TOPOLOGICALLY STABLE HOMEOMORPHISMS
OF THE CIRCLE

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Introduction

In this paper, we study topologically stable homeomorphisms of the circle. Our results are the following:

THEOREM 1. A homeomorphism of the circle is topologically stable if and only if it is topologically conjugate to some Morse-Smale diffeomorphism.

THEOREM 2. There exists a homeomorphism of the circle which has the pseudo-orbit-tracing-property but is not topologically stable.

In [1], Bowen introduced the concept of the pseudo-orbit-tracing-property and essentially showed that expansive homeomorphisms with this property are topologically stable. Recently in [2], Morimoto has proved that the topological stability implies the pseudo-orbit-tracing-property. Theorem 2 above shows that expansiveness condition is necessary in Bowen's result.

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Notations and Definitions

Let $X$ be a compact metric space with metric $d$. For continuous maps $h_1, h_2$ from $X$ to itself, we set

$$\bar{d}(h_1, h_2) = \sup_{x \in X} d(h_1(x), h_2(x)).$$

With this metric, the set of all continuous maps from $X$ to itself is a metric space. Let $f$ and $g$ be homeomorphisms of $X$.

DEFINITION 1. We say that $g$ is topologically conjugate (resp. semi-conjugate) to $f$ by $h$, if $h$ is a homeomorphism (resp. a continuous map)
from $X$ onto itself satisfying $h \circ g = f \circ h$.

If $g$ is topologically conjugate (resp. semi-conjugate) to $f$ by $h$, $g^n$ is also topologically conjugate (resp. semi-conjugate) to $f^n$ by $h$ for any integer $n$.

**DEFINITION 2.** We call $f$ **topologically stable** if for any $\varepsilon > 0$, there exists $\delta > 0$ such that every homeomorphism $f'$ with $\bar{d}(f, f') < \delta$ is topologically semi-conjugate to $f$ by some $h$ satisfying $\bar{d}(h, \text{id}_X) < \varepsilon$.

**DEFINITION 3.** A sequence $\{x_n\}_{n \in \mathbb{Z}}$ of points in $X$ is called a $\delta$-**pseudo orbit** of $f$ if $d(f(x_n), x_{n+1}) < \delta$ for any integer $n$, and is said to be $\varepsilon$-**traced** by a point $x$ of $X$ if $d(f^n(x), x_n) < \varepsilon$ for any integer $n$.

**DEFINITION 4.** We say that $f$ has the **pseudo-orbit-tracing-property** if for any $\varepsilon > 0$ there exists $\delta > 0$ such that every $\delta$-pseudo-orbit of $f$ is $\varepsilon$-traced by some point.

Consider $S^1$ as $\mathbb{R}/\mathbb{Z}$ and let $d$ denote the standard metric on $S^1$. The set of all fixed points of $f$ and the set of all periodic points of $f$ are denoted by $\text{Fix}(f)$ and $\text{Per}(f)$, respectively.

**DEFINITION 5.** A diffeomorphism $f$ of $S^1$ is called **Morse-Smale** if $\text{Per}(f)$ is non-empty and every element of $\text{Per}(f)$ is hyperbolic, i.e., the differential of $f^n$ at $x$ is different from $\pm 1$ for every periodic point $x$ of $f$ with period $n$.

**DEFINITION 6.** Let $f$ be an orientation preserving homeomorphism of $S^1$. A fixed point $x$ of $f$ is said to be **topologically hyperbolic** if $x$ is isolated in $\text{Fix}(f)$ and $f(t) - t$ changes its sign at $t = x$. A periodic point $x$ of any homeomorphism $g$ of $S^1$ is called **topologically hyperbolic** if $x$ is a topologically hyperbolic fixed point of $g^n$, where $n$ is a period of $x$.

**Proof of Theorem 1**

By a theorem of Nitecki [3], every Morse-Smale diffeomorphism is topologically stable. Since the topological stability is invariant under the topological conjugacy, every homeomorphism which is topologically conjugate to some Morse-Smale diffeomorphism is topologically stable.

