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COMPACTNESS AND WEAK COMPACTNESS IN SPACES OF COMPACT-RANGE VECTOR MEASURES

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1. Introduction. This paper features strong and weak compactness in spaces of vector measures with relatively compact ranges in Banach spaces. Its tools are the measure-operator identification of [**16**] and [**24**] and the description of strong and weak compactness in spaces of compact operators in [**10**], [**11**], and [**29**].

Given a Banach space X and an algebra $\mathscr{A}$ of sets, it is shown in [**16**] that under the usual identification via integration of X-valued bounded additive measures on $\mathscr{A}$ with X-valued sup norm continuous linear operators on the space $S(\mathscr{A})$ of $\mathscr{A}$-simple scalar functions, the strongly bounded, countably additive measures correspond exactly to those operators which are continuous for the coarser (locally convex) universal measure topology τ on $S(\mathscr{A})$. It is through the latter identification that the results on strong and weak compactness in [**10**], [**11**], and [**29**] can be applied to X-valued continuous linear operators on the generalized DF space $S(\mathscr{A})_\tau$ to yield results on strong and weak compactness in spaces of vector measures. These measure-operator identifications are stated in Section 2.

The main theorem on strong compactness is stated and proved in Section 3 as Theorem 3.1. It gives necessary and sufficient conditions that a subset of X-valued strongly bounded, countably additive (here called strongly countably additive) measures with relatively compact ranges on an algebra of sets be relatively compact in the strong topology of the semi-variation norm. Related results have recently been found by Brooks and Dinculeanu [**9**]. Among the several equivalent conditions given in Theorem 3.1, condition (3.1.6) seems, from the point of view of necessity, stronger than any heretofore known.

It is the generality of the measure-theoretic setting (strongly countably additive measures on algebras of sets) of Theorem 3.1 which permits many applications in Section 4. These include strong compactness results for spaces of Pettis-integrable functions, spaces of unconditionally convergent series, spaces of compact-range bounded additive measures on Boolean

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algebras, and spaces of compact operators on some classical function space.

Theorem 3.1 can also be applied to a single measure and combined with a deep theorem of Rosenthal **[25]** to characterize measures with relatively compact ranges in terms of special sequences of expectation operators. This is done in Section 5.

In Section 6, the measure-operator identifications of Section 2 are combined with results from **[10]** and **[12]** to equate weak sequential convergence in spaces of measures with relatively compact ranges to pointwise weak convergence and to describe various forms of compactness in the weak topology of such spaces. These results are then combined with compactness results from **[17]** and **[22]** to characterize relative weak compactness under the assumption that the measures in question are countably additive on a σ-algebra of sets.

In parallel with the studies of Batt **[1]** and Batt and Hiermeyer **[2]**, more can be said when the dual X^* of the range space X has the Radon Nikdodym property. In this case, the measure-operator identifications of Section 2 are combined in Section 7 with results from **[11]** to derive representations of the duals of spaces of measures with relatively compact ranges.

2. Vector measures as operators. The main results in this paper are established by identifying vector-valued measures with continuous linear operators and applying theorems on compactness in spaces of operators. Accordingly, this section is devoted to explaining the measure-operator identifications and stating some related results for later use.

Without additional specification, X is always a real or complex Banach space, $\mathscr{A}$ is an algebra of subsets of some non-empty set Ω, and $\mathscr{B}$ is a Boolean algebra. The term "measure" is used generically to refer to functions Φ on $\mathscr{A}$ or $\mathscr{B}$ to X which are at least *bounded additive* (ba). Measures may be *strongly bounded* (sb), *countably additive* (ca), or *strongly countably additive* (sca). These and other basic, measure-theoretic notions and notations included here can be found in **[16]**. The prefix "c", as in csca($\mathscr{A}$, X), refers to the subspace of the given space of measures (here, sca ($\mathscr{A}$, X), the strongly countably additive measures on $\mathscr{A}$ to X) consisting of those with relatively compact ranges.

