Bull. Austral. Math. Soc. 81R05, 20C35, 22E70
Vol. 48 (1993) [275–301]

QUANTUM DOUBLE FINITE GROUP ALGEBRAS AND THEIR REPRESENTATIONS

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The quantum double construction is applied to the group algebra of a finite group. Such algebras are shown to be semi-simple and a complete theory of characters is developed. The irreducible matrix representations are classified and applied to the explicit construction of R-matrices: this affords solutions to the Yang-Baxter equation associated with certain induced representations of a finite group. These results are applied in the second paper of the series to construct unitary representations of the Braid group and corresponding link polynomials.

1. Introduction

Recently, there has been renewed interest in the study of Hopf algebras because of their role in obtaining solutions to the Yang-Baxter equation, which arises in areas such as integrable lattice models [3], the quantum inverse scattering method [2, 12] and the theory of knots and links [1, 18, 19, 20]. In particular, it is the so-called quasi-triangular Hopf algebras as defined by Drinfeld [6] that have received most attention since they admit a canonical element known as the universal (that is, representation-independent) R-matrix which automatically provides a solution to the Yang-Baxter equation.

The representation theory of such algebras plays an indispensable role in applications: in particular a representation of the Braid group, and corresponding link polynomial, can be obtained corresponding to each finite dimensional irreducible representation of a quasi-triangular Hopf algebra. Important examples of quasi-triangular Hopf algebras are afforded by quantum groups [6, 11] which are obtained by deforming the universal enveloping algebra of a simple Lie algebra through the introduction of a non-zero deformation parameter q. The representation theory and applications of quantum groups have been extensively investigated recently [6, 7, 8, 11]. However other families of quasi-triangular Hopf algebras have received comparitively little attention although they may be just as important both in physical applications and from the mathematical

Received 7th October, 1992
The author gratefully acknowledges useful discussions with Drs. I. Tsohantjis, B.D. Jones and S. Siller. The author also thanks Dr. R.B. Zhang for bringing to his attention the work of [15].

view point of affording new examples of non-commutative and non-cocommutative Hopf algebras.

Through the quantum double construction of Drinfeld [**6, 9**] one may construct (under certain mild conditions) a quasi-triangular Hopf algebra from any Hopf algebra and its dual. In this series of two papers we are concerned with the important special case of quantum double algebras, denoted $D(G)$, arising from the group algebra of a finite group G. Below all such algebras are shown to be semi-simple and their character theory is developed along traditional lines [**5, 10**]. The latter plays an important role in decomposing tensor product representations and is indispensable for obtaining link polynomials, as will be seen in the second paper of the series.

As well all irreducible $D(G)$-modules are classified. In fact it is shown that representations of a finite group G, induced from an irreducible representation of the centraliser subgroup of an element $g \in G$, gives rise to an irreducible module of the quantum double: moreover all irreducible $D(G)$-modules are obtained in this way. The corresponding matrix representations are determined and applied to obtain an explicit formula for the R-matrix in any irreducible representation.

These results will be applied, in the second paper of the series, to obtain new representations of the Braid group and corresponding link polynomials. Representations of the Braid group constructed in this way are always unitary, unlike those arising from quantum groups. It appears therefore that quantum double group algebras are to play an important role in obtaining unitary representations of the Braid group.

The paper is set up as follows. In Section 2 we outline Drinfeld's quantum double construction. The special case arising from a finite group algebra is investigated in detail in Section 3. In Section 4 an orthogonality relation for matrix elements is derived and applied in Section 5 to develop a complete theory of characters. In Section 6 all irreducible $D(G)$-modules are classified and these results are applied in Section 7 to the explicit construction of R-matrices.

2. Quasi-triangular Hopf algebras and the double construction

Let A be a Hopf algebra with identity $1 \in A$, co-unit $\varepsilon\colon A \to \mathbb{C}$, coproduct $\Delta\colon A \to A \otimes A$ and bijective antipode $S\colon A \to A$. Following the notation of Sweedler [**16**] we write

$$\Delta(a) = \sum_{(a)} a^{(1)} \otimes a^{(2)}, \qquad a \in A. \tag{1}$$

More generally we define

$$\Delta_1 = \Delta, \quad \Delta_n = \left(\Delta \otimes \underbrace{I \otimes \ldots . \otimes I}_{n-1}\right)\Delta_{n-1}, \quad n \geqslant 2$$

with $I\colon A \to A$ the identity map, and write

(2) $$\triangle_n(a) = \sum_{(a)} a^{(1)} \otimes a^{(2)} \ldots\ldots\ldots \otimes a^{(n+1)}.$$

With this notation the counit and antipode properties are expressible

$$a = \sum_{(a)} a^{(1)}\varepsilon\left(a^{(2)}\right) = \sum_{(a)} \varepsilon\left(a^{(1)}\right)a^{(2)}$$
$$\varepsilon(a) = \sum_{(a)} a^{(1)}S\left(a^{(2)}\right) = \sum_{(a)} S\left(a^{(1)}\right)a^{(2)}$$

respectively. Recall [16] that S determines an algebra anti-automorphism $S\colon A \to A$.

Let $T\colon A \otimes A \to A \otimes A$ be the twist map defined by

$$T(a \otimes b) = b \otimes a, \qquad \forall a, b \in A.$$

Then A also constitutes a Hopf algebra under the opposite coproduct $\triangle^T = T \cdot \triangle$ with antipode S^{-1} and counit ε. In the case that $\triangle^T = \triangle$ we call A *cocommutative*: recall [16] that if A is commutative or cocommutative then $S^2 = I$, the identity map on A. Following Drinfeld [6] we have the following:

DEFINITION 2.1: A Hopf algebra A is called *quasi-triangular* if there exists an invertible element

$$R = \sum_i a_i \otimes b_i \in A \otimes A$$

satisfying $$\triangle^T(a)R = R\triangle(a), \qquad \forall a \in A$$

and $$(\triangle \otimes I)R = R_{12}R_{23}, \quad (I \otimes \triangle)R = R_{13}R_{12}$$

where $$R_{12} = \sum_i a_i \otimes b_i \otimes 1, \quad R_{13} = \sum_i a_i \otimes 1 \otimes b_i \quad \text{et cetera}$$

A direct consequence of this definition is that the canonical element R, called the universal R-matrix, satisfies the Yang-Baxter equation

(3) $$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

in $A \otimes A \otimes A$. Thus corresponding to each irreducible A-module V, we obtain a solution to the Yang-Baxter equation on $V \otimes V \otimes V$, of importance in the theory of exactly solvable models in statistical mechanics [3] and knot theory [**1, 18, 19, 20**]. It is important to note that in order to obtain non-trivial R-matrices, it is necessary that A be non-commutative and non-cocommutative.

https://doi.org/10.1017/S0004972700015707 Published online by Cambridge University Press

A large class of such algebras is afforded by the quantum double construction of Drinfeld [**6, 9**] whereby a quasi-triangular Hopf algebra is manufactured from any Hopf algebra A and its dual A^*. Hence we now say something about the dual of a Hopf algebra and the double construction. Throughout, given a vector space V, $\langle\ ,\ \rangle : V^* \otimes V \to \mathbb{C}$ denotes the natural bilinear form defined by

$$\langle f, v\rangle = f(v), \quad \forall f \in V^*, v \in V.$$

We assume that

$$A^0 \equiv \{a^* \in A^* \mid \ker a^* \text{ contains a cofinite two sided ideal of } A\}$$

is *dense* in A^*; that is, the subspace A^0 satisfies [**16**]

$$\left(A^0\right)^{\perp} \equiv \left\{a \in A \mid \langle b^*, a\rangle = 0, \forall b^* \in A^0\right\} = (0)$$

in which case A is called a *proper algebra*: note that every finite dimensional Hopf algebra is proper since, in such a case, $A^* = A^0$. Following Sweedler [**16**] we have

THEOREM 2.1. *A^0 becomes a Hopf algebra with multiplication m^0, unit u^0, coproduct $\triangle^0$, antipode S^0 and counit ε^0 defined respectively by*

$$\begin{aligned} m^0 &= \triangle^*\Big|_{A^0 \otimes A^0} \qquad u^0 = \varepsilon^*\Big|_{A^0} \quad , \quad \triangle^0 = m^*\Big|_{A^0} \\ S^0 &= S^*\Big|_{A^0}, \qquad \varepsilon^0(a^*) = \langle a^*, 1\rangle, \quad \forall a^* \in A^0 \end{aligned} \tag{4}$$

where $m\colon A \otimes A \to A$ is the multiplication map on A and m^*, $\triangle^*$, ε^*, S^* are the natural dual maps of m, $\triangle$, ε, S respectively.

