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APPROXIMATION PROPERTIES OF RANDOM POLYTOPES ASSOCIATED WITH POISSON HYPERPLANE PROCESSES

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Abstract

We consider a stationary Poisson hyperplane process with given directional distribution and intensity in d-dimensional Euclidean space. Generalizing the zero cell of such a process, we fix a convex body K and consider the intersection of all closed halfspaces bounded by hyperplanes of the process and containing K. We study how well these random polytopes approximate K (measured by the Hausdorff distance) if the intensity increases, and how this approximation depends on the directional distribution in relation to properties of K.

Keywords: Poisson hyperplane process; zero polytope; approximation of convex bodies; directional distribution

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1. Introduction

Asymptotic properties of the convex hull of n independent, identically distributed (i.i.d.) random points in $\mathbb{R}^d$, as n tends to ∞, are an actively studied topic of stochastic geometry; see, for example, Subsection 8.2.4 of the book [11] and the more recent survey by Reitzner [6]. Very often, one studies uniform random points in a given convex body and measures the rate of approximation by the volume difference, or the difference of other global functionals, or one investigates the asymptotic behaviour of combinatorial quantities such as face numbers. In contrast, approximation by random polytopes, measured in terms of the Hausdorff metric δ, has been investigated less frequently. We recall that the Hausdorff distance of two nonempty compact sets $K, L \subset \mathbb{R}^d$ is defined by

$$\delta(K, L) = \max\Big\{\max_{x\in K}\min_{y\in L}\|x-y\|, \max_{x\in L}\min_{y\in K}\|x-y\|\Big\}.$$

For results on Hausdorff distances of random polytopes, we refer the reader to Note 5 of [11, Subsection 8.2.4] and mention here only the following. For a convex body K of class C^2_+ (that is, with a twice continuously differentiable boundary with positive Gauss curvature), Bárány [1, Theorem 6] showed that the Hausdorff distance from K to the convex hull K_n of n i.i.d. uniform random points in K satisfies

$$\mathbb{E}\delta(K, K_n) \sim \left(\frac{\log n}{n}\right)^{2/(d+1)} \quad \text{as } n \to \infty$$

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(here $f(n) \sim g(n)$ means that there are constants c_1 and c_2 such that $c_1 g(n) < f(n) < c_2 g(n)$). A result of Dümbgen and Walther [3, Corollary 1] says that, for an arbitrary convex body K,

$$\delta(K, K_n) = \mathrm{O}\left(\left(\frac{\log n}{n}\right)^{1/d}\right) \quad \text{almost surely.}$$

The second standard approach to convex polytopes, generating them as intersections of closed halfspaces instead of convex hulls of points, was, for the case of random polygons in the plane, already considered in the third of the seminal papers by Rényi and Sulanke [7], [8], [9], which initiated this subject. Nevertheless, this approach has later not found equal attention in the study of random polytopes. To learn more about the role that duality, either in an exact or a heuristic sense, can play here, we refer the reader to the introduction of [2]. This alternative approach offers some new aspects, in particular since random hyperplanes naturally come with some directional distribution, which influences the random polytopes that they generate. This aspect is emphasized in the present article, where we consider random polytopes generated by a stationary Poisson hyperplane process, with an arbitrary directional distribution.

Let X be a stationary nondegenerate (see [11, p. 486]) Poisson hyperplane process in Euclidean space $\mathbb{R}^d$, $d \geq 2$ (with scalar product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$). The reader is referred to Chapters 3 and 4 of [11] for an introduction, and also for some notational conventions used here. In particular, we recall the convention that a simple point process X, which is by definition a simple random counting measure, is often identified with its support, which is a locally finite random set.

For a hyperplane H in $\mathbb{R}^d$, not passing through the origin $\boldsymbol{o}$, we denote by $H_{\boldsymbol{o}}^-$ the closed halfspace bounded by H that contains $\boldsymbol{o}$. The random polytope

$$Z_0 := \bigcap_{H \in X} H_{\boldsymbol{o}}^-$$

is called the *zero cell* of X (it is also known as the *Crofton polytope* of X).

A generalization of this notion is obtained as follows. Let $K \subset \mathbb{R}^d$ be a convex body, by which we understand, in the following, a compact convex subset with interior points. For a hyperplane H not intersecting K, we denote by H_K^- the closed halfspace bounded by H that contains K. Then we define the *K-cell* of X as the random polytope

$$Z_K := \bigcap_{H \in X,\, H \cap K = \varnothing} H_K^-.$$

The almost-sure boundedness of Z_K follows as in the proof of [11, Theorem 10.3.2]. In the following we are interested in the question how well K is approximated by Z_K, if the intensity of the process X tends to ∞. Since the intensity is a constant multiple of the expected number of hyperplanes in the process that hit K, the analogy to convex hulls of an increasing number of points is evident.

We consider approximation in the sense of the Hausdorff metric δ on the space $\mathcal{K}^d$ of convex bodies in $\mathbb{R}^d$. Of course, in order that the approximation of K by Z_K is at all possible, the convex body K and the directional distribution of the hyperplane process X must somehow be adapted to each other. For example, a ball K cannot be approximated arbitrarily closely by Z_K if the hyperplane process X has only hyperplanes of finitely many directions. To make this more precise, let N be a closed subset of the unit sphere $\mathbb{S}^{d-1}$, not contained in a closed halfsphere. For a given convex body K, we denote by $\mathcal{P}(K, N)$ the set of all polytopes which are finite intersections of closed halfspaces containing K and with outer unit normal vectors in N.

Proposition 1. *The convex body K can be approximated arbitrarily closely, with respect to the Hausdorff metric, by polytopes from $\mathcal{P}(K, N)$ if and only if* $\operatorname{supp} S_{d-1}(K, \cdot) \subset N$.

Here supp denotes the support of a measure, and $S_{d-1}(K, \cdot)$ is the surface area measure of K (see [10, Section 4.2] for example). We shall give a proof of Proposition 1 in the next section. It serves here only to motivate assumption (2) made below.

The intensity measure $\Theta = \mathbb{E}X(\cdot)$ of X is assumed, as usual, to be locally finite. It can then be represented in the form (see [11, Equation (4.33)])

$$\Theta(A) = 2\gamma \int_{\mathbb{S}^{d-1}} \int_0^\infty \mathbf{1}_A(H(\boldsymbol{u}, t))\, \mathrm{d}t\, \varphi(\mathrm{d}\boldsymbol{u}) \tag{1}$$

for $A \in \mathcal{B}(\mathcal{H}^d)$, where $\gamma > 0$ is the intensity and φ is the spherical directional distribution of X; the latter is an even Borel probability measure on the unit sphere $\mathbb{S}^{d-1}$ which is not concentrated on a great subsphere. Later, when φ is fixed and γ varies, we write Θ_γ instead of Θ. By $\mathcal{H}^d$ we denote the space of hyperplanes in $\mathbb{R}^d$, and $\mathcal{B}(T)$ is the σ-algebra of Borel sets of a topological space T. Furthermore,

$$H(\boldsymbol{u}, t) = \{\boldsymbol{x} \in \mathbb{R}^d : \langle \boldsymbol{x}, \boldsymbol{u} \rangle = t\}$$

for $\boldsymbol{u} \in \mathbb{S}^{d-1}$, where $t > 0$ is the standard parameterization of a hyperplane not passing through the origin $\boldsymbol{o}$. For convenience (in view of some later estimations of constants), we also assume that $\gamma \geq 1$.

For $K \in \mathcal{K}^d$, the Hausdorff distance $\delta(K, P)$ of K from a polytope P containing it is the smallest number $\varepsilon \geq 0$ such that $P \subset K(\varepsilon)$, where $K(\varepsilon) = K + \varepsilon B^d$ (B^d is the unit ball) denotes the outer parallel body of K at distance ε. Thus, for given $\varepsilon > 0$, the probability $\mathbb{P}\{\delta(K, Z_K) > \varepsilon\}$, in which we are interested, is equal to $\mathbb{P}\{Z_K \not\subset K(\varepsilon)\}$. First we give a necessary and sufficient condition that this probability tends to 0 if the intensity of the process X tends to ∞; if the condition is satisfied then the decay is exponential. Under a slightly stronger assumption, this can then be used to derive our main results concerning the rate of convergence.