All spaces of X-valued measures are henceforth assumed to be endowed with the topology of the semi-variation norm

$$\|\Phi\| = \sup \{ \|x^* \circ \Phi\| : \|x^*\| \leqq 1\}.$$

This topology is easily seen to be that of uniform convergence on $\mathscr{A}$ and is called the *strong topology*.

For locally convex spaces E and F, E' (as opposed to X^* for Banach space X) is the continuous dual of E, $\mathscr{L}(E, F)$ is the space of continuous linear operators from E to F and $\mathscr{K}(E, F)$ is its subspace of all compact operators. Implicit in notation such as $\mathscr{L}_b(E, F)$, $\mathscr{K}_b(E, F)$, and E'_b is that in each case the underlying space of continuous linear maps is endowed with the topology of uniform convergence on bounded subsets of E. E_c is E endowed with the topology of uniform convergence on compact convex circled subsets of E'_b.

Playing a key role throughout is a property studied in [**26**], [**27**] and [**28**]: E is a gDF *space* provided E'_b is a Frechet space and every linear operator on E which is continuous on bounded sets is continuous. Measures will now be seen to give rise to operators on a gDF space.

Let $S(\mathscr{A})$ be the space of all scalar-valued $\mathscr{A}$-simple functions on Ω (generated by the characteristic functions χ_A, $A \in \mathscr{A}$). If $\Phi:\mathscr{A} \to X$, then $\Phi = \tilde{\Phi} \circ \chi$ where

$$\chi(A) = \chi_A \quad \text{for } A \in \mathscr{A} \quad \text{and}$$

$$\tilde{\Phi}(f) = \int f d\Phi \quad \text{for } f \in S(\mathscr{A}).$$

It is henceforth assumed that $S(\mathscr{A})$ is endowed with the universal measure topology τ, described in [**16**] and [**24**] and subsequently studied in [**4**]-[**8**] and [**17**]-[**19**], so that the measure χ is sca and the linear operator $\tilde{\Phi}$ is τ-continuous for each Φ which is sca. Let $L(\mathscr{A})$ be the completion of $S(\mathscr{A})$ (with topology also denoted τ but notationally suppressed), and let $\mathscr{B}(\mathscr{A})$ be the completion of $S(\mathscr{A})$ in the sup norm. Then $\tilde{\Phi}$ may be assumed to be defined (by continuous extension) on $L(\mathscr{A})$ for each Φ which is sca. Of course, $\tilde{\Phi}$ is just integration on $\mathscr{B}(\Sigma)$ when $\mathscr{A} = \Sigma$ is a σ-algebra. The following fundamental representation theorem and related facts are proved in [**16**], [**4**], and [**5**] and will be used in this paper.

THEOREM 2.1. (1) *The map* $\Phi \mapsto \tilde{\Phi}$ *is an isometrical isomorphism (onto) giving*

$$\text{sca}(\mathscr{A}, X) \simeq \mathscr{L}_b(S(\mathscr{A}), X) = \mathscr{L}_b(L(\mathscr{A}), X).$$

In particular,

$$\text{sca}(\mathscr{A}) \simeq S(\mathscr{A})'_b = L(\mathscr{A})'_b.$$

(2) *For* $H \subseteq \text{sca}(\mathscr{A}, X)$ *and the corresponding subset* $\tilde{H}$ *of* $\mathscr{L}(L(\mathscr{A}), X)$, *the following are equivalent:* (i) $\tilde{H}$ *is equicontinuous.* (ii) $H(\mathscr{A})$ *is bounded (or* $H(A)$ *is bounded for all* $A \in \mathscr{A}$*), and there is* $\mu \in \text{sca}(\mathscr{A})^+$ *such that* H *(even* $|H|$ *in the scalar case) is uniformly* μ*-continuous. In the scalar case, these are equivalent to* (iii) H *is relatively weakly compact in* sca$(\mathscr{A})$, *and, when* $\mathscr{A} = \Sigma$ *is a* σ*-algebra, to* (iv) H *is relatively* $\sigma(\text{ca}(\Sigma), S(\Sigma))$*-compact (*$w^*$*-relatively compact).*