We note that the identity element of A^0 is given by the counit ε of A. It is crucial to the double construction that we take the opposite Hopf algebra structure on A^0, with coproduct $\Delta_0 = \left(\triangle^0\right)^T$ and antipode $S_0 = \left(S^0\right)^{-1}$ given explicitly by

$$\begin{aligned} &\langle \Delta_0(a^*), b \otimes c\rangle = \langle a^*, cb\rangle \\ &\langle S_0(a^*), b\rangle = \langle a^*, S^{-1}(b)\rangle, \qquad \forall a^* \in A^*, b, c \in A \end{aligned} \tag{5}$$

respectively, while the multiplication, unit and counit remain the same. For uniformity of notation we set

$$u_0 = u^0, \quad m_0 = m^0, \quad \varepsilon_0 = \varepsilon^0.$$

Then $A \otimes A^0$, $A^0 \otimes A$ inherit Hopf algebra structures from those of A and A^0 in a natural way: for example the coproducts $\widehat{\Delta}$, $\widehat{\Delta}'$ on $A \otimes A^0$, $A^0 \otimes A$ are defined respectively by

$$\widehat{\triangle} = (I \otimes \tau \otimes I)(\triangle \otimes \Delta_0), \quad \widehat{\triangle}' = \left(I \otimes \tau^{-1} \otimes I\right)(\Delta_0 \otimes \triangle)$$

where $\tau\colon A\otimes A^0 \to A^0\otimes A$ is the twist isomorphism defined by

$$\tau(a\otimes b^*) = b^*\otimes a, \qquad \forall a\in A,\, b^*\in A^*$$

and where, for ease of notation, I denotes the identity map on both A and A^0.

The quantum double construction affords a method of imbedding A and A^0 in a quasi-triangular Hopf algebra, denoted $D(A)$, which is vector space isomorphic to $A\otimes A^0$ but with a different algebra structure. Explicitly we let $D(A)$ be the vector space spanned by all free products

$$ab^*, \qquad a\in A,\, b^*\in A^*$$

which becomes an algebra by defining all products b^*a, $a\in A$, $b^*\in A^*$, according to the rule

$$b^*a = \mu(b^*\otimes a)$$

where $\mu\colon A^0\otimes A\to D(A)$ is defined to be the composite map

$$A^0\otimes A \xrightarrow{(tr\otimes I^{\otimes 2})(S_0\otimes I^{\otimes 3})\widehat{\triangle}'} A^0\otimes A \xrightarrow{\tau^{-1}} A\otimes A^0 \xrightarrow{(I^{\otimes 2}\otimes tr)\widehat{\triangle}} A\otimes A^0 \longrightarrow D(A).$$

Here $tr\colon A^0\otimes A\to\mathbb{C}$ is defined by

$$tr(a^*\otimes b) = \langle a^*, b\rangle$$

and $A\otimes A^0\to D(A)$ is the natural bijection, $a\otimes b^*\longmapsto ab^*$. In the notation of equation (2) we have explicitly, [9]

$$b^*a = \sum_{(a),(b^*)} \left\langle S_0\left(b^{*(1)}\right), a^{(1)}\right\rangle \langle b^{*(3)}, a^{(3)}\rangle a^{(2)} b^{*(2)}. \tag{6}$$

With this construction $D(A)$ becomes a Hopf algebra under the naturally inherited coproduct $\overline{\triangle}$, counit $\overline{\varepsilon}$ and antipode $\overline{S}$ given respectively by [6, 9]

$$\begin{aligned}
\overline{\triangle}(ab^*) &= \triangle(a)\triangle_0(b^*) = \sum_{(a),(b^*)} a^{(1)}b^{*(1)}\otimes a^{(2)}b^{*(2)}\\
\overline{\varepsilon}(ab^*) &= \varepsilon(a)\varepsilon_0(b^*) = \varepsilon(a)\langle b^*, 1\rangle\\
\overline{S}(ab^*) &= \mu(S_0(b^*)\otimes S(a)) = S_0(b^*)S(a), \quad \forall a\in A,\, b^*\in A^*.
\end{aligned}$$

Moreover if $\{a_s^*\}$ denotes a basis for A^0 with corresponding dual basis $\{a_s\}$ for A defined by

$$\langle a_s^*, a_t\rangle = \delta_{st},$$

https://doi.org/10.1017/S0004972700015707 Published online by Cambridge University Press

then it is easily checked [**9**] that

$$R = \sum_s a_s \otimes a_s^* \in D(A) \otimes D(A) \tag{7}$$

has inverse

$$R^{-1} = (\overline{S} \otimes I)R$$

and satisfies the conditions of Definition 2.1. Thus we arrive at [**6, 9**]

THEOREM 2.2. *$D(A)$ with canonical element R constitutes a quasi-triangular Hopf-algebra called the quantum double of A.*

We remark that here and below we regard A and A^0 as naturally imbedded in $D(A)$ by identifying $1 \cdot a^*$ and $b \cdot \varepsilon$ with a^*, b respectively, for all $a^* \in A^*$, $b \in A$.

Important examples of such quantum double algebras are afforded by quantum groups [**6, 7, 11**] which have been extensively studied recently. Below we apply the double construction to the case A is the group algebra of a finite group which, in turn, we use to construct some link polynomials. We shall need the following generalisation of Maschke's theorem due to Sweedler [**16**]:

THEOREM 2.3. *A finite dimensional Hopf algebra A is semi-simple (as an algebra) if and only if there exists a left integral $x \in A$, where $x \in A$ is called a left integral provided $\varepsilon(x) \neq 0$ and*

$$ax = \varepsilon(a)x, \qquad \forall a \in A.$$

We conclude with some general remarks about the representation theory of a Hopf algebra A. First we note that the counit ε itself gives rise to a representation of A, herein referred to as the *identity representation*. If V, W are A-modules then obviously $V \otimes W$ becomes an A-module under the action determined by the coproduct Δ.

Given any finite dimensional A-module V, V^* becomes an A-module, called the *dual* of V, with the definition [**14**]

$$\langle af, v\rangle = \langle f, S(a)v\rangle, \qquad \forall a \in A, v \in V, f \in V^*. \tag{8}$$

If W is another finite dimensional A-module then $\mathrm{Hom}(V, W)$, the space of linear maps of V to W, becomes an A-module with the definition [**14**]

$$a \circ f = \sum_{(a)} a^{(1)} f\, S\left(a^{(2)}\right), \qquad \forall a \in A,\ f \in \mathrm{Hom}(V, W),$$

where we have adopted the notation of equation (1). With these constructions we have [**14**]:

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Lemma 2.1. *The mapping* $\xi: V^* \otimes W \to \mathrm{Hom}(V, W)$, *defined by*

$$\xi(f \otimes w)(v) = \langle f, v\rangle w, \qquad \forall f \in V^*, w \in W, v \in V$$

determines an A*-module isomorphism.*

Definition 2.2: Given any A-module V, we call $v \in V$ an *invariant* if

$$av = \varepsilon(a)v, \qquad \forall a \in A.$$

If A admits a left integral $x \in A$, then clearly x determines an invariant if we regard A as a module under the left regular representation. If V, W are A-modules then the invariants of $\mathrm{Hom}(V, W)$ are precisely the A-module homomorphisms [**14**]. We thus arrive at the following version of Schur's lemma:

Lemma 2.2. *Let* V, W *be finite dimensional irreducible* A*-modules. Then the space of* A*-invariants of* $\mathrm{Hom}(V, W)$ *has dimension* 1 *when* V *and* W *are isomorphic and otherwise it is trivial.*

3. Quantum double group algebras

Let A be the group algebra of a finite group G over the complex field $\mathbb{C}$. Then A becomes a co-commutative Hopf algebra with coproduct, antipode and counit respectively defined by

$$\Delta(g) = g \otimes g, \quad S(g) = g^{-1}, \quad \varepsilon(g) = 1, \quad \forall g \in G$$

which we extend linearly to all of A in an obvious way: throughout 1 denotes the identity element of G and A.