We assume in the following that the surface area measure of the given convex body K satisfies

$$\operatorname{supp} S_{d-1}(K, \cdot) \subset \operatorname{supp} \varphi. \tag{2}$$

By Proposition 1, this assumption is necessary for an arbitrarily good approximation of K by Z_K. Theorem 1 shows, in a stronger form, that it is also sufficient.

For $\boldsymbol{y} \in \mathbb{R}^d \setminus K$, let $K^{\boldsymbol{y}} := \operatorname{conv}(K \cup \{\boldsymbol{y}\})$. For $\varepsilon > 0$, we define

$$\mu(K, \varphi, \varepsilon) := \min_{\boldsymbol{y} \in \operatorname{bd} K(\varepsilon)} \int_{\mathbb{S}^{d-1}} [h(K^{\boldsymbol{y}}, \boldsymbol{u}) - h(K, \boldsymbol{u})]\, \varphi(\mathrm{d}\boldsymbol{u}), \tag{3}$$

where h denotes the support function. Lemma 1, to be proved in the next section, shows that condition (2) implies that $\mu(K, \varphi, \varepsilon) > 0$.

Theorem 1. *Let $K \in \mathcal{K}^d$ be a convex body. Let X be a stationary Poisson hyperplane process in $\mathbb{R}^d$ with intensity γ and with a directional distribution φ satisfying (2). There are positive constants $C_1(\varepsilon)$ and C_2 (both depending on K, φ, and d) such that the following holds. If $0 < \varepsilon \leq 1$ then*

$$\mathbb{P}\{\delta(K, Z_K) > \varepsilon\} \leq C_1(\varepsilon) \exp[-C_2 \mu(K, \varphi, \varepsilon)\gamma], \tag{4}$$

where $\mu(K, \varphi, \varepsilon) > 0$.

In order to be able to deal with convergence for increasing intensities, we consider an embedding of the stationary Poisson hyperplane processes X_γ with intensity $\gamma > 0$, directional distribution φ, and intensity measure

$$\mathbb{E}X_\gamma(\cdot) = 2\gamma \int_{\mathbb{S}^{d-1}} \int_0^\infty \mathbf{1}\{H(\boldsymbol{u}, t) \in \cdot\} \,\mathrm{d}t\, \varphi(\mathrm{d}\boldsymbol{u}) =: \Theta_\gamma$$

into a Poisson process ξ in $[0, \infty) \times \mathcal{H}^d$ (on a suitable probability space) with intensity measure $\lambda \otimes \Theta_1$, where λ denotes the Lebesgue measure on $[0, \infty)$. Then $\xi([0, \gamma] \times \cdot)$ is a Poisson hyperplane process in $\mathbb{R}^d$ with intensity measure Θ_γ; thus, X_γ is stochastically equivalent to $\xi([0, \gamma] \times \cdot)$ (e.g. by [11, Theorem 3.2.1]). In the following, we can identify X_γ with $\xi([0, \gamma] \times \cdot)$. Let $Z_K^{(\gamma)}$ denote the K-cell associated with $\xi([0, \gamma] \times \cdot)$. Then we have $K \subset Z_K^{(\tau)} \subset Z_K^{(\gamma)}$ for $\tau \geq \gamma > 0$, and, therefore, $\delta(K, Z_K^{(\tau)}) \leq \delta(K, Z_K^{(\gamma)})$. This shows that

$$\mathbb{P}\Big\{\sup_{\tau \geq \gamma} \delta(K, Z_K^{(\tau)}) \geq \varepsilon\Big\} = \mathbb{P}\{\delta(K, Z_K^{(\gamma)}) \geq \varepsilon\} \leq C_1(\varepsilon) \exp[-C_2 \mu(K, \varphi, \varepsilon)\gamma]$$

for all $\varepsilon > 0$, and, thus,

$$\lim_{\gamma \to \infty} \delta(K, Z_K^{(\gamma)}) = 0$$

holds almost surely. We state this as a corollary.

Corollary 1. *If the Poisson hyperplane processes X_γ, $\gamma \geq 1$, are defined as above on a common probability space and if $Z_K^{(\gamma)}$ denotes the K-cell of X_γ for a convex body $K \in \mathcal{K}^d$, then condition (2) is necessary and sufficient in order that*

$$\lim_{\gamma \to \infty} \delta(K, Z_K^{(\gamma)}) = 0 \quad \textit{almost surely}. \tag{5}$$

In the following, we will be interested in the rates of convergence. For this, we consider the sequence $X_1, X_2, \dots$ of Poisson hyperplane processes defined as above, with spherical directional distribution φ, where X_n has intensity n.

Under the sole assumption (2), no statement stronger than (5), involving also a rate of convergence, is possible. In fact, if any decreasing sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ with $\varepsilon_n \to 0$ for $n \to \infty$ is given and if K is a convex body, then the directional distribution φ of the hyperplane processes X_n can be chosen in such a way that (2) is satisfied but

$$\mathbb{P}\{\delta(K, Z_K^{(n)}) \geq \varepsilon_n \text{ for almost all } n\} = 1. \tag{6}$$

We prove this at the end of the paper. Therefore, no assumption on the convex body K alone allows us to estimate the rate of convergence of $\delta(K, Z_K^{(n)})$ for arbitrary directional distributions φ. On the other hand, suitable assumptions on the directional distribution, for example,

$$\varphi \geq b\sigma \tag{7}$$

with a constant $b > 0$, where σ denotes spherical Lebesgue measure, permit us to estimate the rate of convergence for arbitrary convex bodies. This is shown by the first assertion of Theorem 2 below.

If the directional distribution does not satisfy such a strong assumption then the rates of convergence can only be estimated if this distribution is suitably adapted to the given convex body. In this sense, we assume that

$$\varphi \geq b S_{d-1}(K, \cdot) \tag{8}$$

with some constant b.

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If $(Y_n)_{n\in\mathbb{N}}$ is a sequence of real random variables and $f(n)_{n\in\mathbb{N}}$ is a sequence of nonnegative real numbers, we write $Y_n = \mathrm{O}(f(n))$ *almost surely* if there is a constant $C < \infty$ such that, with probability 1, we have $Y_n \le Cf(n)$ for sufficiently large n. Moreover, we write $Y_n \sim f(n)$ *almost surely* if there are constants $0 < c \le C < \infty$ such that, with probability 1, we have $cf(n) \le Y_n \le Cf(n)$ for all sufficiently large n. A 'ball' in the following is a Euclidean ball of positive radius. One says that a convex body M *slides freely* inside a convex body K if K is the union of all translates of M that are contained in K.

Theorem 2. *Let $K \in \mathcal{K}^d$ be a convex body. Let X be a stationary Poisson hyperplane process in $\mathbb{R}^d$ with intensity γ and with a directional distribution φ satisfying (7) or (8). Then*

$$\delta(K, Z_K^{(n)}) = \mathrm{O}\bigg(\bigg(\frac{\log n}{n}\bigg)^{1/d}\bigg) \quad \textit{almost surely} \tag{9}$$

as $n \to \infty$.

Suppose that (8) holds. If a ball slides freely inside K then the exponent $1/d$ in (9) can be replaced by $2/(d+1)$, and if K is a polytope then it can be replaced by 1.

Under stronger assumptions on K and φ, we can determine the exact asymptotic order of approximation.

Theorem 3. *Let the convex body $K \in \mathcal{K}^d$ be such that a ball slides freely inside K and that K slides freely inside a ball. Suppose that the directional distribution φ of the stationary Poisson hyperplane processes X_n satisfies*

$$a\sigma \ge \varphi \ge b\sigma \tag{10}$$

with some positive constants a and b. Then

$$\delta(K, Z_K^{(n)}) \sim \bigg(\frac{\log n}{n}\bigg)^{2/(d+1)} \quad \textit{almost surely}$$

as $n \to \infty$.

Note that Theorem 3 covers, in particular, the case where K is of class C^2_+ and the hyperplane processes X_n are isotropic, that is, their directional distribution φ is invariant under rotations and is thus equal to the normalized spherical Lebesgue measure. If K is of class C^2_+ then the assumptions on K are satisfied by Blaschke's rolling theorem (see Corollary 3.2.13 of [10]).

In the next section we prove some auxiliary results. Theorem 1 is proved in Section 3, and the proofs of Theorems 2 and 3 follow in Section 4.

2. Auxiliary results

Proof of Proposition 1. By [10, Theorem 4.5.3], the support of the area measure $S_{d-1}(K, \cdot)$ is equal to cl extn K, the closure of the set of extreme (unit) normal vectors of K.