It is easy to see that a homeomorphism $f$ of $S^1$ is topologically conjugate to some Morse-Smale diffeomorphism if and only if it satisfies the following two conditions:
(a) $\text{Per}(f)$ is non-empty and finite.
(b) Every element of $\text{Per}(f)$ is topologically hyperbolic.

Hence, to prove Theorem 1, it suffices to show that every topologically stable homeomorphism of $S^1$ satisfies the above conditions (a) and (b). First we prove the following

**Lemma 1.** Suppose $f$ and $g$ are orientation preserving homeomorphisms of $S^1$ and $g$ is topologically semi-conjugate to $f$ by some $h$. Then if $\text{Per}(g)$ is non-empty, so is $\text{Per}(f)$, and there exists a constant $C$ depending only on $f$ such that if $h$ satisfies $d(h, \text{id}_{S^1}) \leq C$, the cardinality of $\text{Per}(g)$ is not less than that of $\text{Per}(f)$.

**Proof.** Since for every homeomorphism $f'$ of $S^1$ there exists an integer $n$ such that $\text{Per}(f') = \text{Fix}(f'^n)$, it is enough to prove Lemma 1 replacing $\text{Per}(\ )$ by $\text{Fix}(\ )$.

The proof of the first part is immediate. To prove the second part, we take a positive constant $C$ satisfying the following two conditions:

1. $C \leq 1/8$.
2. If $I$ is a closed interval in $S^1$ of length not greater than $4C$, then the length of $f(I)$ is not greater than $1/4$.

Suppose $g$ is topologically semi-conjugate to $f$ by a map $h$ with $d(h, \text{id}_{S^1}) \leq C$ and take a fixed point $x$ of $f$. Since $h^{-1}(x)$ is a non-empty closed $g$-invariant subset of $S^1$ contained in $[x - C, x + C]$, we can define $\sup h^{-1}(x)$ and $\inf h^{-1}(x)$ without confusion. Put $B = [\inf h^{-1}(x), \sup h^{-1}(x)]$. Then we have

\[
\text{(length of } B) \leq 2C
\]

and

\[
\text{(length of } g(B)) \leq (\text{length of } h \circ g(B)) + 2d(h, \text{id}_{S^1}) \\
\leq (\text{length of } f \circ h(B)) + 2C \\
\leq 1 + \frac{1}{4} = \frac{5}{4}.
\]

For the last inequality, we used the condition (2) and the estimation:

\[
\text{(length of } h(B)) \leq (\text{length of } B) + 2d(h, \text{id}_{S^1}) \leq 4C.
\]

Hence

\[
\text{(length of } B) + (\text{length of } g(B)) \leq 2C + 1 \leq \frac{5}{2}.
\]

By the invariance of $h^{-1}(x)$, both end points of $g(B)$ are in $B$ and those of $B$ in $g(B)$. Because the total length of $S^1$ is one, the above estimation shows that $B$ is $g$-invariant. Since $g$ is orientation preserving, $\sup h^{-1}(x)$ is a fixed point of $g$. Hence we have an injection from $\text{Fix}(f)$
to \( \text{Fix}(g) \), which maps \( x \) to \( \sup h^{-1}(x) \). This completes the proof of Lemma 1.

Now suppose that \( f \) is topologically stable. Since every homeomorphism of \( S^1 \) can be approximated by a diffeomorphism, by a theorem of Peixoto ([4]; p. 51), \( f \) is approximated by a Morse-Smale diffeomorphism. Thus there exists a Morse-Smale diffeomorphism which is topologically semi-conjugate to \( f \). Therefore, by Lemma 1, \( \text{Per}(f) \) is non-empty and finite. (If \( f \) is orientation reversing, apply Lemma 1 to \( f^{-1} \)).

Next, take a periodic point \( x \) of \( f \). If \( x \) is not topologically hyperbolic, we can eliminate this periodic point by a small perturbation, this contradicts Lemma 1. Therefore \( f \) satisfies the conditions (a) and (b), this completes the proof of Theorem 1.