Recall [**5, 10**] that given any irreducible A-module V_λ, of dimension d_λ, we have the group character defined by

$$\chi_\lambda(g) = tr\pi_\lambda(g), \qquad \forall g \in G$$

where π_λ denotes the representation afforded by V_λ. Then the operators

$$E_\lambda = \frac{d_\lambda}{|G|}\sum_{g\in G}\chi_\lambda(g^{-1})g \tag{9}$$

form a basis for the centre C of A and give rise to an orthogonal set of idempotents adding up to the identity:

$$E_\lambda E_\mu = \delta_{\lambda\mu}E_\mu, \qquad \sum_\lambda E_\lambda = 1$$

https://doi.org/10.1017/S0004972700015707 Published online by Cambridge University Press

where the sum on λ is over all non-isomorphic irreducible A-modules.

From the Hopf algebra viewpoint, the counit ε corresponds to the character χ_ι of the identity representation, herein denoted π_ι. The corresponding central idempotent

$$E_\iota = \frac{1}{|G|}\sum_{g\in G} g \tag{10}$$

constitutes a *left integral* in A from which it follows (Theorem 2.3) that A is a semi-simple algebra. Explicitly the decomposition of A into simple two-sided ideals is given by

$$A = \bigoplus_\lambda A_\lambda, \qquad A_\lambda = E_\lambda A = AE_\lambda.$$

Moreover A_λ is the direct sum of d_λ copies of the irreducible (left) A-module V_λ so that

$$\dim A_\lambda = d_\lambda^2$$

and hence

$$|G| = \sum_\lambda d_\lambda^2$$

where $|G| = \dim A$ is the order of the group G.

We now turn our attention to the dual space $A^* = A^0$ which has a basis of elements g^*, $g\in G$, defined by

$$\langle g^*, h\rangle = \delta(g, h).$$

Then A^* inherits the structure of a Hopf algebra from that of A: explicitly, from equation (4), A^* becomes an algebra with product

$$\begin{aligned} g^*h^* &= \sum_{k\in G}\langle g^*\otimes h^*, \Delta(k)\rangle k^* \\ &= \sum_{k\in G}\langle g^*, k\rangle\langle h^*, k\rangle k^* = \delta(g, h)h^*, \qquad \forall g, h\in G \end{aligned} \tag{11}$$

so that A^* is simply the algebra generated by $|G|$ orthogonal idempotents. In particular A^* is a commutative algebra, as expected (since A is cocommutative).

To complete the Hopf algebra structure on A^*, the coproduct Δ_0, antipode S_0 and counit ε_0 are given, from equation (5), respectively by

$$\begin{aligned} \Delta_0(g^*) &= \sum_{h\in G}\left(h^{-1}g\right)^*\otimes h^* = \sum_{h\in G} h^*\otimes\left(gh^{-1}\right)^*, \\ S_0(g^*) &= \left(g^{-1}\right)^*, \quad \varepsilon_0(g^*) = \delta(g, 1), \quad \forall g\in G. \end{aligned} \tag{12}$$

As noted previously, the identity on A^* is given by the counit ε on A which is expressible

$$\varepsilon = \sum_{g \in G} g^*.$$

From equation (12) we have

$$\Delta_0(\varepsilon) = \varepsilon \otimes \varepsilon, \quad S_0(\varepsilon) = \varepsilon, \quad \varepsilon_0(\varepsilon) = 1.$$

In view of equations (11, 12) 1^*, 1 the identity element of G, satisfies

$$g^* 1^* = \delta(g, 1) 1^* = \varepsilon_0(g^*) 1^*, \qquad \forall g \in G$$

and thus qualifies as a left integral in A^*. It follows immediately from Theorem 2.3 that:

Theorem 3.1. *A^* is a semi-simple algebra.*

Since A^* is commutative we thus obtain

Corollary. *Every finite dimensional A^*-module is a direct sum of irreducible one-dimensional modules.*

Remark. It is worth noting that with the above structure A is a quasi-triangular Hopf algebra with trivial canonical element $R = 1 \otimes 1$. However A^* is *not* quasi-triangular unless G is an abelian group: in that case $R = \varepsilon \otimes \varepsilon$ is the canonical element of A^*.

Following the double construction, the quantum double of the group algebra A, herein denoted $D(G)$, is the $|G|^2$-dimensional algebra spanned by all free products

$$gh^*, \ g, h \in G$$

where, according to equation (6), the products $h^* g$ are to be computed as follows:

$$h^* g = g\left(g^{-1} h g\right)^*.$$

Equivalently, $D(G)$ may be regarded as spanned by all free products $h^* g$, $h, g \in G$, where gh^* is computed according to

$$gh^* = \left(ghg^{-1}\right)^* g.$$

Then, from Theorem 2.2, $D(G)$ gives rise to a quasi-triangular Hopf algebra with coproduct $\overline{\Delta}$, antipode $\overline{S}$ and counit $\overline{\varepsilon}$ given respectively by

$$\begin{aligned} \overline{\Delta}(gh^*) &= \Delta(g)\Delta_0(h^*) = \sum_{k \in G} g\left(k^{-1}h\right)^* \otimes gk^* \\ \overline{S}(gh^*) &= S_0(h^*)S(g) = \left(h^{-1}\right)^* g^{-1} = g^{-1}\left(gh^{-1}g^{-1}\right)^* \\ \overline{\varepsilon}(gh^*) &= \varepsilon(g)\varepsilon_0(h^*) = \delta(h, 1), \quad \forall g, h \in G. \end{aligned} \tag{13}$$

The corresponding canonical element R is given, from equation (7), by

$$R = \sum_{g \in G} g \otimes g^*, \tag{14}$$

which can be shown directly to satisfy the Yang-Baxter equation (3).

As before we here regard A and A^* as subalgebras of $D(G)$ by identifying $g \cdot \varepsilon$ and $1 \cdot g^*$ with g, g^* respectively, for all $g \in G$. We note that the antipode $\overline{S}$ satisfies

$$\overline{S}^2 = I, \tag{15}$$

I the identity map on $D(G)$, and that the identity element of $D(G)$ is given by $1 \cdot \varepsilon$ which we identify with $1 \in G$.

Now let V_Λ be a (finite dimensional) irreducible $D(G)$-module of dimension $d[\Lambda]$ and let π_Λ be the representation of $D(G)$ afforded by V_Λ. Then it follows immediately that the matrix

$$R_\Lambda = \sum_{g \in G} \pi_\Lambda(g) \otimes \pi_\Lambda(g^*)$$

satisfies the Yang-Baxter equation (3) on $V_\Lambda \otimes V_\Lambda \otimes V_\Lambda$. Thus in order to obtain interesting new solutions of the Yang-Baxter equation and, in turn, to construct link polynomials, it is necessary to determine explicitly the irreducible matrix representations of $D(G)$, which we investigate below.

We note here that

$$x = E_\iota 1^* = 1^* E_\iota, \tag{16}$$

with E_ι as in equation (10), satisfies $\varepsilon(x) = 1$ and

$$x h^* g = h^* g \cdot x = \overline{\varepsilon}(h^* g) x, \qquad \forall g, h \in G$$

so that, by linearity,

$$xa = ax = \varepsilon(a) x, \qquad \forall a \in D(G). \tag{17}$$

Thus x constitutes a left integral in $D(G)$ from which it follows, in view of Theorem (2.3), that

Theorem 3.2. *$D(G)$ is a semi-simple algebra.*

This result implies that $D(G)$ is a direct sum of simple two-sided ideals

$$D(G) = \bigoplus_\Lambda D(G)_\Lambda \tag{18}$$

where $D(G)_\Lambda$ is the direct sum of $d[\Lambda]$ copies of the irreducible (left) $D(G)$-module V_Λ and the sum on Λ is over all non-isomorphic irreducible $D(G)$-modules. In particular

$$\dim D(G)_\Lambda = d[\Lambda]^2$$

so that

$$|G|^2 = \dim D(G) = \sum_\Lambda d[\Lambda]^2. \tag{19}$$

It is worth noting, from equation (17), that the left integral x spans a one-dimensional simple two sided ideal of $D(G)$ which must therefore occur in the decomposition (18). The two-sided ideal $D(G)x = \mathbb{C}x$ also gives rise, as we have seen, to an irreducible (left) $D(G)$-module whose corresponding representation is simply the identity representation. In view of equation (17), x satisfies

$$x^2 = \bar{\varepsilon}(x)x = x$$

and thus x determines a central primitive idempotent which projects onto the identity representation. Thus, following Definition 2.2, given any finite dimensional $D(G)$-module V, xV is the subspace of invariants of V; namely.

