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THE TRANSIENT BLOCKING PROBABILITIES IN $M/M/N$ LOSS SYSTEMS VIA LARGE DEVIATIONS

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Abstract

By using large devaitions theory, we give asymptotic formulas for the transient blocking probabilities of $M/M/N/N$ and M (with finite Poissonian sources) $M/N/N$ queues.

LARGE DEVIATIONS; LOSS SYSTEM

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1. Introduction and motivation

This paper follows the short communication [7], based on [3], in which the author gives an intuitive method for the derivation of the transient blocking probability in a $M[\lambda N]/M[\mu]/N/N$ loss system, where λN and μ represent the parameters of the exponential interarrival and service distributions. In this paper we denote this system by Σ_1. In the references mentioned, the probability $P_N(t, N)$ that all the N servers are busy at time t was approximated by the following formula:

$$P_N(t, N) \approx C[P_1(t, 1)]^N = C[1 - \exp(-(\lambda + \mu)t)]^N, \tag{1}$$

where $P_1(t, 1)$ is the blocking probability of an $M[\lambda]/M[\mu]/1/1$ system at time t, supposed to be initially free. We prove here that this probability (1) is rather related to the transient blocking probability of a $M[\lambda(N - Q(t))]/M[\mu]/N/N$ queue, where $\lambda(N - Q(t))$ means that the interarrival time between two successive customers entering the system has an exponential distribution with parameter $\lambda(N - Q(t))$, when the number of busy servers just before the arrival of the new customer is $Q(t)$. We denote this second system by Σ_2. We also give the corrected formula for the system $M[\lambda N]/M[\mu]/N/N$. The proofs are based on the principle of large deviations.

2. Main results

Let us denote by $(Q^N(t))_{t \geqq 0}$ the queue length process which is a pure jump Markov process for both systems Σ_1 and Σ_2. We denote by 1_E the indicator of event E; it is equal to 1 if E is true and 0 otherwise. When $Q^N(t) = q$, the jumps $e_+ = +1$ occur with the intensities $\lambda^N(q) = N\lambda I_{q \leqq N}$ and $\lambda^N(q) = \lambda(N - q)$ for Σ_1 and Σ_2 respectively, and the jumps $e_- = -1$ occur with the intensity $\mu^N(q) = \mu(q \wedge N)$ for both systems, where $q \wedge N$ is the minimum of q and N. The sequence of processes $(X^N(t)) = (Q^N(t)/N)$ satisfies the principle of large deviations (cf. [4], [5], [6]). Let A^N denote the infinitesimal generator of X^N. The instantaneous Laplace transform of $(X^N(t))$ is defined by $G^N(x, z) = (A^N f_z/f_z)(x)$ where

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$f_z(x) = \exp(zx)$. Direct computations give

(2) $$G^N(x, z) = NG(x, z/N)$$

where

(3) $$G(x, z) = \lambda(x)[e^z - 1] + \mu(x \wedge 1)[e^{-z} - 1]$$

with $\lambda(x) = \lambda I_{x \leqq 1}$ for Σ_1 and $\lambda(x) = \lambda(1-x)$ for Σ_2. Its Cramer–Legendre transform (cf. [5]) is defined by

(4) $$H(r, r') = \sup_z (zr' - G(r, z)).$$

Let $B_{0,T}$ be the set of piecewise differentiable right-continuous functions ϕ defined on $(0, T)$, having left limits and such that $\phi(0) = X_{\text{eq}}$ and $\phi(T) = 1$, where X_{eq} is the equilibrium point of $X^\infty(t) = \lim_{N\to+\infty} X^N(t)$. The action functional (cf. [2]) of $B_{0,T}$ is $S(B_{0,T}) = \inf_{\phi \in B_{0,T}} \int_0^T H(\phi(t), \phi'(t))\, dt$. By the principle of large deviations, we have

(5) $$S(B_{0,T}) = \lim_{N\to+\infty} -\frac{1}{N} \log \boldsymbol{P}((X^N(t))_{0 \leqq t \leqq T} \in B_{0,T}).$$

The Euler–Lagrange equation associated to the dynamic programming problem (cf. [1], [6])

(6) $$S(B) = \inf_{0 \leqq T} S(B_{0,T}) = \inf_{T \geqq 0, \phi \in B_{0,T}} \int_0^T H(\phi(t), \phi'(t))\, dt$$

is:

(7) $$\frac{\partial H}{\partial t} - \frac{d}{dt}\left(\frac{\partial H}{\partial r'}\right) = 0$$

(8) $$r(0) = X_{\text{eq}} \quad \text{and} \quad r(T) = 1.$$

An integration by parts gives

$$H(r, r') - r' \frac{\partial H}{\partial r'} = 0,$$

$$r(0) = X_{\text{eq}} \quad \text{and} \quad r(T) = 1.$$

Thus, for a solution r of (7, 8), we obtain

(9) $$S(B) = \int_0^T r'(t) \frac{\partial H}{\partial r'}\, dt = \int_{r(0)}^{r(T)} \frac{\partial H}{\partial r'}\, dr.$$

From the definition (4) we get $\partial H/\partial r' = z(r')$, where $z(r')$ maximizes $zr' - G(r, z)$. This, together with the Euler–Lagrange equation (7), gives $G(r, z(r')) = 0$; from which we can solve $z(r')$ as a function of r and get a tractable expression of $\int_{r(0)}^{r(T)} (\partial H/\partial r')\, dr$.

