(S), \ell^\infty(S))$ with $\|\delta\psi\| < 1$. We show that there exists a constant M and $\hat{\psi} \in \mathcal{C}^1(\ell^1(S), \ell^\infty(S))$ such that $||\hat{\psi}|| < M$ and $\delta\hat{\psi} = \delta\psi$, which proves the result.

For every $g \in S$ and $e \in E$, we have

$$|\delta\psi(e,e)(g)| = |2\psi(e)(ge) - \psi(e)(g)| < 1. \tag{4.1}$$

For ge instead of g in (4.1), we obtain $|\psi(e)(ge)| < 1$. Thus

$$|\psi(e)(g)| < 3. \tag{4.2}$$

For $g \in G_{e_1}$, $h \in G_{e_2}$ and $e' = e_1e_2$, we have

$$|\delta\psi(e_1, g)(h)| = |\psi(g)(he_1) - \psi(g)(h) + \psi(e_1)(gh)| < 1,$$
$$|\delta\psi(e_2, g)(he_1)| = |\psi(g)(he') - \psi(ge_2)(he_1) + \psi(e_2)(ghe_1)| < 1.$$

From (4.2), we have, using respectively $he' = he_1$ and $ge_2 = ge'$,

$$|\psi(g)(h) - \psi(g)(he')| < 4 \quad \text{and} \quad |\psi(g)(he') - \psi(ge')(he')| < 4,$$

and therefore

$$|\psi(g)(h) - \psi(ge')(he')| < 8. \tag{4.3}$$

Now for $g \in G_{e_1}$, $h \in G_{e_2}$ and $e' = e_1e_2$, we define $\psi_0(g)(h) = \psi(e'g)(e'h)$. Then by (4.3), we have $\|\psi - \psi_0\| < 8$.

For every $e \in E$ let us define $\psi_e \in \mathcal{C}^1(\ell^1(G_e), \ell^\infty(G_e))$ by $\psi_e(g)(h) = \psi_0(g)(h)$. It is clear that $\|\delta\psi_e\| \leq \|\delta\psi_0\| < 1$. Thus by Theorem 1.1 there exists a $\psi'_e \in \mathcal{C}^1(\ell^1(G_e), \ell^\infty(G_e))$ such that $\|\psi'_e\| \leq 2$ and $\delta\psi'_e = \delta\psi_e$.

Let $\psi' \colon \ell^1(S) \longrightarrow \ell^\infty(S)$ be given by $\psi'(g)(h) = \psi_{e'}(ge')(he')$, where $g \in G_{e_1}, h \in G_{e_2}$ and $e' = e_1e_2$. Now we need to show that $\delta\psi' = \delta\psi_0$. By Lemma 4.5, for $g \in G_{e_1}$, $h \in G_{e_2}$, $k \in G_{e_3}$ and $e' = e_1e_2e_3$, we have

$$\delta\psi'(g,h)(k) = \delta\psi'_{e'}(ge', he')(ke') = \delta\psi_0(ge', he')(ke') = \delta\psi_0(g,h)(k).$$

Now define $\hat{\psi}$ by $\hat{\psi} = \psi - \psi_0 + \psi'$. We have

$$\delta\psi = \delta(\psi - \psi_0) + \delta\psi_0 = \delta(\psi - \psi_0) + \delta\psi' = \delta(\psi - \psi_0 + \psi') = \delta\hat{\psi},$$

and

$$\|\hat{\psi}\| = \|\psi - \psi_0 + \psi'\| \leq \|\psi - \psi_0\| + \|\psi'\| \leq 10. \qquad \blacksquare$$

5 Rees Semigroup Algebra

In this section we show that the second simplicial cohomology group for the regular Rees semigroups are Banach spaces. We conjecture that it is actually isomorphic to the second simplicial cohomology of the underlying group.

Let G be a group, I and Λ be index sets, and $G^0 = G \cup \{0\}$ be the group with zero arising from G by adjunction of a zero element. Let $P = (p_{\lambda i})$ be a regular sandwich matrix over G^0, so each row and each column of P contains at least one nonzero entry. The associated Rees semigroup is defined by $S_\varnothing = I \times G \times \Lambda \cup \{\varnothing\}$, where $\varnothing$ acts as the zero element of S and $(i, g, \lambda)(j, h, \mu) = (i, gp_{\lambda j}h, \mu)$, if $p_{\lambda, j} \neq 0$ and $\varnothing$ otherwise.