Suppose now that K can be approximated arbitrarily closely by polytopes from $\mathcal{P}(K, N)$. Let $\boldsymbol{x}$ be a regular boundary point of K, and let $(\boldsymbol{x}_i)_{i\in\mathbb{N}}$ be a sequence of points in $\mathbb{R}^d \setminus K$ converging to $\boldsymbol{x}$. To each i, there exists a polytope $P_i \in \mathcal{P}(K, N)$ not containing $\boldsymbol{x}_i$; hence, there is a closed halfspace H_i^- with outer normal vector $\boldsymbol{u}_i \in N$ containing K but not $\boldsymbol{x}_i$. For $i \to \infty$, the sequence of hyperplanes H_i bounding H_i^- has a convergent subsequence; its limit is the unique supporting hyperplane of K at $\boldsymbol{x}$. It follows that the outer unit normal vector of K at $\boldsymbol{x}$ belongs to the closed set N. A normal vector at a regular boundary point of K is

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a 0-exposed normal vector. Since $\boldsymbol{x}$ was an arbitrary regular boundary point of K, the set N contains the set of 0-exposed normal vectors of K. The closure of the 0-exposed normal vectors is equal to the closure of the extreme normal vectors (see Theorem 2.2.9 of [10], also for the terminology used here). Hence, cl extn $K \subset N$.

Conversely, suppose that cl extn $K \subset N$. The body K is the intersection of its supporting halfspaces with a regular point of K in the boundary (see [10, Theorem 2.2.5]). The outer unit normal vector of such a halfspace is extreme and, hence, belongs to N. Thus, denoting by $H^-(K, \boldsymbol{u})$ the supporting halfspace of K with outer unit normal vector $\boldsymbol{u}$, we have $K = \bigcap_{\boldsymbol{u}\in N} H^-(K, \boldsymbol{u})$. Therefore, if $\varepsilon > 0$ then

$$\bigcap_{\boldsymbol{u}\in N} \mathrm{bd}(K + \varepsilon B^d) \cap H^-(K, \boldsymbol{u}) = \varnothing.$$

By compactness, there is a finite subset $F \subset N$ such that the corresponding intersection is empty, which implies that

$$P := \bigcap_{\boldsymbol{u}\in F} H^-(K, \boldsymbol{u}) \subset \mathrm{int}(K + \varepsilon B^d).$$

Thus, P is a polytope in $\mathcal{P}(K, N)$ with $\delta(K, P) < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, this shows that K can be approximated arbitrarily closely by polytopes from $\mathcal{P}(K, N)$.

In the rest of this paper, $c_1, c_2, \dots$ denote positive constants that depend only on K, φ, and the dimension d.

Lemma 1. *Let $K \in \mathcal{K}^d$, and let φ be a probability measure on $\mathbb{S}^{d-1}$. Let $0 < \varepsilon \le 1$.*

(a) *If (2) holds then $\mu(K, \varphi, \varepsilon) > 0$.*

(b) *If (7) holds then there exists a constant c_1 such that*

$$\mu(K, \varphi, \varepsilon) \ge c_1 \varepsilon^d. \tag{11}$$

In the following statements it is assumed that (8) is satisfied.

(c) *For $\varepsilon \le D(K)$, where $D(K)$ denotes the diameter of K, there exists a constant c_2 such that*

$$\mu(K, \varphi, \varepsilon) \ge c_2 \varepsilon^d. \tag{12}$$

(d) *If a ball slides freely inside K then there exists a constant c_3 such that*

$$\mu(K, \varphi, \varepsilon) \ge c_3 \varepsilon^{(d+1)/2}. \tag{13}$$

(e) *If K is a polytope then there exists a constant c_4 such that*

$$\mu(K, \varphi, \varepsilon) \ge c_4 \varepsilon. \tag{14}$$

Proof. (a) Let (2) be satisfied. Let $\boldsymbol{y} \in \mathbb{R}^d \setminus K$. Let V_d denote the volume and V the mixed volume in $\mathbb{R}^d$. Using a formula for mixed volumes ([10, Equation (5.19)]) and Minkowski's inequality (e.g. [10, Equation (7.18)]), we obtain

$$\begin{aligned}\frac{1}{d}\int_{\mathbb{S}^{d-1}} [h(K^{\boldsymbol{y}}, \boldsymbol{u}) - h(K, \boldsymbol{u})]\, S_{d-1}(K, \mathrm{d}\boldsymbol{u}) &= V(K^{\boldsymbol{y}}, K, \dots, K) - V_d(K)\\ &\ge V_d(K^{\boldsymbol{y}})^{1/d} V_d(K)^{(d-1)/d} - V_d(K)\\ &= V_d(K)^{(d-1)/d}[V_d(K^{\boldsymbol{y}})^{1/d} - V_d(K)^{1/d}]\\ &> 0.\end{aligned}$$

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The integrand is nonnegative and continuous as a function of $\boldsymbol{u}$. Since the integral is positive, there exists a neighbourhood (in $\mathbb{S}^{d-1}$) of some point $\boldsymbol{u}_0 \in \operatorname{supp} S_{d-1}(K, \cdot)$ on which the integrand is positive. By (2), $\boldsymbol{u}_0 \in \operatorname{supp} \varphi$, and, hence,

$$g(\boldsymbol{y}) := \int_{\mathbb{S}^{d-1}} [h(K^{\boldsymbol{y}}, \boldsymbol{u}) - h(K, \boldsymbol{u})]\, \varphi(\mathrm{d}\boldsymbol{u}) > 0.$$

The function g is continuous and, hence, on each compact subset of $\mathbb{R}^d \setminus K$ it attains a minimum. This proves that $\mu(K, \varphi, \varepsilon) > 0$.

(b) Suppose that (7) holds. For the proof of (11), let $K \in \mathcal{K}^d$ be given. Let $\boldsymbol{y} \in \operatorname{bd} K(\varepsilon)$, and let $\boldsymbol{x}$ be the point in K nearest to $\boldsymbol{y}$. Then $N(\boldsymbol{x}) := (\boldsymbol{y} - \boldsymbol{x})/\varepsilon$ is an outer unit normal vector of K at $\boldsymbol{x}$. We denote by H^- the closed halfspace bounded by the hyperplane through $\boldsymbol{x}$ and orthogonal to $N(\boldsymbol{x})$ and containing K. If $D(K)$ denotes the diameter of K then $K \subset H^- \cap (\boldsymbol{x} + D(K)B^d)$. Define $\beta = \beta(\varepsilon) \in [0, \pi/2)$ by $\cos\beta = D(K)/\sqrt{D(K)^2 + \varepsilon^2}$, and let $S(\boldsymbol{y}, \varepsilon)$ be the set of all $\boldsymbol{u} \in \mathbb{S}^{d-1}$ such that $\angle(\boldsymbol{u}, N(\boldsymbol{x})) \le \beta/2$. Then

$$\sigma(S(\boldsymbol{y}, \varepsilon)) \ge c_5 \sin^{d-1}\left(\frac{\beta}{2}\right) \ge c_6 \varepsilon^{d-1}. \tag{15}$$

For $\boldsymbol{u} \in S(\boldsymbol{y}, \varepsilon) \setminus \{N(\boldsymbol{x})\}$, there is a unique unit vector $\boldsymbol{e}$ orthogonal to $N(\boldsymbol{x})$ such that $\boldsymbol{u} = \tau N(\boldsymbol{x}) + \sqrt{1-\tau^2}\,\boldsymbol{e}$ with $0 < \tau < 1$. With $\boldsymbol{z} := D(K)\boldsymbol{e}$ we then obtain

$$\begin{aligned} h(K^{\boldsymbol{y}}, \boldsymbol{u}) - h(K, \boldsymbol{u}) &\ge \langle \boldsymbol{y}, \boldsymbol{u}\rangle - \langle \boldsymbol{z}, \boldsymbol{u}\rangle \\ &= \langle \boldsymbol{y} - \boldsymbol{z}, \boldsymbol{u}\rangle \\ &\ge D(K)\left\langle \frac{\boldsymbol{y} - \boldsymbol{z}}{\|\boldsymbol{y} - \boldsymbol{z}\|}, \boldsymbol{u}\right\rangle \\ &\ge D(K)\sin\left(\frac{\beta}{2}\right) \\ &\ge c_7 \varepsilon \end{aligned} \tag{16}$$

for all $\boldsymbol{u} \in S(\boldsymbol{y}, \varepsilon)$. Combining (7), (15), and (16), we obtain

$$\begin{aligned} \frac{1}{b}\int_{\mathbb{S}^{d-1}} [h(K^{\boldsymbol{y}}, \boldsymbol{u}) - h(K, \boldsymbol{u})]\, \varphi(\mathrm{d}\boldsymbol{u}) &\ge \int_{\mathbb{S}^{d-1}} [h(K^{\boldsymbol{y}}, \boldsymbol{u}) - h(K, \boldsymbol{u})]\, \sigma(\mathrm{d}\boldsymbol{u}) \\ &\ge \sigma(S(\boldsymbol{y}, \varepsilon)) c_7 \varepsilon \\ &\ge c_8 \varepsilon^d, \end{aligned}$$

which completes the proof of (b).

