NUMERICAL RANGE ESTIMATES FOR THE NORMS OF ITERATED OPERATORS

by M. J. CRABB

(Received 8 February, 1969)

Let X be a complex normed space, with dual space X', and T a bounded linear operator on X. The numerical range $V(T)$ of T is defined as $\{f(Tx): x \in X, f \in X', \| x \| = \| f \| = f(x) = 1\}$. Let $| V(T) |$ denote $\sup \{ | \lambda | : \lambda \in V(T) \}$. Our purpose is to prove the following theorem.

THEOREM.
$$\|T^n\| \leqq n!\left(\frac{e}{n}\right)^n |V(T)|^n \qquad (n=1,2,\ldots). \tag{1}$$

From the proof of Stirling's formula, it is known that

$$\frac{n!\,e^n}{n^n} \leqq en^{\frac{1}{2}} \qquad (n = 1,2,\ldots).$$

The estimate for $\| T^n \|$ given in the present theorem is therefore very much better than the estimate $\| T^n \| \leqq e^n | V(T) |^n$ given by the case $n = 1$.

When X is a complex Hilbert space, $V(T) = \{ (Tx, x): \| x \| = 1 \}$. In this case, Berger [**1**] proves that $| V(T) | \leqq 1$ implies that $| V(T^n) | \leqq 1$, and so $\| T^n \| \leqq 2$, for positive integers n. An elementary proof of this is given by Pearcy [**8**]. For a general normed space, $V(T)$ is the union of all possible numerical ranges $W(T)$ in the sense of Lumer [**6**]. For each such $W(T)$, $| W(T) | = | V(T) |$, and so we may replace V by W in (1). The theorem for the case $n = 1$ was proved by Bohnenblust and Karlin [**3**]; a simplified proof was given by Glickfeld [**5**], and the present result is based on his argument.

We shall require the following elementary result from Lumer [**6**].

LEMMA. *Let T be a bounded linear operator on a Banach space, with $| V(T) | < 1$. Then $(I-T)^{-1}$ exists, $\| (I-T)^{-1} \| \leqq (1 - | V(T) |)^{-1}$, and $\rho(T) \leqq | V(T) |$, where $\rho(T)$ denotes the spectral radius of T.*

Proof of Theorem. By [**6**], we have

$$\sup \operatorname{Re} V(T) = \lim_{\alpha \to 0+} \frac{\|I + \alpha T\| - 1}{\alpha}$$

and therefore $| V(T) |$ is unchanged if X is replaced by its completion. We assume therefore that X is complete.

Let ω_k $(k = 1, 2, \ldots, m)$ be the mth roots of unity. Let n and p be positive integers. Assume first that $| V(T) | = \mu < 1$, and that $m > n$. Then

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$$\frac{1}{m}\sum_{k=1}^{m}\omega_k^n(I-\omega_k^{-1}T)^{-p}=\frac{1}{m}\sum_{k=1}^{m}\omega_k^n\left\{I+p\omega_k^{-1}T+\ldots+\frac{p(p+1)\ldots(p+n-1)}{1.2\ldots n}\omega_k^{-n}T^n+\ldots\right\}$$

$$=\frac{p(p+1)\ldots(p+n-1)}{1.2\ldots n}T^n+\frac{p(p+1)\ldots(p+n+m-1)}{1.2\ldots(n+m)}T^{n+m}+\ldots.$$

By the lemma,

$$\|(I-\omega_k^{-1}T)^{-1}\|\leqq(1-\mu)^{-1}.$$

Therefore

$$\|(I-\omega_k^{-1}T)^{-p}\|\leqq(1-\mu)^{-p},$$

and

$$\left\|\frac{1}{m}\sum_{k=1}^{m}\omega_k^n(I-\omega_k^{-1}T)^{-p}\right\|\leqq(1-\mu)^{-p}.$$

Letting $m\to\infty$, we deduce that

$$\frac{p(p+1)\ldots(p+n-1)}{1.2\ldots n}\|T^n\|\leqq(1-\mu)^{-p}. \tag{2}$$

If $\mu=0$, (2) gives $p\|T\|\leqq 1$ $(p=1,2,\ldots)$, so that $T=0$. So assume that $\mu\neq 0$.

Now let T be any bounded linear operator on X, and apply (2) to $nT/(p+1)\mu$ for $p>n-1$. Then

$$\|T^n\|\leqq\frac{1.2\ldots n}{p(p+1)\ldots(p+n-1)}\left(\frac{p+1}{n}\right)^n\left(\frac{p-n+1}{p+1}\right)^{-p}\mu^n.$$

Letting $p\to\infty$, we have

$$\|T^n\|\leqq n!\left(\frac{e}{n}\right)^n\mu^n.$$

Remark. We do not know of any operator T such that $|V(T)|\leqq 1$ and $\{\|T^n\|\}$ is unbounded. It is quite easy to prove that, if T is an operator on a finite-dimensional space, or, more generally, is a meromorphic operator (Taylor [9], Caradus [4]), then $\{\|T^n\|\}$ is bounded whenever $|V(T)|\leqq 1$.

To prove this, let T be a bounded linear meromorphic operator with $|V(T)|=1$. Let $\mathrm{sp}(T)$ denote the spectrum of T. Suppose that $\lambda\in\mathrm{sp}(T)$ with $|\lambda|=1$. Then there exists an idempotent P such that $TP=PT$, $(\lambda I-T)P$ is nilpotent, and $(\lambda I-T)(I-P)$ is invertible in $(I-P)X$. Since λ is a boundary point of the convex hull of $V(T)$, Theorem 4 of Nirschl and Schneider [7], extended to the case of linear operators on general normed linear spaces, is applicable. This shows that $(\lambda I-T)^2x=0$ implies that $(\lambda I-T)x=0$. It follows that $(\lambda I-T)P=0$. Therefore

$$T=TP+T(I-P)=\lambda P+T(I-P).$$

Since the non-zero points of $\mathrm{sp}(T)$ are isolated, T may therefore be written

$$T=\sum_{i=1}^{m}\lambda_iP_i+S,$$

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where $|\lambda_i| = 1$, $P_i^2 = P_i$, $P_i S = SP_i = 0$, $P_i P_j = 0$ $(i \neq j)$, and $\rho(S) < 1$. Then

$$T^n = \sum_{i=1}^{m} \lambda_i^n P_i + S^n,$$

so that $\{\|T^n\| : n = 1, 2, \ldots\}$ is bounded.

Added in proof. The author has found an example of a non-zero operator T for which equality holds in (1) for every integer n; details of this will be published elsewhere.

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