For Σ_1:

$$r(0) = X_{\text{eq}} = \frac{\lambda}{\mu},$$

$$z(r') = \log \frac{\mu r}{\lambda}.$$

Thus,

$$S(B) = \log\left(\frac{\mu}{\lambda}\right) - \left(1 - \frac{\lambda}{\mu}\right).$$

We can easily check that

$$\lim_{N\to+\infty} -\frac{1}{N} \log\left(B\left(N, \rho = \frac{\lambda}{\mu}\right)\right) = S(B)$$

where $B(N, \rho)$ is the Erlang loss probability.

For Σ_2:

$$r(0) = X_{eq} = \frac{\lambda}{\mu + \lambda},$$

$$z(r') = \log \frac{\mu r}{\lambda(1-r)}.$$

Thus,

$$S(B) = \log\left(1 + \frac{\mu}{\lambda}\right).$$

For finite T, $S(B_{0,T})$ is computed by using the optimal function $(r(t))$ realizing the minimum of $S(B)$. The optimal function verifies

$$\begin{aligned} r'(t) &= \mu(r(t)) - \lambda(r(t)) \\ r(T) &= 1. \end{aligned} \tag{10}$$

The initial condition $r(0)$ depends on T and is computed by using the final condition $r(T) = 1$. Therefore,

$$S(B_{0,T}) = S(B) - \int_{X_{eq}}^{r(0)} z(r')\,dr = S(B) - I(T), \tag{11}$$

where

$$I(T) = \int_{X_{eq}}^{r(0)} z(r')\,dr. \tag{12}$$

The solution of the linear differential equation (10) added to the condition $r(T) = 1$ give the initial value of $r(t)$. This yields for Σ_1 and Σ_2, respectively:

$$r_{\Sigma_1}(0) = \left\{1 - \frac{\lambda}{\mu}[1 - \exp(\mu T)]\right\} \exp(-\mu T) \tag{13}$$

$$r_{\Sigma_2}(0) = \left\{1 - \frac{\lambda}{\mu + \lambda}[1 - \exp((\mu + \lambda)T)]\right\} \exp(-(\mu + \lambda)T). \tag{14}$$

Let us determine $I(T)$ and compute its asymptotic value when T grows to infinity. By replacing, for Σ_1, expression (13) of $r_{\Sigma_1}(0)$ into the definition of $I(T)$, we obtain

$$\begin{aligned} I(T) &= \int_{\lambda/\mu}^{r_{\Sigma_1}(0)} z(r')\,dr \\ &= \int_1^{\mu/\lambda r_{\Sigma_1}(0)} \log(r)\,dr \\ &= \{r \log(r) - r\}_1^{[\mu/\lambda - (1 - \exp(\mu T))]} \exp(-\mu T). \end{aligned}$$

Using the Maclaurin expansion of $\log(1 + y)$ for $y = (\mu/\lambda - 1)\exp(-\mu T)$ and large values of T, we obtain that $I(T)$ is equivalent (denoted by $\sim$) to $(\mu/\lambda) - 1)^2 \exp(-2\mu T)$ when T grows to infinity. The expression of $I(T)$ and its equivalent for Σ_2 are derived in a similar way. These results are summarized as follows:

$$\begin{aligned} I_{\Sigma_1}(T) &= (1 + y)\log(1 + y) - y - 1, \quad \text{where} \quad y = ((\mu/\lambda) - 1)\exp(-\mu T), \\ I_{\Sigma_1}(T) &\sim_{T\to\infty} \frac{1}{2}\left(\frac{\mu}{\lambda} - 1\right)^2 \exp(-2\mu T), \\ I_{\Sigma_2}(T) &= \alpha(1 + y')\log(1 + y') + [1 - \alpha(1 + y')]\log\left(1 - \frac{\alpha}{1-\alpha} y'\right), \\ &\text{where} \quad \alpha = \lambda/(\lambda + \mu) \text{ and } y' = (\mu/\lambda)\exp(-(\lambda + \mu)T), \end{aligned} \tag{15}$$

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$$I_{\Sigma_2}(T) \sim_{T\to\infty} \frac{1}{2}\frac{\mu}{\lambda} \exp(-2(\lambda+\mu)T). \tag{16}$$

Remark that the asymptotic value of $I_{\Sigma_1}(T)$ depends exponentially only on μ while $I_{\Sigma_2}(T)$ depends on μ and λ. The coefficient of T in (15) and (16), which we call relaxation time, represents the convergence rate of X^N. It is larger for Σ_2 than for Σ_1 due to the dynamic of the customers' arrivals. The relaxation time $(\lambda+\mu)$ of Σ_1, derived in [7] (formula 1) is, in fact, related to that of Σ_2 given here by formula (16).

This method can be applied to more general queueing networks than $M/M/N/N$ loss systems studied in this paper. Since this technique is based on large deviations theory, the time-transient behavior result is asymptotic when the network is large.

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