A QUESTION OF VALDIVIA ON QUASINORMABLE FRÉCHET SPACES

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ABSTRACT. It is proved that a Fréchet space is quasinormable if and only if every null sequence in the strong dual converges equicontinuously to the origin. This answers positively a question raised by Valdivia. As a consequence a positive answer to a problem of Jarchow on Fréchet Schwartz spaces is obtained.

The class of quasinormable Fréchet spaces was studied by Grothendieck in [2] as a class “containing the most usual Fréchet functions spaces” (cf. [2, p. 107]). This class received recently much attention in the context of the structure theory of Fréchet spaces and Köthe echelon spaces (see [1,6,8,9,10]). Valdivia in 1981 [8] asked if every separable Fréchet space such that its strong dual verifies the Mackey convergence condition is quasinormable. This question was also collected in the problem list of [7, problem 13.5.1]. Here we present a positive answer to this problem, even without the assumption of the separability of the Fréchet space.

Let $F$ be a Fréchet space with an increasing fundamental sequence of seminorms $(\| \cdot \|_n)_{n \in \mathbb{N}}$ such that $U_n := \{ x \in F; \| x \|_n \leq 1 \}$ $(n \in \mathbb{N})$ form a basis of 0-neighbourhoods in $F$. The system of all closed absolutely convex bounded subsets of $F$ is denoted by $\mathcal{B}(F)$. The dual seminorms are defined by $\| u \|_n^* := \sup \{ \| (u,x) \| ; x \in U_n \}$, if $u \in F^*$. We denote by $F_n^* := \{ u \in F^*; \| u \|_n^* < \infty \}$ the linear span of $U_n^*$ endowed with the normed topology defined by $\| \cdot \|_*$. The symbols $F_b^*$ and $F'_{nb}$ stand for the strong and the inductive dual of $F$ respectively, i.e., $F_b^* := \text{ind} F_{nb}$ is the bornological space associated with $F_{nb}$. According to Grothendieck [2], we say that $F_{nb}$ satisfies the Mackey convergence condition if every null sequence in $F_{nb}$ is contained in some $F_{nb}$ and converges to the origin in $F_{nb}$. The quasinormable spaces were introduced by Grothendieck [2]. The Fréchet $F$ is called quasinormable if the following condition holds:

\[(QN) \quad \forall n \quad \exists m > n \quad \forall \varepsilon > 0 \quad \exists B \in \mathcal{B}(F) : U_m \subset B + \varepsilon U_n.\]

The positive solution to Valdivia's problem is contained in the following theorem.

THEOREM. Let $F$ be a Fréchet space. The following conditions are equivalent:

(1) $F$ is quasinormable.

(2) $\forall n \quad \exists m > n \quad \forall k > m \quad \forall \varepsilon > 0 \quad \exists \lambda > 0 : U_m \subset \lambda U_k + \varepsilon U_n$ (cf. [6])

(3) $F_{nb}$ satisfies the Mackey convergence condition.
(4) $F'_n = \text{ind} F'_n$ is a sequentially retractive inductive limit (i.e., every null sequence in $F'_n$ is contained in some $F'_n$ and converges to the origin in $F'_n$).

PROOF. It is a direct matter to check that (1) implies (2). The fact that (1) implies (3) follows from the original definition of quasinormable Fréchet spaces (cf. [2]). Conditions (3) and (4) are equivalent since $F'_b$ and $F'_n$ have the same convergent sequences. Indeed, let $(x_j)_{j \in \mathbb{N}}$ be a null sequence in $F'_b$ and let $L$ denote the linear span of this sequence. By [7,8.2.18], $F'_b$ and $(F', \beta(F', F''))$ induce the same topology on $L$. The conclusion follows since $F'_n = (F', \beta(F', F''))$ (see e.g. [4, 29,4(2)]).