(3) *Except when* $\mathscr{A}$ *is finite,* τ *is strictly coarser than the topology of the sup norm on* $S(\mathscr{A})$, *yet both topologies have the same bounded sets.* $L(\mathscr{A})$ *is a*

semi-reflexive Mackey space giving the following sequence of linear spaces:

$$S(\mathscr{A}) \subseteq B(\mathscr{A}) \subseteq L(\mathscr{A}) = L(\mathscr{A})'' \cong \mathrm{sca}(\mathscr{A})^*.$$

With the identification $L(\mathscr{A}) = \mathrm{sca}(\mathscr{A})^*$, *the* τ*-bounded and norm bounded subsets of* $L(\mathscr{A})$ *coincide, and the norm closed unit ball in* $L(\mathscr{A})$ *is the* τ*-closure of the sup norm closed unit ball in* $S(\mathscr{A})$. *For* $\Phi \in \mathrm{sca}\,(\mathscr{A}, X)$ *and each extreme point* P *in the closed unit ball in* $L(\mathscr{A})$, *there are* $A, B \in \mathscr{A}$ *such that*

$$\tilde{\Phi}(P) = \Phi(A) - \Phi(B).$$

If additionally $\mathscr{A} = \Sigma$ *is a* σ*-algebra, then* $S(\Sigma)$ *is a Mackey space and for each* $\Phi \in \mathrm{ca}(\Sigma, X)$ *and each* F *in the closed unit ball in* $L(\Sigma)$, *there exists* f *in the closed unit ball of the space* $B(\Sigma)$ *of bounded* Σ*-measurable functions such that* $\tilde{\Phi}(F) = \int f d\Phi$.

(4) *Both* $S(\mathscr{A})$ *and* $L(\mathscr{A})$ *are* gDF *spaces.*

As a result of the fact [**28**, Theorem 3.1 (4)] that continuous linear operators from gDF spaces into Banach spaces which transform bounded sets into relatively compact sets are compact (i.e., transform some neighborhood of 0 into a relatively compact set), the basic representation of (2.1.1) adapts immediately to a representation of measures with relatively compact ranges.

THEOREM 2.2. *The map* $\Phi \mapsto \tilde{\Phi}$ *is an isometrical isomorphism (onto) giving*

$$\mathrm{csca}(\mathscr{A}, X) \simeq \mathscr{L}_b(S(A)_c, X) = \mathscr{K}_b(S(\mathscr{A}), X)$$
$$= \mathscr{K}_b(L(\mathscr{A}), X) = \mathscr{L}_b(L(\mathscr{A})_c, X).$$

The theory of sca measures on algebras of sets is sufficiently general to yield as a special case a seemingly more general theory. Let $\mathscr{A}$ be the algebra of all closed and open (clopen) subsets of the Stone space Ω of a Boolean algebra $\mathscr{B}$. Let $M(\Omega, X)$ be the space of all regular ca X-valued Borel measures on Ω. Stonean considerations and the isometries above are shown in [**4**] and [**19**] to lead to natural isometries

$$M(\Omega, X) \simeq \mathrm{sb}(\mathscr{B}, X) \simeq \mathrm{sca}(\mathscr{A}, X) \simeq \mathscr{L}_b(L(\mathscr{A}), X).$$

These combine with the fact that

$$\mathrm{csb}(\mathscr{B}, X) = \mathrm{cba}(\mathscr{B}, X) \simeq \mathscr{K}_b(C(\Omega), X)$$

and the fact that $C(\Omega)$ may be considered to be subspace of $L(\mathscr{A})$ via the Stone-Weierstrass isometry of $B(\mathscr{A})$ and $C(\Omega)$ to yield the following special case of Theorem 2.2 in which $\tilde{\Phi}$ is classical integration on $C(\Omega)$:

THEOREM 2.3. *The map* $\Phi \mapsto \tilde{\Phi}$ *is an isometrical isomorphism (onto) giving*

$$\text{cba}(\mathscr{B}, X) = \text{c}M(\Omega, X) \simeq \mathscr{L}_b(C(\Omega)_c, X)$$
$$= \mathscr{K}_b(C(\Omega)_\tau, X) = \mathscr{K}_b(C(\Omega), X).$$