$$xV = \{v \in V \mid av = \bar{\varepsilon}(a)v, \quad \forall a \in D(G)\}.$$

REMARK. The left integral x is of independent interest for constructing representations of the Temperley-Lieb algebra [17] and hence for obtaining R-matrices with a spectral parameter. In fact given any irreducible self-dual $D(G)$-module $V = V_\Lambda$ we have the canonical generator

$$T = d[\Lambda]\overline{\Delta}(x) \in \operatorname{End} V \otimes V$$

where

$$\overline{\Delta}(x) = \frac{1}{|G|} \sum_{g,h\in G} (h^{-1})^* g \otimes h^* g. \tag{20}$$

Then T gives rise to a representation of the Temperley-Lieb algebra on $V^{\otimes N}$ according to

$$T_i = \underbrace{I \otimes \ldots\ldots \otimes I}_{i-1} \otimes T \otimes \underbrace{I \otimes \ldots\ldots \otimes I}_{N-i-1}, \qquad 1 \leqslant i < N$$

where I denotes the identity map on V. It can be shown that indeed the T_i satisfy the following defining relations of the Temperley-Lieb algebra [17]

$$\begin{aligned} T_i{}^2 &= \sqrt{Q}\, T_i \\ T_i T_{i\pm1} T_i &= T_i \\ T_i T_j &= T_j T_i, \qquad |i-j| > 1 \end{aligned}$$

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where, for the case at hand, $Q = d[\Lambda]^2$. It is then possible to construct from T an R-matrix $R(u)$, u a spectral parameter, which satisfies the parameter-dependent Yang-Baxter equation of interest in exactly solvable models in statistical mechanics [4].

4. Matrix elements and unitary modules

We call a finite dimensional $D(G)$-module V *unitary* if V can be equipped with an inner product (,) such that for all $g, h \in G$

$$(gh^*v,\, w) = \big(v,\, h^*g^{-1}w\big), \qquad \forall v, w \in V.$$

Equivalently, if π is the representation of $D(G)$ afforded by V, then V is called unitary if it can be equipped with an inner product such that

$$\pi(h^*g) = \pi\big(g^{-1}h^*\big)^\dagger$$

where $\dagger$ denotes Hermitian conjugate. We have

Lemma 4.1. *Every finite dimensional $D(G)$-module is equivalent to a unitary one.*

Proof: Let (,) be any inner product on a finite dimensional $D(G)$-module V. Then it is easily seen that $(\;,\;)_0$ on V defined by

$$(v,\, w)_0 = \sum_{g,h\in G} (h^*gv,\, h^*g\,w), \qquad \forall v, w \in V$$

determines an inner product with the required properties. $\square$

We may therefore assume, without loss of generality, that all finite-dimensional $D(G)$-modules are unitary. It follows, from standard arguments, that every such module is an orthogonal direct sum of irreducible submodules. In particular every finite dimensional $D(G)$-module is completely reducible: this affords an alternative direct proof of Theorem 3.2.

Throughout we let $\{v_j^\Lambda\}$ be a fixed *orthonormal basis* for the irreducible $D(G)$-module V_Λ. Since V_Λ is assumed unitary, we have in this basis

$$\pi_\Lambda(h^*g)_{ij} = \overline{[\pi_\Lambda(g^{-1}h^*)_{ji}]}$$

where the over-bar denotes complex conjugation. From equation (8), the dual space V_Λ^* also gives rise to an irreducible $D(G)$-module, herein denoted V_{Λ^*}. We let $\{v_i^{\Lambda^*}\}$ be the basis for $V_{\Lambda^*} = V_\Lambda^*$ dual to $\{v_i^\Lambda\}$ so that

$$\langle v_i^{\Lambda^*},\, v_j^\Lambda\rangle = \delta_{ij}.$$

Then it is easily seen that $\{v_i^{\Lambda^*}\}$ defines an inner product on V_{Λ^*} with respect to which this basis is orthonormal and V_{Λ^*} is unitary. The corresponding matrix representation in this basis is given by

$$\pi_{\Lambda^*}(a)_{ij} = \pi_\Lambda\left(\overline{S}(a)\right)_{ji}$$

or

$$\pi_{\Lambda^*}(a) \quad = \pi_\Lambda\left(\overline{S}(a)\right)^t, \qquad \forall a \in D(G) \tag{21}$$

where t denotes matrix transposition.

We now demonstrate that the above matrix elements satisfy certain orthogonality relations. From Lemma 2.1 the $D(G)$-modules $V_\Lambda^* \otimes V_\mu$ and $\mathrm{Hom}(V_\Lambda, V_\mu)$ are isomorphic. On the other hand Lemma 2.2 implies that the space of invariants of $\mathrm{Hom}(V_\Lambda, V_\mu)$ is (0) for $\Lambda \neq \mu$ in which case the identity module cannot occur in $V_\Lambda^* \otimes V_\mu$; that is,

$$\overline{\Delta}(x) v_i^{\Lambda^*} \otimes v_j^\mu = 0, \qquad \forall i, j, \text{ when } \mu \neq \Lambda. \tag{22}$$

In the case $\mu = \Lambda$ we have:

Lemma 4.2. *The identity module occurs exactly once in $V_\Lambda^* \otimes V_\Lambda$ and is spanned by the vector*

$$v^\Lambda = \sum_i v_i^{\Lambda^*} \otimes v_i^\Lambda.$$

Proof: Lemma 2.2 implies that the space of invariants of $\mathrm{Hom}(V_\Lambda, V_\Lambda)$ is one dimensional so that the identity module occurs exactly once in $V_\Lambda^* \otimes V_\Lambda$. With v^Λ as above we have, in the notation of equation (1)

$$\begin{aligned} av^\Lambda &= \sum_i \overline{\Delta}(a)\left(v_i^{\Lambda^*} \otimes v_i^\Lambda\right) \\ &= \sum_{i,j,k} \sum_{(a)} \pi_{\Lambda^*}\left(a^{(1)}\right)_{ji} \pi_\Lambda\left(a^{(2)}\right)_{ki} v_j^{\Lambda^*} \otimes v_k^\Lambda, \quad \forall a \in D(G). \end{aligned}$$

Using equation (21) we thus obtain

$$av^\Lambda = \sum_{j,k} \pi_\Lambda\left(\sum_{(a)} a^{(2)} \overline{S}\left(a^{(1)}\right)\right)_{kj} v_j^{\Lambda^*} \otimes v_k^\Lambda. \tag{*}$$

On the other hand from the antipode property and equation (15) we have

$$\begin{aligned} \sum_{(a)} a^{(2)} \overline{S}\left(a^{(1)}\right) &= \sum_{(a)} \overline{S}\left(a^{(1)} \overline{S}\left(a^{(2)}\right)\right) \\ &= \overline{S}(\overline{\varepsilon}(a)) = \overline{\varepsilon}(a). \end{aligned}$$

Substituting into equation (*) we arrive at

$$av^{\Lambda} = \overline{\varepsilon}(a)v^{\Lambda}, \qquad \forall a \in D(G)$$

which is sufficient to prove the result. ∎

In the notation above, we note that the vectors

$$v_i^{\Lambda^*} \otimes v_j^{\Lambda}, \quad i \neq j$$
$$v_i^{\Lambda^*} \otimes v_i^{\Lambda} - v^{\Lambda}/d[\Lambda], \quad 1 \leqslant i \leqslant d[\Lambda]$$

are orthogonal to the vector v^{Λ} of Lemma 4.2 so that

$$\overline{\triangle}(x)v_i^{\Lambda^*} \otimes v_j^{\Lambda} = \frac{\delta_{ij}}{d[\Lambda]}v^{\Lambda}. \tag{23}$$

Using the explicit form for $\overline{\triangle}(x)$, as determined by equation (20), together with equations (22, 23) above, we obtain the following orthogonality relation

THEOREM 4.1.

$$\sum_{g,h\in G} \pi_{\Lambda}\left(g^{-1}h^*\right)_{ik}\pi_{\mu}(h^*g)_{lj} = \frac{|G|}{d[\Lambda]}\delta_{\Lambda\mu}\delta_{ij}\delta_{kl}.$$

PROOF: Equations (22, 23) immediately imply the orthogonality relation

$$\sum_{g,h\in G} \pi_{\Lambda^*}[(h^{-1})^*g]_{ki}\pi_{\mu}(h^*g)_{lj} = \frac{|G|}{d[\Lambda]}\delta_{\Lambda\mu}\delta_{ij}\delta_{kl}.$$

The result is then seen to follow from equation (21). ∎

Theorem 4.1 is clearly a generalisation of a well known [**10**] orthogonality relation for the matrix elements of a finite group. We now apply this result to the characters of $D(G)$.