We prove now that (4) implies (2). If $F'_n = \text{ind} F'_n$ is sequentially retractive, we can apply a theorem of Neus to conclude that it is even strongly boundedly retractive (see e.g. [9, p. 169] or [7,8.5.48]). This means precisely

$$\forall n \exists m > n : F'_m \text{ and } F'_n \text{ induce the same topology on } U_n^\circ.$$  

This implies at once

$$\forall n \exists m > n \forall k > m : F'_k \text{ and } F'_m \text{ induce the same topology on } U_n^\circ,$$

or equivalently

$$\forall n \exists m > n \forall k > m \forall \alpha > 0 \exists \beta > 0 : \beta U_k \cap U_n^\circ \subset \alpha U_m^\circ.$$

Taking polars in $F$ and using the bipolar theorem, it is easy to see that this implies (2).

Now it is a direct matter to check that condition (2) is equivalent to the fact that $F$ satisfies the property $(\Omega_\alpha)$ of Vogt and Wagner (see [6] and [11]) for some strictly increasing function $\varphi : (0, \infty) \to (0, \infty)$. By [6, Theorem 7], this implies that $F$ is quasinormable. The proof is already complete, but, since the proof of [6, Theorem 7] is rather involved, we present now a simple and direct proof of (2) implies (1) by use of a Mittag-Leffler procedure.

Without loss of generality, we may assume that $m = n+1$ in (2). Our assumption may be then formulated as follows

$$(\ast) \forall n \forall k \forall \varepsilon > 0 \exists \lambda > 0 : U_{n+1} \subset \lambda U_k + \varepsilon U_n.$$

To prove that condition $(QN)$ is satisfied we only do it for the first neighbourhood in the basis. For simplicity in the notation we call it $U_0$. We fix $n = 0$ and $\varepsilon > 0$. By $\ast$ for "$n" = 0, "k" = 2, "\varepsilon" := \varepsilon / 2, we have $U_1 \subset \lambda_1 U_2 + (\varepsilon / 2) U_0$. Applying $\ast$ to "$n" := 1, "k" := 3, "\varepsilon" := \varepsilon / (\lambda_1 2^2)" we get $U_2 \subset \lambda_2 U_3 + (\varepsilon / \lambda_1 2^3) U_1$, hence $\lambda_1 U_2 \subset \lambda_2 U_3 + (\varepsilon / \lambda_1 2^3) U_1$ with $\lambda_2 := \lambda_1 \lambda_2$.

Proceeding by recurrence we determine $(\lambda_k)_{k \in \mathbb{N}}$, $\lambda_0 := 1$, such that

$$(\ast\ast) \forall k \lambda_{k-1} U_k \subset \lambda_k U_{k+1} + \varepsilon 2^{-k} U_{k-1}.$$  

Fix $z \in U_1$. We have $z = \lambda_1 u_2 + \varepsilon 2^{-1} v_1$, where $u_2 \in U_2$ and $v_1 \in U_0$. If $k \in \mathbb{N}$, we have, from $(\ast\ast)$, $\lambda_{k-1} u_k = \lambda_k u_{k+1} + \varepsilon 2^{-k} v_k$, $u_{k+1} \in U_{k+1}$ and $v_k \in U_{k-1}$. Since $F$ is a

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Fréchet space and \( v_k \in U_{k-1} \), the series \( \sum_{k=1}^{\infty} \varepsilon 2^{-k}v_k \) converges to an element \( x \) of \( F \) which belongs to \( \varepsilon U_0 \). The set \( B := \cap_{k \in \mathbb{N}} (\lambda_k + \varepsilon)U_k \) is bounded in \( F \) (and independent of \( z \)). We prove that \( z - x \in B \). Indeed, fix \( k \in \mathbb{N} \),

\[
\begin{align*}
z - x &= \left( z - \sum_{j=1}^{k} \varepsilon 2^{-j}v_j \right) - \sum_{j=k+1}^{\infty} \varepsilon 2^{-j}v_j = \lambda_k U_{k+1} \\
&\quad - \sum_{j=k+1}^{\infty} \varepsilon 2^{-j}v_j \in \lambda_k U_{k+1} + \varepsilon 2^{-k}U_k \subset (\lambda_k + \varepsilon)U_k.
\end{align*}
\]

Consequently, \( \forall \varepsilon > 0 \exists B \in \mathcal{B}(F) : U_1 \subset B + \varepsilon U_0 \). The proof is complete.