Now suppose that (8) holds. From the estimate in the proof of (a) we obtain

$$\begin{aligned} \frac{1}{bd}\int_{\mathbb{S}^{d-1}} [h(K^{\boldsymbol{y}}, \boldsymbol{u}) - h(K, \boldsymbol{u})]\, \varphi(\mathrm{d}\boldsymbol{u}) &\ge \frac{1}{d}\int_{\mathbb{S}^{d-1}} [h(K^{\boldsymbol{y}}, \boldsymbol{u}) - h(K, \boldsymbol{u})]\, S_{d-1}(K, \mathrm{d}\boldsymbol{u}) \\ &\ge V_d(K)^{(d-1)/d}[V_d(K^{\boldsymbol{y}})^{1/d} - V_d(K)^{1/d}] \\ &\ge c_9[V_d(K^{\boldsymbol{y}}) - V_d(K)]. \end{aligned}$$

(c) For the proof of (12), let $\boldsymbol{y} \in \operatorname{bd} K(\varepsilon)$ and let C be the cone with apex $\boldsymbol{y}$ spanned by K. Let $\boldsymbol{y}'$ be the point in K nearest to $\boldsymbol{y}$. The vector $\boldsymbol{y} - \boldsymbol{y}'$ has length ε, and the hyperplane H'

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orthogonal to it and passing through $\boldsymbol{y}'$ supports K. Let H be the other supporting hyperplane of K parallel to H'. Let Δ be the convex hull of $\boldsymbol{y}$ and $H \cap C$, and let Δ' be the convex hull of $\boldsymbol{y}$ and $H' \cap C$. Denoting by $D(K)$ the diameter of K and assuming that $\varepsilon \leq D(K)$, we have

$$V_d(K^{\boldsymbol{y}}) - V_d(K) \geq V_d(\Delta') \geq \left(\frac{\varepsilon}{D(K)+\varepsilon}\right)^d V_d(\Delta) \geq \left(\frac{\varepsilon}{2D(K)}\right)^d V_d(K).$$

This gives (12).

(d) Suppose that a ball of radius $r > 0$ slides freely inside K. Since $\mu(\cdot, \varphi, \varepsilon)$ is translation invariant, we can assume that K contains the ball $B(\boldsymbol{o}, r)$ of radius r centred at $\boldsymbol{o}$. Let $R > 0$ be such that $K \subset B(\boldsymbol{o}, R)$. For $s > 0$, the convex body

$$K^s := \{\boldsymbol{x} \in \mathbb{R}^d : V_d(K^{\boldsymbol{x}}) - V_d(K) \leq s\}$$

is known as an illumination body of K (cf. [12, p. 258]; the convexity follows from Satz 4 of [4]). Now let $\boldsymbol{y} \in \operatorname{bd} K(\varepsilon)$, and put $\nu := V_d(K^{\boldsymbol{y}}) - V_d(K)$. Then $\boldsymbol{y} \in \operatorname{bd} K^{\nu}$. Let $\boldsymbol{x} \in \operatorname{bd} K$ be determined by $\{\boldsymbol{x}\} = [\boldsymbol{o}, \boldsymbol{y}] \cap \operatorname{bd} K$, and denote by $N(\boldsymbol{x})$ the unique exterior unit normal vector of K at $\boldsymbol{x}$ (the normal vector is unique since, by assumption, there is a ball B' of radius $r > 0$ with $\boldsymbol{x} \in B' \subset K$). Since $B(\boldsymbol{o}, r) \subset K$, we have

$$\langle \boldsymbol{x}, N(\boldsymbol{x})\rangle \geq r, \qquad \left\langle \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}, N(\boldsymbol{x})\right\rangle \geq \frac{r}{R}.$$

From $\|\boldsymbol{y}\| - \|\boldsymbol{x}\| \geq \varepsilon$ we obtain $\|\boldsymbol{y}\|^d - \|\boldsymbol{x}\|^d \geq dr^{d-1}\varepsilon$. Therefore, Lemma 2 of [12] yields

$$\nu^{2/(d+1)} \geq c_{10} r r^{(d-1)/(d+1)} \left(\left(\frac{\|\boldsymbol{y}\|}{\|\boldsymbol{x}\|}\right)^d - 1\right) \geq c_{11} R^{-d}(\|\boldsymbol{y}\|^d - \|\boldsymbol{x}\|^d) \geq c_{12}\varepsilon;$$

hence,

$$V_d(K^{\boldsymbol{y}}) - V_d(K) \geq c_{13}\, \varepsilon^{(d+1)/2},$$

which gives (13).

(e) Now suppose that K is a polytope. Let $\boldsymbol{y} \in \operatorname{bd} K(\varepsilon)$, and let $\boldsymbol{y}'$ be the point in K nearest to $\boldsymbol{y}$. Put $\boldsymbol{v} := (\boldsymbol{y} - \boldsymbol{y}')/\|\boldsymbol{y} - \boldsymbol{y}'\|$, and let F denote the unique (proper) face of K which contains $\boldsymbol{y}'$ in its relative interior. Let $F_1, \dots, F_m$ be the facets of K that contain F, and let $\boldsymbol{u}_1, \dots, \boldsymbol{u}_m$ be their outer unit normal vectors. By [10, p. 85 and Theorem 2.4.9], we have

$$\boldsymbol{v} \in N(K, F) = N(K, \boldsymbol{y}') = \operatorname{pos}\{\boldsymbol{u}_i : i = 1, \dots, m\},$$

where $N(K, F)$ and $N(K, \boldsymbol{y}')$ are the normal cones of K at F and $\boldsymbol{y}'$, respectively, and pos denotes the positive hull. For any unit vector $\boldsymbol{w} \in N(K, F)$, there is some $i \in \{1, \dots, m\}$ such that $\langle \boldsymbol{w}, \boldsymbol{u}_i\rangle > 0$; in particular,

$$a(F, \boldsymbol{w}) := \max\{\langle \boldsymbol{w}, \boldsymbol{u}_i\rangle : i = 1, \dots, m\} > 0$$

and $a(F, \boldsymbol{v}) = \langle \boldsymbol{v}, \boldsymbol{u}_{i_0}\rangle > 0$ for some $i_0 \in \{1, \dots, m\}$. Since $N(K, F) \cap \mathbb{S}^{d-1}$ is compact, we have

$$a(F) := \min\{a(F, \boldsymbol{w}) : \boldsymbol{w} \in N(K, F) \cap \mathbb{S}^{d-1}\} > 0$$

and, thus,

$$c_{14} := \min\{a(F) : F \text{ is a proper face of } K\} > 0.$$

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Therefore, with $c_{15} := \min\{V_{d-1}(F) \colon F \text{ is a facet of } K\} > 0$, where V_{d-1} denotes the $(d-1)$-dimensional volume, we obtain

$$\begin{aligned}\int_{\mathbb{S}^{d-1}} [h(K^{\boldsymbol{y}}, \boldsymbol{u}) - h(K, \boldsymbol{u})]\, S_{d-1}(K, \mathrm{d}\boldsymbol{u}) &\geq \langle \boldsymbol{y} - \boldsymbol{y}', \boldsymbol{u}_{i_0}\rangle V_{d-1}(F_{i_0}) \\ &\geq \|\boldsymbol{y} - \boldsymbol{y}'\| c_{14} c_{15} \\ &= c_{16}\varepsilon.\end{aligned}$$

This yields (14).