We wish to compute the simplicial cohomology of the algebra $\ell^1(S_\varnothing)$. This algebra includes the one-dimensional (closed) ideal generated by $1_\varnothing$, *i.e.,* the element of $\ell^1(S_\varnothing)$ which is 1 at $\varnothing$ and zero elsewhere. It will be convenient to consider the quotient algebra obtained by mapping $1_\varnothing$ to 0. We will denote this quotient algebra by $\ell^1(S)$ even if it is no longer a semigroup algebra. Note that in this algebra, $1_s 1_t = 0$ whenever $st = \varnothing$ in the semigroup, where 1_s is point mass at s.

In the following, it will be convenient to set $q_{\alpha i} = p_{\alpha i}^{-1}$ for those $(\alpha, i) \in \Lambda \times I$ such that $p_{\alpha i} \neq 0$. For other indices, we set $q_{\alpha i} = 1$. Also, we set $\Delta_i^\alpha = 0$ or 1, depending on whether $p_{\alpha i}$ is zero or non-zero.

Theorem 5.1 *Let $S_\varnothing$ be a regular Rees semigroup. Then the cohomology groups $\mathcal{H}^2(\ell^1(S_\varnothing), \ell^\infty(S_\varnothing))$ and $\mathcal{HC}^2(\ell^1(S_\varnothing))$ are Banach spaces.*

Proof To show that $\mathcal{H}^2(\ell^1(S_\varnothing), \ell^\infty(S_\varnothing))$ is a Banach space, we must show that the space $\mathcal{B}^2(\ell^1(S_\varnothing), \ell^\infty(S_\varnothing))$ is closed. We do this by showing that the map

$$\delta\colon \mathcal{C}^1(\ell^1(S_\varnothing), \ell^\infty(S_\varnothing)) \to \mathcal{C}^2(\ell^1(S_\varnothing), \ell^\infty(S_\varnothing))$$

is an open map onto its range and hence has closed range.

Throughout the proof, one should think of the special case in which $\Lambda = I = \{0, \dots, n\}$ and the sandwich matrix is the identity. In a first reading one might also like to consider the group G to be the trivial group. In this much simplified case, the algebra $\ell^1(S)$ is just the $n \times n$ matrices with norm given by the sum of the absolute values of the entries.

We pick a pair $(\omega, z) \in \Lambda \times I$ such that $p_{\omega z} \neq 0$. (Then $(z, q_{\omega z}, \omega)$ plays the role of the elementary matrix E_{11}.)

First step: We begin with an element ϕ in $\mathcal{B}^2(\ell^1(S_\varnothing), \ell^\infty(S_\varnothing))$ with $\|\phi\| < 1$. This is the image $\phi_\varnothing = \delta D$ of some element D in $\mathcal{C}^1(\ell^1(S_\varnothing), \ell^\infty(S_\varnothing))$ which may have a large norm. We think of D as being an approximate derivation. It is our task to show that we can choose an element of $\mathcal{C}^1(\ell^1(S_\varnothing), \ell^\infty(S_\varnothing))$ with small norm (that is, smaller than some absolute constant) that has the same image under δ.

It is much easier to work with the algebra $\ell^1(S)$, so we note that $1_\varnothing$ acts centrally and so the value of the approximate derivation is small on $1_\varnothing$, *i.e.,* $D(1_\varnothing)$ is small. Precisely, we have

$$\phi(1_\varnothing, 1_\varnothing) = \delta D(1_\varnothing, 1_\varnothing) = (1_\varnothing \cdot D)(1_\varnothing) - D(1_\varnothing) + (D \cdot 1_\varnothing)(1_\varnothing)$$

and therefore $D(1_\varnothing) = 2(1_\varnothing \cdot D)(1_\varnothing) - \phi(1_\varnothing, 1_\varnothing)$. Acting on the left by $1_\varnothing$ yields $(1_\varnothing \cdot D)(1_\varnothing) = (1_\varnothing \cdot \phi)(1_\varnothing, 1_\varnothing)$. These two equations give

$$D(1_\varnothing) = 2(1_\varnothing \cdot \phi)(1_\varnothing, 1_\varnothing) - \phi(1_\varnothing, 1_\varnothing).$$

Therefore $\|D(1_\varnothing)\| < 3$.