**Proof of Theorem 2**

First consider a homeomorphism \( f_0 \) of \([0,1]\) defined by

\[
f_0(t) = \begin{cases} 
\frac{1}{2}t, & 0 \leq t \leq \frac{1}{4}, \\
\frac{3}{8}t - \frac{1}{4}, & \frac{1}{4} \leq t \leq \frac{3}{4}, \\
\frac{1}{2}t + \frac{1}{4}, & \frac{3}{4} \leq t \leq 1.
\end{cases}
\]

Let \( p_n = 1/2^n, p_{-n} = 1 - 1/2^n \), \( q_n = (1/2^{n+1})(1+1/2) \) and \( q_{-n} = 1 - (1/2^{n+1})(1+1/2) \) for a positive integer \( n \). Then the desired homeomorphism \( f \) of \( S^1 = [0,1]/\sim \) is given as follows:

\[
f(0) = 0,
\]

\[
f(x) = \begin{cases} 
p_{n+1} + (1/2^{n+1})f_n(2^{n+1}(x - p_{n+1})), & p_{n+1} \leq x \leq p_n, \\
p_{-n} + (1/2^{n+1})f_n(2^{n+1}(x - p_{-n})), & p_{-n} \leq x \leq p_{n-1}.
\end{cases}
\]

Since \( \text{Fix}(f) \) is an infinite set, Theorem 1 implies that \( f \) is not topologically stable. So we have only to show that \( f \) has the pseudo-orbit-tracing-property.

**Lemma 2.** For a real number \( k \), let \( L_k \) be the linear map from \( \mathbb{R} \) to itself defined by \( L_k(x) = kx \). Suppose \( \{x_n\} \) is a \( \delta \)-pseudo-orbit of \( L_k \) with \( x_0 \in [-M, M] \).

(i) If \( 0 < k < 1 \) and \( \delta \leq (1 - k)M \), then \( x_n \in [-M, M] \) for every \( n \geq 0 \).

(ii) If \( k > 1 \) and \( \delta \leq (k - 1)M \), then \( x_n \in [-M, M] \) for every \( n \leq 0 \).

The proof is immediate and is omitted.
Fix an arbitrary positive number $\varepsilon$ and choose a positive integer $n$ satisfying $1/2^n < \varepsilon$. Let $I = [p_n, p_{n+1}]$, $J = [p_{n+1}, p_{n-1}]$ and $J' = [p_{n-1}, p_{n-1}]$. Then $J \subset J'$ and $I \cup J = S'$. By the definition of $f$, there exists a homeomorphism $\tilde{f}$ of $S'$, which is topologically conjugate to some Morse-Smale diffeomorphism and satisfies $\tilde{f}|_{J'} = f|_{J'}$.

Now we take a constant $\delta$ satisfying the following two conditions:

1. Every $\delta$-pseudo-orbit of $\tilde{f}$ can be $1/2^{n+2}$-traced by some point.
2. $\delta \leq 1/2^{n+4}$.

Suppose $\{x_i\}$ is a $\delta$-pseudo-orbit of $f$. Then we can show that $\{x_i\}$ is $\varepsilon$-traced by some point as follows:

**Case 1.** For every integer $n$, $x_n$ is in $I$.

It is evident that $0 \in S'$ $\varepsilon$-traces this sequence with respect to $f$.

**Case 2.** There exists $m$ such that $x_m \not\in I$.

By Lemma 2 together with the condition (2), this sequence does not jump over the intervals $[p_n - 1/2^{n+3}, p_{n+1} + 1/2^{n+3}]$ and $[p_{n+1} - 1/2^{n+3}, p_{n+1} + 1/2^{n+3}]$ in the positive direction and the intervals $[q_n - 1/2^{n+3}, q_{n+1} + 1/2^{n+3}]$ and $[q_{n+1} - 1/2^{n+3}, q_{n+1} + 1/2^{n+3}]$ in the negative direction. Therefore this sequence always stays in $J$, and is a $\delta$-pseudo-orbit of $\tilde{f}$. By the condition (1), there exists a point $x$ of $S'$ which $1/2^{n+2}$-traces this sequence with respect to $\tilde{f}$. In particular $x$ is in the $1/2^{n+2}$-neighborhood of $x_0$, hence in $J'$. It follows that $x$ $\varepsilon$-traces this sequence with respect to $f$.

Thus we have shown that every $\delta$-pseudo-orbit of $f$ can be $\varepsilon$-traced by some point. This completes the proof of Theorem 2.

**References**


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