5. CHARACTERS

Following the usual definition for finite groups, we may now define the character χ_{Λ}, corresponding to the irreducible $D(G)$ module V_{Λ}, by

$$\chi_{\Lambda}(a) = tr\,\pi_{\Lambda}(a), \qquad \forall a \in D(G)$$

where π_{Λ} is the representation afforded by V_{Λ}. If Ad denotes the adjoint representation [**14**] of $D(G)$ defined by

$$Ad\, a \circ b = \sum_{(a)} a^{(1)}b\overline{S}\left(a^{(2)}\right), \qquad \forall a, b \in D(G)$$

https://doi.org/10.1017/S0004972700015707 Published online by Cambridge University Press

then it is easily verified that

$$\chi_\Lambda(Ad\, a \circ b) = \bar{\varepsilon}(a)\chi_\Lambda(b), \qquad \forall a, b \in D(G).$$

Now setting $i = k$, $l = j$ in Theorem 4.1 and summing we obtain the following result, herein referred to as the *first orthogonality relation* for characters:

THEOREM 5.1.

$$\sum_{g,h \in G} \chi_\Lambda(g^{-1}h^*)\chi_\mu(h^*g) = |G|\,\delta_{\Lambda\mu}.$$

REMARKS. Our definition of character and Theorem 5.1 are a particular case of a far more general theory due to Larson [**13**]. In the language of [**13**], equation (15) is equivalent to the statement that $D(G)$ is an *involutary* Hopf algebra. Below we extend the theory of characters for the special case of the algebras $D(G)$.

If we set $i = k$ in Theorem 4.1 and sum, we obtain

$$\sum_{g,h \in G} \chi_\Lambda(g^{-1}h^*)\pi_\mu(h^*g)_{ij} = \frac{|G|}{d[\Lambda]}\delta_{ij}\delta_{\Lambda\mu}$$

from which we deduce that the operator

$$E_\Lambda = \frac{d[\Lambda]}{|G|}\sum_{g,h \in G} \chi_\Lambda(g^{-1}h^*)h^*g \tag{24}$$

is a projection operator onto irreducible modules V_Λ. Moreover the first orthogonality relation guarantees that these projections are orthogonal:

$$E_\Lambda E_\mu = \delta_{\Lambda\mu}E_\mu.$$

Thus the E_Λ constitute the central primitive idempotents of the algebra $D(G)$ which therefore form a basis for the centre $\overline{C}$ of $D(G)$ and yield the following resolution of the identity:

$$1 = \sum_\Lambda E_\Lambda.$$

The simple two-sided ideals occurring in the decomposition (18) are then given explicitly by

$$D(G)_\Lambda = E_\Lambda D(G) = D(G)E_\Lambda.$$

These results imply the following important property of the characters χ_Λ:

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Lemma 5.1.

$$\chi_{\Lambda}\left(g^{-1}h^*\right) = 0, \qquad \textit{unless } g \textit{ and } h \textit{ commute.}$$

Proof: Since E_Λ belongs to the centre of $D(G)$ we have

$$\begin{aligned}\sum_{g\in G}\chi_{\Lambda}\left(g^{-1}h^*\right)h^*g &= \frac{|G|}{d[\Lambda]}h^*E_\Lambda \\ &= \frac{|G|}{d[\Lambda]}E_\Lambda h^* \\ &= \sum_{k,g\in G}\chi_{\Lambda}\left(g^{-1}k^*\right)k^*gh^* \\ &= \sum_{g\in G}\chi_{\Lambda}\left(g^{-1}h^*\right)\left(ghg^{-1}\right)^*g.\end{aligned}$$

Equating coefficients of g we arrive at

$$\chi_{\Lambda}\left(g^{-1}h^*\right)h^* = \chi_{\Lambda}\left(g^{-1}h^*\right)\left(ghg^{-1}\right)^*, \qquad \forall h \in G.$$

Since the h^* form an orthogonal set of idempotents, we must have $\chi_{\Lambda}\left(g^{-1}h^*\right) = 0$, unless $h = ghg^{-1}$ as required. ∎

We are now in a position to determine a second orthogonality relation for characters. It is convenient to introduce the set

$$(25) \qquad Q = \{gh^* \mid g, h \in G \quad \text{with} \quad gh = hg\}$$

the linear span of which, denoted $Q_{\mathbb{C}}$, is the centraliser of A^* in $D(G)$; that is,

$$Q_{\mathbb{C}} = \{a \in D(G) \mid h^*a = ah^*, \quad \forall h \in G\}.$$

We note that Q is stable under the adjoint action of G; namely

$$gQg^{-1} = Q.$$

It follows that we may partition Q into G-conjugacy classes, herein denoted $Q_1, Q_2, \ldots\ldots, Q_N$. We observe that $gh^* \in Q$ if and only if $g^{-1}h^* \in Q$. If $gh^* \in Q_i$ we denote by $Q_{\bar{i}}$ the conjugacy class of $g^{-1}h^*$: note that $|Q_i| = \left|Q_{\bar{i}}\right|$. Throughout we let $q_i \in Q_i$ be a fixed conjugacy class representative.

Associated with each conjugacy class Q_i we have the central element

$$\sigma_i = \sum_{a\in Q_i} a.$$

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By construction, σ_i centralises G and thus belongs to the centre $\overline{C}$ of $D(G)$. Clearly the σ_i are linearly independent; we claim they form a basis for $\overline{C}$. To see this suppose

$$c = \sum_{g,h\in G} \alpha(g,h)gh^* \in \overline{C}, \quad \alpha(g,h) \in \mathbb{C}.$$

Since $h^*c = ch^*$, for all $h \in G$, we have

$$\alpha(g, h) = 0, \quad \text{unless } g \text{ and } h \text{ commute.}$$

Hence c is a linear combination of elements from Q, so we may write

$$c = \sum_{a\in Q} \alpha(a)a = \sum_{i=1}^{N} \sum_{a\in Q_i} \alpha(a)a, \quad \alpha(a) \in \mathbb{C}. \tag{26}$$

Also, since $c \in \overline{C}$, we must have, for all $g \in G$

$$\begin{aligned} c = g^{-1}cg &= \sum_{i=1}^{N} \sum_{a\in Q_i} \alpha(a)g^{-1}ag \\ &= \sum_{i=1}^{N} \sum_{a\in Q_i} \alpha(gag^{-1})a. \end{aligned} \tag{26'}$$

Comparing coefficients of a in equations (26, 26$'$) it follows that

$$\alpha(a) = \alpha(gag^{-1}), \qquad \forall g \in G.$$

Thus α takes a constant value, α_i say, on a given conjugacy class Q_i so that

$$c = \sum_{i=1}^{N} \alpha_i \sum_{a\in Q_i} a = \sum_{i=1}^{N} \alpha_i\sigma_i$$

which is sufficient to prove the result.