Remark 1. Although in the case of a general convex body K, the derivation of estimate (12) may seem rather crude, the order of ε^d cannot be improved. In fact, if (12) was replaced by $\mu(K, \varphi, \varepsilon) \geq c_2\varepsilon^\alpha$ with $1 < \alpha < d$, then a counterexample would be provided by a body K which in a neighbourhood of some boundary point is congruent to a suitable part of a body of revolution with meridian curve given by $\mu(t) = |t|^r$ with $1 < r < (d-1)/(\alpha - 1)$.

Lemma 2. *Let the convex body $K \in \mathcal{K}^d$ be such that a ball slides freely inside K. Assume further that*

$$a\sigma \geq \varphi \tag{17}$$

with some positive constant a. Then

$$\int_{\mathbb{S}^{d-1}} [h(K^{\boldsymbol{y}}, \boldsymbol{u}) - h(K, \boldsymbol{u})]\, \varphi(\mathrm{d}\boldsymbol{u}) \leq c_{17}\varepsilon^{(d+1)/2}$$

for $\varepsilon > 0$ and $\boldsymbol{y} \in \operatorname{bd} K(\varepsilon)$.

Proof. Let $\boldsymbol{y} \in \operatorname{bd} K(\varepsilon)$. From (17) we obtain

$$\int_{\mathbb{S}^{d-1}} [h(K^{\boldsymbol{y}}, \boldsymbol{u}) - h(K, \boldsymbol{u})]\, \varphi(\mathrm{d}\boldsymbol{u}) \leq c_{18} \int_{\mathbb{S}^{d-1}} [h(K^{\boldsymbol{y}}, \boldsymbol{u}) - h(K, \boldsymbol{u})]\, \sigma(\mathrm{d}\boldsymbol{u}).$$

Let $\boldsymbol{x}$ be the point in K nearest to $\boldsymbol{y}$; then $\boldsymbol{y} = \boldsymbol{x} + \varepsilon N(\boldsymbol{x})$, where $N(\boldsymbol{x})$ is the outer unit normal vector of K at $\boldsymbol{x}$. By assumption, a ball, say of radius $r > 0$, slides freely inside K. In particular, some ball B of radius r satisfies $\boldsymbol{x} \in B \subset K$. Let

$$\operatorname{Cap}(\boldsymbol{y}, \varepsilon) := \left\{\boldsymbol{u} \in \mathbb{S}^{d-1} \colon \langle \boldsymbol{u}, N(\boldsymbol{x})\rangle \geq \frac{r}{r+\varepsilon}\right\}.$$

For $\boldsymbol{u} \in \mathbb{S}^{d-1} \setminus \operatorname{Cap}(\boldsymbol{y}, \varepsilon)$, we have $h(K^{\boldsymbol{y}}, \boldsymbol{u}) - h(K, \boldsymbol{u}) = 0$. If $h(K^{\boldsymbol{y}}, \boldsymbol{u}) - h(K, \boldsymbol{u}) \neq 0$ then

$$h(K^{\boldsymbol{y}}, \boldsymbol{u}) - h(K, \boldsymbol{u}) \leq \langle \boldsymbol{y} - \boldsymbol{x}, \boldsymbol{u}\rangle \leq \varepsilon.$$

With $\alpha(\varepsilon) := \arccos r/(r+\varepsilon)$ this gives

$$\begin{aligned}\int_{\mathbb{S}^{d-1}} [h(K^{\boldsymbol{y}}, \boldsymbol{u}) - h(K, \boldsymbol{u})]\, \sigma(\mathrm{d}\boldsymbol{u}) &\leq \int_{\operatorname{Cap}(\boldsymbol{y},\varepsilon)} \varepsilon\, \sigma(\mathrm{d}\boldsymbol{u}) \\ &\leq c_{18}\varepsilon \sin^{d-1}\alpha(\varepsilon) \\ &= c_{18}\varepsilon \sqrt{1 - \left(\frac{r}{r+\varepsilon}\right)^2}^{\,d-1} \\ &\leq c_{19}\varepsilon^{(d+1)/2}.\end{aligned}$$

This yields the assertion.

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The following lemma is sufficient for our purpose; it does not aim at an optimal order.

Lemma 3. *Let* $K \in \mathcal{K}^d$ *be a convex body which slides freely in some ball. There are constants* $c_{20}, c_{21} > 0$ *such that the following holds. For* $0 < \varepsilon < c_{20}$*, let* $m(\varepsilon)$ *be the largest number* m *such that there are* m *points in* $\operatorname{bd} K(\varepsilon)$ *with the property that each segment connecting any two of them intersects the interior of* K*. Then*

$$m(\varepsilon) \geq c_{21}\varepsilon^{-1/2}.$$

Proof. The convex body K (which has interior points, by our general assumption) contains some ball, without loss of generality the ball rB^d. Let R be such that K slides freely in a ball of radius R. We put $c_{20} := \min\{2R, (\pi r)^2/64R\}$, and assume that $0 < \varepsilon < c_{20}$.

For points $\boldsymbol{x}, \boldsymbol{y} \in \operatorname{bd} K(\varepsilon)$, we assert that

$$\|\boldsymbol{x} - \boldsymbol{y}\| \geq 4\sqrt{R\varepsilon} \quad \Longrightarrow \quad [\boldsymbol{x}, \boldsymbol{y}] \cap \operatorname{int} K \neq \varnothing. \tag{18}$$

For the proof, let $\boldsymbol{x}, \boldsymbol{y} \in \operatorname{bd} K(\varepsilon)$ and suppose that $[\boldsymbol{x}, \boldsymbol{y}] \cap \operatorname{int} K = \varnothing$. Let $\boldsymbol{p} \in K$ and $\boldsymbol{q} \in \operatorname{aff}\{\boldsymbol{x}, \boldsymbol{y}\}$ be points of smallest distance. If $\boldsymbol{p} \neq \boldsymbol{q}$ then the hyperplane H through $\boldsymbol{p}$ orthogonal to $\boldsymbol{q} - \boldsymbol{p}$ supports K. If $\boldsymbol{p} = \boldsymbol{q}$ then the line through $\boldsymbol{x}$ and $\boldsymbol{y}$ touches K, and we choose H as a supporting hyperplane of K containing that line. The body K slides freely in a ball, say B, of radius R; hence, K is a summand of B (see [10, Theorem 3.2.2]). This means that there exists a compact convex set $M \subset \mathbb{R}^d$ such that $K + M = B$.

Let $\boldsymbol{u}$ denote the outer unit normal vector of the supporting hyperplane H of K at $\boldsymbol{p}$, so that $h(K, \boldsymbol{u}) = \langle \boldsymbol{p}, \boldsymbol{u}\rangle$. There is a point $\boldsymbol{t} \in M$ with $h(M, \boldsymbol{u}) = \langle \boldsymbol{t}, \boldsymbol{u}\rangle$, and the point $\boldsymbol{z} := \boldsymbol{p} + \boldsymbol{t}$ satisfies $\boldsymbol{z} \in B$ and $h(B, \boldsymbol{u}) = \langle \boldsymbol{z}, \boldsymbol{u}\rangle$. It follows that $K \subset B - \boldsymbol{t}$ and that H is a supporting hyperplane of $B - \boldsymbol{t}$ at $\boldsymbol{p}$.

The ball $(B - \boldsymbol{t}) + \varepsilon B^d$ contains $K(\varepsilon)$ and, hence, the segment $[\boldsymbol{x}, \boldsymbol{y}]$. The line parallel to $[\boldsymbol{x}, \boldsymbol{y}]$ through $\boldsymbol{p}$ lies in H and intersects the ball $(B - \boldsymbol{t}) + \varepsilon B^d$ in a segment S, which is not shorter than $[\boldsymbol{x}, \boldsymbol{y}]$. Thus, $\|\boldsymbol{x} - \boldsymbol{y}\| \leq \operatorname{length}(S) = 2\sqrt{2R\varepsilon + \varepsilon^2} < 4\sqrt{R\varepsilon}$, since $\varepsilon < 2R$. This proves (18).

Let m be the largest integer with

$$m \leq \frac{\pi r}{4\sqrt{R}}\varepsilon^{-1/2}.$$

Then $m \geq 2$ (by the choice of c_{20}), and there is a constant c_{21} with $m \geq c_{21}/\sqrt{\varepsilon}$. Let C be an arbitrary great circle of the ball rB^d. On C, we choose m equidistant points $\boldsymbol{y}_1, \dots, \boldsymbol{y}_m$. For $i \neq j$, we have $\|\boldsymbol{y}_i - \boldsymbol{y}_j\| \geq 2r\sin(\pi/m) > r\pi/m$. Let $\boldsymbol{x}_i = \lambda_i \boldsymbol{y}_i \in \operatorname{bd} K(\varepsilon)$ with $\lambda_i > 0$. Then $\lambda_i > 1$ for $i = 1, \dots, m$ and, hence, $\|\boldsymbol{x}_i - \boldsymbol{x}_j\| > r\pi/m \geq 4\sqrt{R\varepsilon}$ for $i \neq j$. By (18), this completes the proof.