**Remark.** Let \( E \) be a (DF)-space with a fundamental sequence of bounded sets \((B_n)_{n \in \mathbb{N}}\). We consider the following two conditions on \( E \).

(a) \( \forall n \exists m > n \forall \alpha > 0 \exists \beta = 0 \text{-neighbourhood } U \in E : B_n \cap U \subset \alpha B_m \).

(b) \( \forall n \exists m > n \forall k \alpha > 0 \exists \beta > 0 : B_n \cap \beta B_k \subset \alpha B_m \).

Property (a) is precisely the strict Mackey condition introduced by Grothendieck in [2]. Property (b) means exactly that the inductive limit \( \text{ind} E_{B_n} \) satisfies the condition \((M)\) of Retakh (see e.g. [8, p. 164]). Clearly condition (a) implies condition (b). The converse implication holds if \( E \) is the strong dual of a Fréchet space according to our previous theorem, or if \( E \) is bornological (i.e., if \( E = \text{ind} E_{B_n} \) holds topologically) by a result of Retakh (see [9, p. 164(2)]). In general (b) does not imply (a), which shows that our theorem can not be deduced from a more general result about (DF)-spaces using duality. Here is the example: let \( X \) be a Banach space such that \((X, \sigma(X', X))\) is not separable and denote by \( E \) the linear space \( X \) endowed with the topology of uniform convergence on the countable bounded subsets of \((X', \sigma(X', X))\). Then \( E \) is a (DF)-space which does not satisfy the strict Mackey condition (cf. [8, Prop. p. 79]). But if \( B \) is the unit ball of the Banach space \( X \), then \((nB)_{n \in \mathbb{N}}\) is a fundamental sequence of bounded subsets of \( E \). Property (b) is then certainly satisfied.

Our next corollary contains one of the possible extensions to Fréchet spaces of what is known as the Josefson-Nissenzweig theorem (if \( X \) is a Banach space in the dual of which all weak* convergent sequences are norm convergent, then \( X \) is finite-dimensional). The corollary is the version of [3, 11.6.3] without the assumption of separability on the Fréchet space, and constitutes the precise positive solution to Jarchow question in [3, 11.10] about the characterization of Fréchet Schwartz spaces. Our next result is obtained by combining the theorem with results of Lindström [5]. These latter results depend heavily on a version of Bourgain and Diestel of the Josefson-Nissenzweig theorem (see [5]), so that the corollary extends but not reproves the theorem.

**Corollary.** A Fréchet space \( F \) is Schwartz if and only if every \( \sigma(F', F) \)-convergent sequence in \( F' \) is contained in some \( F'_n \) and converges there (i.e. converges equicontinuously).

**Proof.** Assume that every \( \sigma(F', F) \)-convergent sequence converges equicontinuously. This implies that \( F'_n \) satisfies the Mackey convergence condition. By our theorem \( F \) is quasinormable. Now the conclusion follows from [5, Cor. 3].

(2) As a direct consequence of our theorem it follows that a Fréchet space $F$ is quasi-normable if and only if the space of germs $H(K)$ is strongly boundedly retractive for one (or for all) compact subset(s) $K \neq \emptyset$ of $F$. This is a positive answer to Problem 14 in K. D. Bierstedt, R. Meise, *Aspects of inductive limits in spaces of germs of holomorphic functions on locally convex spaces and applications to a study of $(H(U), \tau_w)$*, p. 111–178 in *Advances in Holomorphy*, North-Holland Math. Studies 34, Amsterdam 1979.

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