On the other hand the centre $\overline{C}$ of $D(G)$ is spanned by the central idempotents E_Λ which are in $1-1$ correspondence with the non-isomorphic irreducible $D(G)$-modules. We thus arrive at

Theorem 5.2. *The number of non-isomorphic irreducible $D(G)$-modules equals the number of G-equivalence classes of Q.*

With the above notation the orthogonality relation of Theorem 5.1 may be expressed, in view of Lemma 5.1, as

$$\sum_{i=1}^{N} |Q_i|\, \chi_\Lambda(q_{\bar{i}})\chi_\mu(q_i) = |G|\, \delta_{\Lambda\mu}$$

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where we have used the fact that χ_Λ takes the same value on all elements of a conjugacy class Q_i. It follows that the $N \times N$ matrix ($N =$ number of conjugacy classes of $Q =$ number of irreducible $D(G)$-modules)

$$U_{i\mu} = \left(\frac{|Q_i|}{|G|}\right)^{1/2} \chi_\mu(q_i)$$

satisfies $$U^t g U = I, \qquad \text{the identity matrix}$$

where g is the symmetric matrix

$$g_{ij} = \delta_{i\bar{j}}$$

satisfying $g^2 = I$. We must therefore have $U^{-1} = U^t g$, from which we deduce $UU^t = g$, or

$$\sum_\Lambda \left(\frac{|Q_i|}{|G|}\right)^{1/2} \chi_\Lambda(q_i) \left(\frac{|Q_j|}{|G|}\right)^{1/2} \chi_\Lambda(q_j) = \delta_{i\bar{j}}.$$

We thus arrive at the *second orthogonality relation* for characters:

THEOREM 5.3.
$$\sum_\Lambda \chi_\Lambda(q_i)\chi_\Lambda\left(q_{\bar{j}}\right) = \frac{|G|}{|Q_i|}\delta_{ij}.$$

In this way we obtain a complete theory of characters for the quantum double algebras $D(G)$, parallelling the well known theory [5, 10] of finite group characters. As for the case of finite groups, characters are very useful for decomposing finite-dimensional representations. In particular the problem of decomposing tensor product modules is of importance for obtaining link polynomials, as will be seen in the second paper of the series. Since $D(G)$ is semi-simple, given any two irreducible $D(G)$-modules V_Λ, V_μ we have the decomposition

$$V_\Lambda \otimes V_\mu = \bigoplus_\nu m_\nu V_\nu$$

where the sum is over all irreducible $D(G)$-modules V_ν with m_ν the multiplicity of V_ν in the tensor product module. Using standard arguments we arrive at

LEMMA 5.2.

$$m_\nu = \frac{1}{|G|} \sum_{g,h,k \in G} \chi_\nu(g^{-1}h^*)\chi_\Lambda[(k^{-1}h)^* g]\chi_\mu(k^* g).$$

PROOF: We clearly have

$$m_\nu = \frac{1}{d[\nu]} tr_{V_\Lambda \otimes V_\mu}\left[\overline{\triangle}(E_\nu)\right].$$

On the other hand using equation (24) together with the action of the coproduct $\overline{\Delta}$, as given in equation (13), we may write

$$\overline{\Delta}(E_\nu) = \frac{d[\nu]}{|G|} \sum_{g,h,k\in G} \chi_\nu(g^{-1}h^*)(k^{-1}h)^* g \otimes k^* g$$

from which the result follows. ∎

We now turn to the problem of explicitly constructing matrix representations and classifying all irreducible $D(G)$-modules.

6. Classification of irreducible $D(G)$-modules

We recall that G may be partitioned into conjugacy classes

$$G = \bigcup_{k=1}^{n} C_k$$

with $C_1 = \{1\}$ the conjugacy class of the identity $1 \in G$. It is worth noting that the set Q of equation (25) may be written

$$Q = \bigcup_{h\in G} Z(h)h^*$$

where

$$Z(h) = \{g \in G \mid gh = hg\}$$

is the centraliser subgroup of $h \in G$. We note that

$$gZ(h)g^{-1} = Z(ghg^{-1})$$

and if $h \in C_k$, then [5]

(27) $$|Z(h)| = |G| \,/\, |C_k|\,.$$

Throughout we let $g_k \in C_k$ $(1 \leqslant k \leqslant n)$ be a fixed conjugacy class representative and write $Z_k = Z(g_k)$ for the centraliser subgroup of g_k: we denote the group algebra of Z_k by A_k. For $s \in C_k$ we choose a fixed $\tau_s \in G$ such that

$$s = \tau_s g_k \tau_s^{-1}\ ;$$

for simplicity when $s = g_k$ we take $\tau_s = 1$. Some of the properties of the τ_s are listed below.

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Lemma 6.1.

(i) $G = \bigcup_{s \in \mathcal{C}_k} \tau_s Z_k$ *(disjoint union).*

(ii) *Given* $g \in G$, $s \in \mathcal{C}_k$, *there exists* $t \in \mathcal{C}_k$ *unique with the property*

$$\tau_t^{-1} g \tau_s \in Z_k ;$$

explicitly $t = g\, s\, g^{-1}$.

Proof: (i) Since, from equation (27),

$$|\tau_s Z_k| = |Z_k| = |G| \,/\, |\mathcal{C}_k|$$

it suffices to show that

$$\tau_t Z_k \cap \tau_s Z_k = \emptyset, \quad s \neq t$$

where $\emptyset$ is the empty set. Suppose therefore that there exists h, $h' \in Z_k$ such that

$$\tau_t h = \tau_s h' \in \tau_t Z_k \cap \tau_s Z_k .$$

Then
$$\tau_s^{-1} \tau_t = h' h^{-1} \in Z_k$$
so that
$$\tau_s^{-1} t\, \tau_s = \tau_s^{-1} \tau_t g_k \left(\tau_s^{-1} \tau_t\right)^{-1} = g_k$$
which implies
$$t = \tau_s g_k \tau_s^{-1} = s.$$

This is sufficient to prove (i).

As to (ii), given $g \in G$, $s \in \mathcal{C}_k$ there exists, from part(i), a unique $t \in \mathcal{C}_k$ with the property

$$g\tau_s \in \tau_t Z_k \quad \text{or} \quad \tau_t^{-1} g\, \tau_s \in Z_k .$$

It remains to show that $t = g\, s\, g^{-1}$. To this end write

$$h = \tau_t^{-1} g\, \tau_s \in Z_k .$$

Then
$$\begin{aligned} g\, s\, g^{-1} &= g\left(\tau_s\, g_k\, \tau_s^{-1}\right) g^{-1} \\ &= \tau_t\, h\, g_k h^{-1} \tau_t^{-1} = \tau_t\, g_k\, \tau_t^{-1} = t \end{aligned}$$

as required. ∎

In dealing with irreducible representations of $D(G)$ a fundamental role is played by the representations of G induced from those of the centraliser subgroups Z_k. Hence let V_α^k denote an irreducible A_k-module. Then we have the corresponding induced A-module [5]

$$V_{k,\alpha} = A \otimes_{A_k} V_\alpha^k \tag{28}$$

which is spanned by vectors

(29) $$v(s) = \tau_s \otimes v, \qquad v \in V_\alpha^k,\ s \in \mathcal{C}_k$$

so that $\dim V_{k,\alpha} = |\mathcal{C}_k| \,.\, \dim V_\alpha^k$, on which the action of G is given by

$$g(\tau_s \otimes v) = \tau_{gsg^{-1}} \otimes \left(\tau_{gsg^{-1}}^{-1}\, g\, \tau_s\right) v$$

or

(30) $$gv(s) = \left(\tau_{gsg^{-1}}^{-1}\, g\, \tau_s\, v\right)\left(gsg^{-1}\right).$$

Using Lemma 6.1 it is easily checked that this gives rise to a consistent A-module structure as required.