3. Proof of Theorem 1

We assume that X and K are as in Theorem 1 and satisfy the assumptions mentioned above, that is, φ is not concentrated on a great subsphere, $\gamma \geq 1$, and inclusion (2) holds. Without loss of generality, we may assume that $\boldsymbol{o} \in \operatorname{int} K$. Recalling that Z_0 denotes the zero cell of X, we note that by the independence properties of the Poisson process we have

$$\mathbb{P}\{Z_K \not\subset K(\varepsilon)\} = \mathbb{P}\{Z_0 \not\subset K(\varepsilon) \mid K \subset Z_0\}.$$

The conditional probability involving the zero cell is slightly more convenient to handle.

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For a compact convex set $L \subset \mathbb{R}^d$, we define

$$\mathcal{H}_L := \{H \in \mathcal{H}^d : H \cap L \neq \varnothing\}$$

and

$$\Phi(L) := \Theta(\mathcal{H}_L).$$

By (1) we have

$$\Phi(L) = 2\gamma \int_{\mathbb{S}^{d-1}} h(L, \boldsymbol{u})\, \varphi(\mathrm{d}\boldsymbol{u}). \tag{19}$$

The following two lemmas use ideas from the proofs of Lemmas 3 and 5 of [5], but the present situation is simpler. As there, we use the abbreviation

$$H_1^- \cap \cdots \cap H_n^- =: P(H_{(n)}),$$

where $H_1, \ldots, H_n$ are hyperplanes not passing through $\boldsymbol{o}$ and H_i^- is the closed halfspace bounded by H_i that contains $\boldsymbol{o}$.

Let $\|\boldsymbol{x}\|_K = \min\{\lambda \geq 0 : \boldsymbol{x} \in \lambda K\}$ for $\boldsymbol{x} \in \mathbb{R}^d$. For a nonempty compact convex set L, we define $\|L\|_K := \max\{\|\boldsymbol{x}\|_K : \boldsymbol{x} \in L\}$. For $\varepsilon \geq 0$ and $m \in \mathbb{N}$, let

$$\mathcal{K}_\varepsilon^d(m) := \{L \in \mathcal{K}^d : K \subset L \not\subset K(\varepsilon),\ \|L\|_K \in (m, m+1]\}$$

and

$$q_\varepsilon(m) := \mathbb{P}\{Z_0 \in \mathcal{K}_\varepsilon^d(m)\}.$$

We abbreviate

$$(m+1)K =: K_m.$$

We have

$$q_\varepsilon(m) = \sum_{N=d+1}^{\infty} \mathbb{P}\{X(\mathcal{H}_{K_m}) = N\} p(N, m, \varepsilon) \tag{20}$$

with

$$\begin{aligned} p(N, m, \varepsilon) &:= \mathbb{P}\{Z_0 \in \mathcal{K}_\varepsilon^d(m) \mid X(\mathcal{H}_{K_m}) = N\} \\ &= \Phi(K_m)^{-N} \int_{\mathcal{H}_{K_m}^N} \mathbf{1}\{P(H_{(N)}) \in \mathcal{K}_\varepsilon^d(m)\}\, \Theta^N(\mathrm{d}(H_1, \ldots, H_N)), \end{aligned}$$

the latter by a well-known property of Poisson processes (see, e.g. [11, Theorem 3.2.2(b)]), and

$$\mathbb{P}\{X(\mathcal{H}_{K_m}) = N\} = \frac{\Phi(K_m)^N}{N!} \exp[-\Phi(K_m)]. \tag{21}$$

Lemma 4. *There exists a number m_0, depending only on K, φ, and d, such that*

$$q_0(m) \leq c_{22} \exp[-\Phi(K) - c_{23}\gamma m]$$

for $m \geq m_0$.

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Proof. We modify and adapt the proof of Lemma 3 of [5]. If $H_1, \dots, H_N \in \mathcal{H}_{K_m}$ and if $P := P(H_{(N)}) \in \mathcal{K}_0^d(m)$, then P has a vertex $\boldsymbol{v}$ with $m < \|\boldsymbol{v}\|_K \le m+1$. Since $\boldsymbol{v}$ is the intersection of some d facets of P, there exists a d-element set $J \subset \{1, \dots, N\}$ with

$$\{\boldsymbol{v}\} = \bigcap_{j \in J} H_j.$$

We denote the segment $[\boldsymbol{o}, \boldsymbol{v}]$ by $S = S(H_i,\ i \in J)$ (where it is assumed that the hyperplanes $H_i,\ i \in J$, have linearly independent normal vectors) and note that

$$H_i \cap \operatorname{relint} S = \varnothing \quad \text{for } i = 1, \dots, N.$$

For any segment $S = [\boldsymbol{o}, \boldsymbol{v}]$ with $\|\boldsymbol{v}\|_K \ge m$, we have (writing $a^+ := \max\{a, 0\}$)

$$\Phi(S) = 2\gamma \int_{\mathbb{S}^{d-1}} \langle \boldsymbol{v}, \boldsymbol{u}\rangle^+ \varphi(\mathrm{d}\boldsymbol{u}) \ge 2c_{24}\gamma m$$

with a positive constant c_{24}. This follows from the fact that the function

$$\boldsymbol{v}_1 \mapsto \int_{\mathbb{S}^{d-1}} \langle \boldsymbol{v}_1, \boldsymbol{u}\rangle^+ \varphi(\mathrm{d}\boldsymbol{u}), \qquad \boldsymbol{v}_1 \in \mathbb{S}^{d-1},$$

is positive (since φ is not concentrated on a great subsphere) and continuous. Let m_0 be the smallest integer greater than or equal to $(2/c_{24})\int_{\mathbb{S}^{d-1}} h(K, \boldsymbol{u})\, \varphi(\mathrm{d}\boldsymbol{u})$. For $m \ge m_0$, we then have

$$\Phi(S) \ge \Phi(K) + c_{24}\gamma m,$$

and, hence,

$$\int_{\mathcal{H}_{K_m}} \mathbf{1}\{H \cap S = \varnothing\}\, \Theta(\mathrm{d}H) = \Phi(K_m) - \Phi(S) \le \Phi(K_m) - \Phi(K) - c_{24}\gamma m,$$

where we used the fact that $S \subset K_m$, since $\|\boldsymbol{v}\|_K \le m+1$. Now we obtain

$$\begin{aligned} p(N, m, \varepsilon) &\le \binom{N}{d} \Phi(K_m)^{-N} \\ &\quad \times \int_{\mathcal{H}_{K_m}^d} \mathbf{1}\{\|S(H_j,\ j \in \{1, \dots, d\})\|_K \ge m\} \\ &\quad \times \int_{\mathcal{H}_{K_m}^{N-d}} \mathbf{1}\{H_i \cap S(H_j,\ j \in \{1, \dots, d\}) = \varnothing \text{ for } i = d+1, \dots, N\} \\ &\qquad \times \Theta^{N-d}(\mathrm{d}(H_{d+1}, \dots, H_N))\, \Theta^d(\mathrm{d}(H_1, \dots, H_d)) \\ &\le \binom{N}{d} \Phi(K_m)^{-N} \int_{\mathcal{H}_{K_m}^d} [\Phi(K_m) - \Phi(K) - c_{24}\gamma m]^{N-d}\, \Theta^d(\mathrm{d}(H_1, \dots, H_d)) \\ &= \binom{N}{d} \Phi(K_m)^{d-N} [\Phi(K_m) - \Phi(K) - c_{24}\gamma m]^{N-d}. \end{aligned}$$

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With (20) (for $\varepsilon = 0$) and (21) this gives

$$\begin{aligned} q_0(m) &\le \sum_{N=d+1}^{\infty} \frac{\Phi(K_m)^N}{N!} \exp[-\Phi(K_m)] \binom{N}{d} \Phi(K_m)^{d-N} [\Phi(K_m) - \Phi(K) - c_{24}\gamma m]^{N-d} \\ &= \frac{1}{d!} \Phi(K_m)^d \exp[-\Phi(K_m)] \sum_{N=d+1}^{\infty} \frac{1}{(N-d)!} [\Phi(K_m) - \Phi(K) - c_{24}\gamma m]^{N-d} \\ &\le \frac{1}{d!} \Phi(K_m)^d \exp[-\Phi(K) - c_{24}\gamma m] \\ &= \frac{1}{d!} \left(2\gamma(m+1) \int_{\mathbb{S}^{d-1}} h(K, \boldsymbol{u})\, \varphi(\mathrm{d}\boldsymbol{u}) \right)^d \exp[-\Phi(K) - c_{24}\gamma m] \\ &\le c_{22} \exp[-\Phi(K) - c_{23}\gamma m] \end{aligned}$$

with $c_{23} = c_{24}/2$, say.