As usual, let s stand for the element of $\ell^1(S_\varnothing)$ which is 1 at s and zero elsewhere, and recall that $1_\varnothing s = s 1_\varnothing = 1_\varnothing$. Then $\phi(1_\varnothing, s)(1_\varnothing) = \delta D(1_\varnothing, s)(1_\varnothing) = D(s)(1_\varnothing) - D(1_\varnothing)(1_\varnothing) + D(1_\varnothing)(s)$ and we easily deduce that $|D(s)(1_\varnothing)| < 7$.

Thus we have obtained a bound on $|D(x)(y)|$ when one of x, y is $1_\varnothing$. We now define $\tilde{D} \in \mathcal{C}^1(\ell^1(S_\varnothing), \ell^\infty(S_\varnothing))$ by setting $\tilde{D}$ to be zero if either argument is in the linear span of $1_\varnothing$, and $\tilde{D} = D$ otherwise. Let $D_\varnothing = D - \tilde{D}$ so that $||D_\varnothing|| < 7$. Then $\phi = \delta D = \delta\tilde{D} - \delta D_\varnothing$ and we get $||\delta\tilde{D}|| < 1 + 3 \times 7 = 22$. This may not be a pretty estimate but it will do fine!

Second step: This new cochain $\tilde{D}$ factors through $\ell^1(S)$ and so defines an element D of $\mathcal{C}^1(\ell^1(S), \ell^\infty(S))$ with a coboundary of norm at most 22, which is cyclic if the original $D \in \mathcal{C}^1(\ell^1(S_\varnothing), \ell^\infty(S_\varnothing))$ was. We now argue with D and the algebra $\ell^1(S)$. We will show that we can modify D by a coboundary $\delta\psi$ in such a way that $||D - \delta\psi||$ is bounded by some universal constant, which will prove that $\mathcal{B}^2(\ell^1(S), \ell^\infty(S))$ is closed. Step one will then imply the same for $\mathcal{B}^2(\ell^1(S_\varnothing), \ell^\infty(S_\varnothing))$.

Observe that the map $g \mapsto (z, q_{\omega z} g, \omega)$ is a homomorphism from the group G into the Rees semigroup, which induces an algebra homomorphism and hence a homomorphism of cohomology groups.

In particular, $D_G(g_1)(g_2) = D((z, q_{\omega z} g_1, \omega))((z, q_{\omega z} g_2, \omega))$ defines an element of $\mathcal{C}^1(\ell^1(G), \ell^\infty(G))$ and $\|\delta D_G\| < 22$ so by Theorem 1.1 there exist a $\psi_G \in \ell^\infty(G)$ and $K > 0$ (which is independent of the group and cocycle) such that

$$|D_G(g_1)(g_2) - \delta\psi_G(g_1)(g_2)| < K. \tag{5.1}$$

We now define a function $\tilde{\psi}_G$ in $\mathcal{C}^1(\ell^1(S), \ell^\infty(S))$ by $\tilde{\psi}_G((z, q_{\omega z} g, \omega)) = \psi_G(g)$ and $\tilde{\psi}_G = 0$ otherwise, where (z, ω) is fixed as we noted before step one. We consider the function $D_1 = D - \delta\tilde{\psi}_G$. It is clear that $\delta(D_1) = \delta(D - \delta\tilde{\psi}_G) = \delta D$. Also, we have $|D_1((z, q_{\omega z} g_1, \omega))((z, q_{\omega z} g_2, \omega))| = |D_G(g_1)(g_2) - \delta\psi_G(g_1)(g_2)| < K$. Thus we have found D_1 such that $\delta D_1 = \delta D$ with the property that $|\delta D_1(x)(y)| < K$ when both x and y are in the image of $\ell_1(G)$ in $\ell_1(S)$.

We complete the proof by finding a coboundary which we add to D_1, making the resulting function small in $\mathcal{C}^1(\ell^1(S), \ell^\infty(S))$. To do so, we define a function

$$\psi_M(i, q_{\alpha i} g, \alpha) = D_1((i, q_{\alpha i}, \omega))((z, q_{\omega z} g, \alpha)).$$

We wish to show that the boundary of ψ_M controls the growth of D. Given two linear operators T and S, we will write $T = S + O(N)$ if $\|T - S\| \le N$. First observe that as $\|\delta D_1\| < 22$ then $D_1(st) = sD_1(t) - D_1 t(s) + O(22)$. Using this approximate