The induced module (28) admits a vector space direct sum decomposition

$$V_{k,\alpha} = \bigoplus_{s \in \mathcal{C}_k} V_{k,\alpha}(s)$$

where

$$V_{k,\alpha}(s) = \{v(s) \mid v \in V_\alpha^k\}.$$

This latter space gives rise to an irreducible module over (the group algebra of) $Z(s) = \tau_s Z_k \tau_s^{-1}$: in the case $s = g_k$, so that $Z(s) = Z_k$, this module is isomorphic to V_α^k. It is worth noting, from equation (30), that $gV_{k,\alpha}(s)$, $g \in G$, determines a non-zero $Z\left(g\, s\, g^{-1}\right)$-submodule of $V_{k,\alpha}\left(g\, s\, g^{-1}\right)$ from which it follows, in view of irreducibility, that

(31) $$gV_{k,\alpha}(s) = V_{k,\alpha}\left(g\, s\, g^{-1}\right).$$

We now turn $V_{k,\alpha}$ into a $D(G)$-module by setting

(30′) $$h^* v(s) = \delta(h,\, s) v(s), \qquad \forall h \in G.$$

With the definition of equations(30, 30′) it is easily seen that $V_{k,\alpha}$ in fact becomes a $D(G)$-module. Throughout we denote the dimension of this module by $d[k\,\alpha]$: clearly, by construction

(32) $$d[k, \alpha] = |\mathcal{C}_k|\, d_\alpha^k$$

where $d_\alpha^k = \dim V_\alpha^k$. With the action of equation (30′), we note that

$$c_k^* = \sum_{g \in \mathcal{C}_k} g^*$$

acts as the identity on $V_{k,\alpha}$; that is, $c_k^* w = w$, for all $w \in V_{k,\alpha}$. We are now in a position to prove

Theorem 6.1. *$V_{k,\alpha}$ is an irreducible $D(G)$-module.*

Proof: Let $w \neq 0$ be an arbitrary vector in $V_{k,\alpha}$: we show that the (left) $D(G)$-module $D(G)w$ generated by w must equal $V_{k,\alpha}$. Since $c_k^* w = w$ there must exist $s \in C_k$ such that $s^* w \neq 0$. Then we may write

$$0 \neq s^* w = v(s)$$

where $v(s)$ is a non-zero vector in $V_{k,\alpha}(s)$. Since the latter is an irreducible module over the group algebra $A(s)$ of $Z(s)$ we must have

$$V_{k,\alpha}(s) = A(s)v(s) \subseteq D(G)w.$$

Also from equation (31) we have, for all $g \in G$

$$g V_{k,\alpha}(s) = V_{k,\alpha}(g\,s\,g^{-1}) \subseteq D(G)w$$

from which it follows that

$$V_{k,\alpha} = \bigoplus_{s \in C_k} V_{k,\alpha}(s) \subseteq D(G)w$$

and hence $D(G)w = V_{k,\alpha}$. Since $0 \neq w \in V_{k,\alpha}$ was chosen arbitrarily, $V_{k,\alpha}$ must be an irreducible $D(G)$-module. □

Below we shall show that every irreducible $D(G)$-module is of the above form. We first need (notation as above).

Theorem 6.2. *The irreducible $D(G)$-modules $V_{k,\alpha}$, $V_{l,\beta}$ are isomorphic if and only if $k = l$ and the A_k-modules V_α^k, V_β^k, are isomorphic.*

Proof: Suppose we have a $D(G)$-module isomorphism

$$\xi \colon V_{k,\alpha} \longrightarrow V_{l,\beta}.$$

Then for $\quad 0 \neq v(s) \in V_{k,\alpha}(s), \qquad s \in C_k$

we have

$$\begin{aligned} 0 \neq \xi(v(s)) &= \xi(s^* v(s)) \\ &= s^* \xi(v(s)) \in V_{l,\beta} \end{aligned}$$

which can only occur if $s \in C_l$ in which case $k = \ell$. We then have

$$(0) \neq \xi[V_{k,\alpha}(s)] \subseteq V_{k,\beta}(s)$$

and, in particular, when $s = g_k$

$$(0) \neq \xi(1 \otimes V_\alpha^k) \subseteq 1 \otimes V_\beta^k.$$

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Then ξ gives rise to an A_k-module homomorphism, and hence isomorphism, $\xi_0 : V_\alpha^k \longrightarrow V_\beta^k$, defined by

$$\xi(1 \otimes v) = 1 \otimes \xi_0(v), \qquad \forall v \in V_\alpha^k.$$

This shows that if $V_{k,\alpha} \cong V_{l,\beta}$ then $k = l$ and the A_k-modules V_α^k, V_β^k are isomorphic.

Conversely suppose we have an A_k-module isomorphism $\xi_0 : V_\alpha^k \longrightarrow V_\beta^k$. Then ξ_0 extends to a $D(G)$-module isomorphism $\xi : V_{k,\alpha} \longrightarrow V_{k,\beta}$ given by

$$\xi(v(s)) = (\xi_0(v))(s),$$

which is sufficient to prove the result. ∎

We are now in a position to show that the $V_{k,\alpha}$ exhaust all irreducible $D(G)$-modules as k runs through the conjugacy classes of G and α through the non-isomorphic irreducible A_k-modules.

THEOREM 6.3. *Every irreducible $D(G)$-module is isomorphic to one of the $V_{k,\alpha}$.*

PROOF: Using a simple counting argument we have, in the notation of equation (32),

$$(*) \qquad \sum_k \sum_\alpha d[k,\alpha]^2 = \sum_k |\mathcal{C}_k|^2 \sum_\alpha \left(d_\alpha^k\right)^2$$

where the sum on k is over the conjugacy classes of G and the sum on α is over the non-isomorphic irreducible A_k-modules. Since Z_k is a finite group we have, in view of equation (27),

$$\sum_\alpha \left(d_\alpha^k\right)^2 = |Z_k| = |G| \,/\, |\mathcal{C}_k| \,.$$

Substituting into equation (*) we arrive at

$$\begin{aligned} \sum_k \sum_\alpha d[k,\alpha]^2 &= |G| \sum_k |\mathcal{C}_k| \\ &= |G|^2 \,. \end{aligned}$$

By comparison with equation (19), it follows that the $V_{k,\alpha}$ must exhaust all irreducible $D(G)$-modules. ∎

7. Representations and R-matrices

Here we clarify the construction of the previous section by explicitly determining the irreducible matrix representations of $D(G)$: throughout we adopt the notation of Section 6. We let $\pi_{k,\alpha}$ be the representation of $D(G)$ afforded by $V_{k,\alpha}$ and we let $\{v_i\}$ be a fixed orthonormal basis for the irreducible A_k-module V_α^k with π_α^k the representation of A_k afforded by V_α^k. In view of equation (29) the vectors

$$v(s)_i = \tau_s \otimes v_i, \qquad s \in \mathcal{C}_k \tag{33}$$

form a basis for the irreducible $D(G)$-module $V_{k,\alpha}$.

According to equations (30, 30′) the action of $D(G)$ in the basis (33) is given by

$$\left.\begin{aligned} g^* v(s)_i &= \delta(g,\, s) v(s)_i \\ g\, v(s)_i &= \pi_\alpha^k \left[\tau^{-1}_{gsg^{-1}}\, g\, \tau_s\right]_{ji} v\left(g\, s\, g^{-1}\right)_j \end{aligned}\right\} \quad \forall g \in G$$

where here and below the summation convention over repeated indices is assumed. We thereby arrive at the following construction for the matrix representation $\pi_{k,\alpha}$ in the basis (33):

$$\begin{aligned} \pi_{k,\alpha}(g^*)_{jt,is} &= \delta(g,\, s)\delta(t,\, s)\delta_{ji} \\ \pi_{k,\alpha}(g)_{jt,is} &= \delta\left(t,\, g\, s\, g^{-1}\right)\pi_\alpha^k\left(\tau_t^{-1}\, g\, \tau_s\right)_{ji} \end{aligned}$$

so that

$$\pi_{k,\alpha}(gh^*)_{jt,is} = \delta(h,\, s)\delta\left(t,\, g\, s\, g^{-1}\right)\pi_\alpha^k\left(\tau_t^{-1}\, g\, \tau_s\right)_{ji} \tag{34}$$

for $1 \leqslant i,\, j \leqslant d_\alpha^k$, $s,\, t \in \mathcal{C}_k$. It is easily checked that if we take the basis (33) to be orthonormal; that is,

$$\left(v(s)_i,\, v(t)_j\right) = \delta(s,\, t)\delta_{ij}$$

thus defining an inner product on $V_{k,\alpha}$, then the above representation is unitary, provided π_α^k is assumed unitary for Z_k, in agreement with Lemma 4.1.

It follows from equation (34) that the irreducible matrix representations of $D(G)$ can be determined from those of the centraliser subgroups Z_k together with a knowledge of the conjugacy classes $\mathcal{C}_k$. In particular we note that $\pi_{k,\alpha}$ when restricted to $G \subseteq D(G)$, coincides with the representation of G induced from π_α^k.