Lemma 5. *Let* $0 < \varepsilon \le 1$. *Then, for* $m \in \mathbb{N}$,

$$q_\varepsilon(m) \le c_{25} (\gamma m)^d \exp[-\Phi(K) - 2\gamma \mu(K, \varphi, \varepsilon)].$$

Proof. With $H_1, \dots, H_N \in \mathcal{H}_{K_m}$ and $P = P(H_{(N)}) \in \mathcal{K}_0^d(m)$ as in the previous proof, the polytope P has a vertex $\boldsymbol{x} \in K_m \setminus K(\varepsilon)$. This vertex is the intersection of d facets of P. Hence, there exists an index set $J \subset \{1, \dots, N\}$ with d elements such that

$$\{\boldsymbol{x}\} = \bigcap_{j \in J} H_j.$$

There exists a point $\boldsymbol{y} \in \operatorname{bd} K(\varepsilon)$ such that

$$\Phi(\operatorname{conv}(K \cup \{\boldsymbol{x}\})) \ge \Phi(\operatorname{conv}(K \cup \{\boldsymbol{y}\})) = \Phi(K^{\boldsymbol{y}}) \ge \Phi(K) + 2\gamma \mu(K, \varphi, \varepsilon),$$

where the last inequality follows from (19) and (3), together with the monotonicity of Φ. This gives

$$\begin{aligned} \int_{\mathcal{H}_{K_m}} \mathbf{1}\{H \cap \operatorname{conv}(K \cup \{\boldsymbol{x}\}) = \varnothing\}\, \Theta(\mathrm{d}H) &= \Phi(K_m) - \Phi(\operatorname{conv}(K \cup \{\boldsymbol{x}\})) \\ &\le \Phi(K_m) - \Phi(K) - 2\gamma \mu(K, \varphi, \varepsilon). \end{aligned}$$

We write $\boldsymbol{x} = \boldsymbol{x}(H_1, \dots, H_d)$ for the intersection point of the hyperplanes $H_1, \dots, H_d$ (supposed in general position) and obtain

$$\begin{aligned} p(N, m, \varepsilon) &\le \binom{N}{d} \Phi(K_m)^{-N} \\ &\quad \times \int_{\mathcal{H}_{K_m}^d} \mathbf{1}\{\boldsymbol{x}(H_1, \dots, H_d) \in K_m \setminus K(\varepsilon)\} \\ &\quad \times \int_{\mathcal{H}_{K_m}^{N-d}} \mathbf{1}\{H_i \cap \operatorname{conv}(K \cup \{\boldsymbol{x}(H_1, \dots, H_d)\}) = \varnothing \text{ for } i = d+1, \dots, N\} \\ &\quad \times \Theta^{N-d}(\mathrm{d}(H_{d+1}, \dots, H_N))\, \Theta^d(\mathrm{d}(H_1, \dots, H_d)) \\ &\le \binom{N}{d} \Phi(K_m)^{d-N} [\Phi(K_m) - \Phi(K) - 2\gamma \mu(K, \varphi, \varepsilon)]^{N-d}. \end{aligned}$$

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Similarly as in the proof of Lemma 4, summation over N gives

$$
\begin{aligned}
q_\varepsilon(m) &\le \sum_{N=d+1}^{\infty} \frac{\Phi(K_m)^N}{N!} \exp[-\Phi(K_m)] \binom{N}{d} \Phi(K_m)^{d-N} \\
&\qquad \times [\Phi(K_m) - \Phi(K) - 2\gamma\mu(K,\varphi,\varepsilon)]^{N-d} \\
&\le \frac{1}{d!} \Phi(K_m)^d \exp[-\Phi(K) - 2\gamma\mu(K,\varphi,\varepsilon)] \\
&\le c_{25}(\gamma m)^d \exp[-\Phi(K) - 2\gamma\mu(K,\varphi,\varepsilon)].
\end{aligned}
$$

Proof of Theorem 1. We have

$$
\begin{aligned}
\mathbb{P}\{\delta(K, Z_K) > \varepsilon\} &= \mathbb{P}\{Z_0 \not\subset K(\varepsilon) \mid K \subset Z_0\} \\
&= \frac{\mathbb{P}\{K \subset Z_0,\ Z_0 \not\subset K(\varepsilon)\}}{\mathbb{P}\{K \subset Z_0\}} \\
&= \frac{\sum_{m=1}^{\infty} q_\varepsilon(m)}{\exp[-\Phi(K)]}.
\end{aligned}
$$

To estimate the last numerator, we choose m_0 according to Lemma 4, and use Lemma 5 for $m \le m_0$ and Lemma 4 together with $q_\varepsilon(m) \le q_0(m)$ for $m > m_0$. By the assumptions of Theorem 1, relation (2) is satisfied. We obtain

$$
\mathbb{P}\{Z_0 \not\subset K(\varepsilon) \mid K \subset Z_0\} \le \sum_{m=1}^{m_0} c_{25}(\gamma m)^d \exp[-2\gamma\mu(K,\varphi,\varepsilon)] + \sum_{m>m_0} c_{22} \exp[-c_{23}\gamma m].
$$

The first sum can be estimated by

$$
\begin{aligned}
\sum_{m=1}^{m_0} c_{25}(\gamma m)^d \exp[-2\gamma\mu(K,\varphi,\varepsilon)] &\le c_{25} m_0^{d+1} \gamma^d \exp[-\gamma\mu(K,\varphi,\varepsilon)] \exp[-\gamma\mu(K,\varphi,\varepsilon)] \\
&\le c_{26}(\varepsilon) \exp[-\gamma\mu(K,\varphi,\varepsilon)], \qquad (22)
\end{aligned}
$$

since $\mu(K,\varphi,\varepsilon) > 0$ by condition (2) and Lemma 1.

The second sum can be estimated by

$$
\sum_{m>m_0} c_{22} \exp[-c_{23}\gamma m] \le c_{22} \exp[-c_{23}\gamma] \sum_{m>m_0} \exp[-c_{23}(m-1)] \le c_{27} \exp[-c_{23}\gamma],
$$

where we have used the facts that $\gamma \ge 1$ (by assumption) and the last sum converges. Both estimates together yield (4).

4. Proofs of Theorems 2 and 3

Under assumptions (7) or (8), we can conclude from Lemma 1 that $\mu(K,\varphi,\varepsilon) \ge c_{28}\varepsilon^\alpha$ with suitable $\alpha \le d$. Therefore, in estimating (22) we can use the fact that

$$
\gamma^d \exp[-\gamma\mu(K,\varphi,\varepsilon)] \le \gamma^d \exp(-\gamma c_{28}\varepsilon^\alpha) \le c_{29}\varepsilon^{-d\alpha}.
$$

This gives

$$
\sum_{m=1}^{m_0} c_{25}(\gamma m)^d \exp[-2\gamma\mu(K,\varphi,\varepsilon)] \le c_{30}\varepsilon^{-d\alpha} \exp[-c_{31}\gamma\varepsilon^\alpha].
$$

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The estimation of the second sum above remains unchanged. Hence, under the assumptions of Theorem 2 and with $\gamma = n$, we can conclude that

$$\mathbb{P}\{\delta(K, Z_K^{(n)}) > \varepsilon\} \le c_{32}\varepsilon^{-d\alpha} \exp[-c_{33} n \varepsilon^{\alpha}].$$

We choose

$$C > \frac{d+1}{c_{33}}$$

and put

$$\varepsilon_n := \left(\frac{C \log n}{n}\right)^{1/\alpha}.$$

Then

$$\begin{aligned}\sum_{n=1}^{\infty} \mathbb{P}\{\delta(K, Z_K^{(n)}) > \varepsilon_n\} &\le \sum_{n=1}^{\infty} c_{32}\left(\frac{n}{C \log n}\right)^{d} \exp(-c_{33} C \log n) \\ &= c_{34} \sum_{n=1}^{\infty} (\log n)^{-d} n^{d - c_{33} C} \qquad (23) \\ &< \infty. \qquad (24)\end{aligned}$$

The Borel–Cantelli lemma gives

$$\mathbb{P}\{\delta(K, Z_K^{(n)}) > \varepsilon_n \text{ for infinitely many } n\} = 0;$$

hence,

$$\mathbb{P}\left\{\delta(K, Z_K^{(n)}) \le \left(\frac{C \log n}{n}\right)^{1/\alpha} \text{ for sufficiently large } n\right\} = 1.$$

This completes the proof of Theorem 2.