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derivation identity twice, we have

$$\begin{aligned} D_1((i,q_{\alpha i}g_1,\beta))((j,q_{\beta j}g_2,\alpha)) &= D_1((i,q_{\alpha i},\omega)(z,q_{\omega z}g_1,\omega)(z,q_{\omega z},\beta))((j,q_{\beta j}g_2,\alpha)) \\ &= D_1((i,q_{\alpha i},\omega))((z,q_{\omega z}g_1g_2,\alpha)) \cdot \Delta_j^\beta \\ &\quad + D_1((z,q_{\omega z}g_1,\omega))((z,q_{\omega z}g_2,\omega)) \cdot \Delta_j^\beta \cdot \Delta_i^\alpha \\ &\quad + D_1((z,q_{\omega z},\beta))((j,q_{\beta j}g_2g_1,\omega)) \cdot \Delta_i^\alpha + O(44). \end{aligned}$$

The first of these last three summands can immediately be identified as

$$D_1((i,q_{\alpha i},\omega))((z,q_{\omega z}g_1g_2,\alpha)) \cdot \Delta_j^\beta = \psi_M((i,q_{\alpha i}g_1,\beta)(j,q_{\beta j}g_2,\alpha)).$$

The second summand is $D_1((z,q_{\omega z}g_1,\omega))((z,q_{\omega z}g_2,\omega)) = (D_1)_G(g_1)(g_2)$, and we know that $\|(D_1)_G\| < K$.

The final summand needs a little more calculation. Observe that

$$\begin{aligned} D_1((z,q_{\omega z},\omega))((z,q_{\omega z}g,\omega)) \cdot \Delta_j^\beta &= D_1((z,q_{\omega z},\beta)(j,q_{\beta j},\omega))((z,q_{\omega z}g,\omega)) \\ &= D_1((z,q_{\omega z},\beta))((j,q_{\beta j}g,\omega)) \\ &\quad + D_1((j,q_{\beta j},\omega))((z,q_{\omega z}g,\beta)) + O(22). \end{aligned}$$

As $|D_1((z,q_{\omega z},\omega))((z,q_{\omega z}g,\omega))| < K$, this allows us to obtain that the last of the three summands is

$$\begin{aligned} D_1((z,q_{\omega z},\beta))((j,q_{\beta j}g_2g_1,\omega)) \cdot \Delta_i^\alpha &= -D_1((j,q_{\beta j},\omega))((z,q_{\omega z}g_2g_1,\beta))\Delta_i^\alpha + O(K') \\ &= -\psi((j,q_{\beta j}g_2,\alpha)(i,q_{\alpha i}g_1,\beta)) + O(K'), \end{aligned}$$

where $K' = 22 + K$.

Pulling together these three results shows that $D_1 = \delta\psi_M + O(K + K')$, and thus $D_1 - \delta\psi_M = O(K'')$.

Summing up, we have shown that D_2 in $\mathcal{C}^1(\ell^1(S),\ell^\infty(S))$ defined by $D_2 = D - \delta\tilde{\psi}_G - \delta\psi_M$ is such that $\delta D_2 = \delta D$ and $\|D_2\| = \|D - \delta\tilde{\psi}_G - \delta\psi_M\| < K'''$ for some absolute constant K'''. Thus the map $\delta\colon \mathcal{C}^1(\ell^1(S),\ell^\infty(S)) \to \mathcal{C}^2(\ell^1(S),\ell^\infty(S))$ has closed range.

Finally we note that these cochains $\tilde{\psi}_G$ and ψ_M naturally induce cochains on $\ell^1(S_\varnothing)$ (by the quotient map) with the same norm estimates as above, showing that the original D in $\mathcal{C}^1(\ell^1(S_\varnothing),\ell^\infty(S_\varnothing))$ is also close to a derivation and so $\mathcal{B}^2(\ell^1(S_\varnothing),\ell^\infty(S_\varnothing))$ is also closed.

For the cyclic case it remains only to note that, as all inner derivations are cyclic, then D_2 is cyclic if D is cyclic. ■

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Département de Mathématiques et de Statistique
Université Laval
Cité Universitaire, QC
e-mail: Frederic.Gourdeau@mat.ulaval.ca

Faculty of Mathematics and Computer Science
Amirkabir University of Technology
424 Hafez Avenue
Tehran 15914
Iran
e-mail: arpabbas@aut.ac.ir

Department of Mathematics
University of Newcastle
Newcastle upon Tyne, NE1 7RU
England
e-mail: Michael.White@ncl.ac.uk