As to the corresponding characters, herein denoted $\chi_{k,\alpha}$, we have directly from equation (34) that

$$\chi_{k,\alpha}(gs^*) = 0, \qquad s \notin \mathcal{C}_k \tag{35_a}$$

otherwise

$$\chi_{k,\alpha}(gs^*) = \delta\left(s,\, g\, s\, g^{-1}\right)\chi_\alpha^k\left(\tau_s^{-1}\, g\, \tau_s\right) \tag{35_b}$$

where χ_α^k denotes the character of the A_k-module V_α^k. Thus the characters of the irreducible $D(G)$-modules may be determined explicitly from those of the centraliser subgroups Z_k. We note from equation (35_b) that $\chi_{k,\alpha}\ (gs^*)$ vanishes unless g and s commute, in agreement with Lemma 5.1.

In the special case that $\alpha = \iota_0$, corresponding to be identity representation of Z_k, the matrix representation (34) reduces to

$$\pi_{k,\iota_0}(gh^*)_{ts} = \delta(h,\, s)\delta\big(t,\, g\,s\,g^{-1}\big), \tag{36}$$

$s,\, t \in \mathcal{C}_k$, and the corresponding module V_{k,ι_0} has dimension

$$d[k,\, \iota_0] = |\mathcal{C}_k|.$$

In this case the character is given simply by

$$\chi_{k,\iota_0}(gs^*) = \begin{cases} \delta\big(s,\, g\,s\,g^{-1}\big), & s \in \mathcal{C}_k \\ 0, & \text{otherwise.} \end{cases} \tag{37}$$

Given a fixed, but arbitrary, irreducible $D(G)$-module $V = V_{k,\alpha}$, we have seen from Section 2 that the R-matrix

$$\begin{aligned} R &= \sum_{g\in G} \pi_{k,\alpha}(g) \otimes \pi_{k,\alpha}(g^*) \\ &= \sum_{s\in \mathcal{C}_k} \pi_{k,\alpha}(s) \otimes \pi_{k,\alpha}(s^*) \end{aligned} \tag{38_a}$$

satisfies the Yang-Baxter equation (3) on $V \otimes V \otimes V$ and has inverse

$$R^{-1} = \sum_{s\in\mathcal{C}_k} \pi_{k,\alpha}\big(s^{-1}\big) \otimes \pi_{k,\alpha}(s^*) \tag{38_b}$$

as may be verified directly. We are now in a position to determine the R-matrices (38) explicitly. We work in the orthonormal basis (33) and we let E^{is}_{jt} $\big(1 \leqslant i,\, j \leqslant d^k_\alpha,\, s, t \in \mathcal{C}_k\big)$ denote the corresponding elementary matrix with a 1 in the $(is,\, jt)$ position and zeros elsewhere.

In terms of elementary matrices we have the expansions

$$\begin{aligned} \pi_{k,\alpha}(g^*) &= \sum_i E^{ig}_{ig}, \qquad g \in \mathcal{C}_k \\ \pi_{k,\alpha}(g) &= \sum_{t\in\mathcal{C}_k} \pi^k_\alpha\Big(\tau^{-1}_{gtg^{-1}}\, g\, \tau_t\Big)_{ji} E^{j\ gtg^{-1}}_{i\ t} \end{aligned}$$

which follows directly from equation (34). Substituting into equations (38) we arrive at

$$\begin{aligned} R &= \sum_{s,t\in\mathcal{C}_k} \pi^k_\alpha\big(\tau^{-1}_{sts^{-1}}\, s\, \tau_t\big)_{ji} E^{j\ sts^{-1}}_{i\ t} \otimes E^{ls}_{ls} \\ R^{-1} &= \sum_{s,t\in\mathcal{C}_k} \pi^k_\alpha\big(\tau^{-1}_{s^{-1}ts} s^{-1} \tau_t\big)_{ji} E^{j\ s^{-1}ts}_{i\ t} \otimes E^{ls}_{ls}. \end{aligned} \tag{39}$$

Under the assumption that $\pi_{k,\alpha}$ is unitary, note that $R^{-1} = R^{\dagger}$; that is, R is a unitary matrix.

In the special case that $\alpha = \iota_0$, corresponding to the identity representation of Z_k, equations (39) reduce simply to the $|\mathcal{C}_k|^2 \times |\mathcal{C}_k|^2$ matrices given by

$$\begin{aligned} R &= \sum_{s,t\in\mathcal{C}_k} E_t^{sts^{-1}} \otimes E_s^s \\ R^{-1} &= \sum_{s,t\in\mathcal{C}_k} E_t^{s^{-1}ts} \otimes E_s^s \end{aligned} \tag{40}$$

which enables families of R-matrices to be constructed corresponding to each conjugacy class of any finite group. It may be checked directly that the R-matrices (39, 40) indeed satisfy the Yang-Baxter equation (3) as required.

In the next paper of the series we shall consider some explicit examples of the R-matrices (39, 40) and use these to construct unitary representations of the Braid group and correpsonding link polynomials.

NOTE Added in Preparation.

After completion of this paper, independent research of Lusztig [15] was brought to my attention. This reference in particular develops the character and representation theory of quantum double group algebras, along somewhat different lines.

References

[1] Y. Akutsu and M. Wadati, 'Knots, links, braids and exactly solvable models in statistical mechanics', *Comm. Math. Phys.* **117** (1988), 243–259.

[2] L. Alvarez-Gaumé, C. Gomes and G. Sierra, 'Duality and quantum groups', *Phys. Lett. B* **220** (1989), 142–150.

[3] R.J. Baxter, *Exactly solved models in statistical mechanics* (Academic press, 1982).

[4] R.J. Baxter, 'The inversion relation method for some two-dimensional exactly solved models in lattice statistics', *J. Statist. Phys.* **28** (1982), 1–14.

[5] C.W. Curtis and I. Reiner, *Representation theory of finite groups and associative algebras* (Wiley, New York, 1962).

[6] V.G. Drinfeld, 'Quantum groups', in *Proc. ICM Berkeley* **1**, 1986, pp. 798–820.

[7] L.D. Faddeev, N. Yu. Reshetikhin and L.A. Takhtajan, 'Quantum groups', *Algebra and analysis* **1** (1988), 129–139.

[8] M.D. Gould, R.B. Zhang and A.J. Bracken, 'Generalized Gelfand invariants and characteristic identities for Quantum groups', *J. Math. Phys.* **32** (1991), 2298–2303.

[9] M.D. Gould, R.B. Zhang and A.J. Bracken, 'Quantum double construction for graded Hopf algebras', *Bull. Austral. Math. Soc.* **47** (1993), 353–375.

[10] M. Hamermesh, *Group theory* (Addison-Wesley, Cambridge MA, 1962).

[11] M. Jimbo, 'A q-difference analogue of $U(g)$ and the Yang-Baxter equation', *Lett. Math. Phys.* **10** (1985), 63–69.

[12] P.P. Kulish and E.K. Sklyanin, *Integrable Quantum Field Theories*, Lecture notes in Physics **151** (Springer-Verlag, New York, 1982).

[13] R.G. Larson, 'Characters of Hopf algebras', *J. Algebra* **17** (1971), 352-368.

[14] J.R. Links and M.D. Gould, 'Casimir invariants for Hopf algebras', *Rep. Math. Phys.* (to appear).

[15] G. Lusztig, 'Leading coefficients of character values of Hecke algebras', *Proc. Sympos. Pure Math.* **47** (1987), 235–262.

[16] M.E. Sweedler, *Hopf algebras* (Benjamin, New York, 1969).

[17] H.N.V. Temperley and E.H. Lieb, 'Relations between the Perculation and colouring problem and other graph-theoretical problems associated with regular planar lattices: some exact results for the perculation problem', *Proc. Roy. Soc.* **A332** (1971), 251–270.

[18] V.G. Turaev, 'The Yang-Baxter equation and invariant of links', *Invent. Math.* **92** (1988), 527–553.

[19] E. Witten, 'Quantum field theorey and the Jones polynomial', *Comm. Math. Phys.* **121** (1989), 351–399.

[20] N. Yu. Reshetikhin, 'Quantized universal enveloping algebras, the Yang-Baxter equation and invariants of links I, II', L.O.M.I. Preprints (Leningrad) E-4-87, E-17-87 .

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