Proof of Theorem 3. Since K slides freely in some ball, say of radius R, there is a convex body L with $K + L = RB^d$ (see [10, Theorem 3.2.2]). From the polynomial expansion of $S_{d-1}(K + L, \cdot)$ (see [10, Equation (5.18)]), it follows that $S_{d-1}(K, \cdot) \le S_{d-1}(RB^d, \cdot) = R^{d-1}\sigma$. Together with assumption (10) this shows that (8) is satisfied. Therefore, Theorem 2 yields

$$\delta(K, Z_K^{(n)}) = \mathrm{O}\left(\left(\frac{\log n}{n}\right)^{2/(d+1)}\right) \quad \text{almost surely} \qquad (25)$$

as $n \to \infty$.

Let $0 < \varepsilon < c_{20}$ (with c_{20} as in Lemma 3). According to Lemma 3, we can choose

$$m = m(\varepsilon) \ge c_{21}\varepsilon^{-1/2}$$

points $\boldsymbol{x}_1, \dots, \boldsymbol{x}_m \in \operatorname{bd} K(\varepsilon)$ such that the segment joining any two of them intersects the interior of K. Let $n \in \mathbb{N}$. Suppose that $\delta(K, Z_K^{(n)}) < \varepsilon$. Then each point $\boldsymbol{x}_i$ is strictly separated from K by some hyperplane from X_n. Let $\mathcal{A}_i \subset \mathcal{H}^d$ be the set of hyperplanes strictly separating $\boldsymbol{x}_i$ and K. By the choice of the points $\boldsymbol{x}_1, \dots, \boldsymbol{x}_m$, the sets $\mathcal{A}_1, \dots, \mathcal{A}_m$

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are pairwise disjoint. Since X_n is a Poisson process, the processes $X_n \llcorner \mathcal{A}_1, \dots, X_n \llcorner \mathcal{A}_m$ are stochastically independent (see, e.g. [11, Theorem 3.2.2]). It follows that

$$\begin{aligned}\mathbb{P}\{\delta(K, Z_K^{(n)}) < \varepsilon\} &\le \mathbb{P}\{X_n(\mathcal{A}_i) \ge 1 \text{ for } i = 1, \dots, m\}\\ &= \prod_{i=1}^{m} \mathbb{P}\{X_n(\mathcal{A}_i) \ge 1\}\\ &= \prod_{i=1}^{m} [1 - \mathbb{P}\{X_n(\mathcal{A}_i) = 0\}]\\ &= \prod_{i=1}^{m} (1 - \exp[-\Theta_n(\mathcal{A}_i)]),\end{aligned}$$

where Θ_n is the intensity measure of X_n. Since the assumptions on K in Lemma 2 are satisfied, we can conclude that

$$\begin{aligned}\Theta_n(\mathcal{A}_i) &= \Theta_n(\mathcal{H}_{K^{x_i}}) - \Theta_n(\mathcal{H}_K)\\ &= 2n \int_{\mathbb{S}^{d-1}} [h(K^{\boldsymbol{x}_i}, \boldsymbol{u}) - h(K, \boldsymbol{u})]\, \varphi(\mathrm{d}\boldsymbol{u})\\ &\le 2nc_{17}\varepsilon^{(d+1)/2}.\end{aligned}$$

This gives

$$\mathbb{P}\{\delta(K, Z_K^{(n)}) < \varepsilon\} \le (1 - \exp[-2c_{17}n\varepsilon^{(d+1)/2}])^{m(\varepsilon)}.$$

Now we choose

$$\varepsilon_n^{(d+1)/2} = \frac{c \log n}{n}$$

with

$$0 < c < \frac{1}{4c_{17}(d+1)}.$$

Then

$$\mathbb{P}\{\delta(K, Z_K^{(n)}) < \varepsilon_n\} \le (1 - n^{-2c_{17}c})^{m(\varepsilon_n)}$$

with

$$m(\varepsilon_n) \ge c_{21}\varepsilon_n^{-1/2} = c_{21}\left(\frac{n}{c \log n}\right)^{1/(d+1)} > c_{35}n^{1/(2d+2)}$$

for sufficiently large n. With $p := 2c_{17}c$ and $q := 1/(2d+2)$ we have $q > p$ and

$$(1 - n^{-2c_{17}c})^{m(\varepsilon_n)} < \left(1 - \frac{1}{n^p}\right)^{c_{35}n^q} = \left[\left(1 - \frac{1}{n^p}\right)^{n^p n^{q-p}}\right]^{c_{35}} \le (\mathrm{e}^{-c_{35}})^{n^{q-p}}.$$

It follows that

$$\sum_{n=1}^{\infty} \mathbb{P}\left\{\delta(K, Z_K^{(n)}) < \left(\frac{c \log n}{n}\right)^{2/(d+1)}\right\} < \infty.$$

From the Borel–Cantelli lemma we conclude that

$$\mathbb{P}\left\{\delta(K, Z_K^{(n)}) < \left(\frac{c \log n}{n}\right)^{2/(d+1)} \text{ for infinitely many } n\right\} = 0$$

and, hence,

$$\mathbb{P}\left\{\delta(K, Z_K^{(n)}) \geq \left(\frac{c\log n}{n}\right)^{2/(d+1)} \text{ for almost all } n\right\} = 1.$$

Together with (25), this completes the proof of Theorem 3.

Finally, we construct a directional distribution exhibiting property (6) for a given convex body K. We do that at this stage since arguments appearing in the previous proofs are employed.

As explained before (6), we assume that a decreasing sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty}\varepsilon_n = 0$ is given. For $n \in \mathbb{N}$, let X_n be a stationary Poisson hyperplane process with intensity n and directional distribution φ, to be constructed.

The d-dimensional convex body K contains some ball touching the boundary; hence, there exists a number $r > 0$ and a point $\boldsymbol{x} \in \operatorname{bd} K$ such that $\boldsymbol{x}$ is contained in a ball of radius r that is contained in K. Let $N(\boldsymbol{x})$ be the unique outer unit normal vector of K at $\boldsymbol{x}$, and let $\boldsymbol{y} = \boldsymbol{x} + \varepsilon N(\boldsymbol{x})$. Let $n \in \mathbb{N}$, and suppose that $\delta(K, Z_K^{(n)}) < \varepsilon_n$. Then the point $\boldsymbol{y}$ is strictly separated from K by some hyperplane of X_n. Similarly as in the proof of Theorem 3, this yields

$$\mathbb{P}\{\delta(K, Z_K^{(n)}) < \varepsilon_n\} \leq 1 - \exp[-2n\varepsilon_n\varphi(S_n)]$$

with

$$S_n := \left\{\boldsymbol{u} \in \mathbb{S}^{d-1} : \langle \boldsymbol{u}, N(\boldsymbol{x})\rangle \geq \frac{r}{r+\varepsilon_n}\right\}.$$

It is easy to construct an even, positive measurable function g on $\mathbb{S}^{d-1}$ such that the measure φ defined by $\mathrm{d}\varphi = g\,\mathrm{d}\sigma$ is a probability measure and that

$$2n\varepsilon_n\varphi(S_n) < |\log(1-n^{-2})|$$

for all $n \in \mathbb{N}$ (for example, g can be a suitable constant on $S_n \setminus S_{n+1}$). The directional distribution φ then satisfies

$$1 - \exp[-2n\varepsilon_n\varphi(S_n)] < \frac{1}{n^2}$$

and, hence,

$$\sum_{n=1}^{\infty} \mathbb{P}\{\delta(K, Z_K^{(n)}) < \varepsilon_n\} < \infty.$$

As in the proof of Theorem 3, this yields (6).

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