Yet another special case of Theorem 2.2 is used in the sequel. Let **P** be the σ-algebra of all subsets of the set **N** of all positive integers. It is well known that

$$\text{ca}(\mathbf{P}, X) = \text{cca}(\mathbf{P}, X)$$

and that this space may be identified with the space $l_1[X]$ of all unconditionally convergent series in X (the net of partial sums over all finite subsets of **N** converges). The identification is $\Phi \mapsto (\Phi_n)$ where $\Phi_n = \Phi(\{n\})$ for each $n \in \mathbf{N}$. It is an isometry with the semi-variation norm on ca(**P**, X) transforming to a familiar norm on $l_1[X]$:

$$\|(\Phi_n)\| = \sup \{\sum |\langle x^*, \Phi_n\rangle| : \|x^*\| \leqq 1\}.$$

Moreover, $S(\mathbf{P}) = (m_0, \tau)$ and $L(\mathbf{P}) = (l_\infty, \tau)$ where m_0 is the subspace of l_∞ of all sequences which take only finitely many values and τ is Buck's strict topology, the Mackey topology of the $l_\infty - l_1$ pairing. Taking $\mathscr{A} = \mathbf{P}$ in Theorem 2.2 now yields the following special case:

THEOREM 2.4. *If*

$$\Phi = (\Phi_n) \in l_1[X], \lambda = (\lambda_n) \in l_\infty \quad \textit{and}$$
$$\tilde{\Phi}(\lambda) = \sum \lambda_n \Phi_n,$$

then $\Phi \mapsto \tilde{\Phi}$ *is an isometry (onto) giving*

$$l_1[X] \simeq \mathscr{K}_b((m_0, \tau), X) = \mathscr{L}_b((m_0, \tau), X)$$
$$= \mathscr{L}_b((l_\infty, \tau), X) = \mathscr{K}_b((l_\infty, \tau), X).$$

3. Relative strong compactness in csca($\mathscr{A}$, X). Before stating and proving the main, general theorem on strong compactness in spaces of compact-range measures, the notion of expectation operator must first be recalled.

The family $\mathscr{P}(\mathscr{A})$ of all finite collections of pairwise disjoint members of $\mathscr{A}$ is directed by refinement. Associated to every $\pi \in \mathscr{P}(\mathscr{A})$ and every scalar measure $\mu \geqq 0$ is the operator

$$E(\pi, \mu)(\Phi) = \sum \{\mu(A)^{-1}\mu(A \cap \cdot)\Phi(A) : A \in \pi\},$$

an operator of norm $\leqq 1$ called the conditional expectation of Φ by μ. The basic role of conditional expectations on spaces of measures and related function spaces is established in [**14**, IV.8.18 and IV.13.19] in the context of characterizing strong compactness in spaces of scalar-valued measures. Thus, the role of expectation operators in the present discussion is a natural one, and its importance will become clear upon examination of

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Example 6.8 in Section 6. Yet Theorem 3.1 below adds new dimensions to previous theorems [**9**, Theorem 11] of its type. Condition 6 reveals that strong compactness of H is indeed collective compactness, and the related construction of $\mathscr{U}(H)$ in the σ-algebra case, set in $B(\Sigma)$ rather than $L(\Sigma)$ without loss of generality according to Theorem 2.1.3 and so that $\tilde{H}(\mathscr{U}(H))$ has an integral representation, is easily seen to yield a grossly unbounded neighborhood of 0 transformed by H to a relatively compact set. Moreover, the generality of the theorem allows its application, in the next section, to more explicit measure theoretic examples.

THEOREM 3.1. *The following are equivalent for* $H \subseteq \text{csca}(\mathscr{A}, X)$.

(1) *H is relatively compact in the strong topology.*

(2) *$H(\mathscr{A})$ is relatively compact in X, and for each $x^* \in X^*$ and each sequence in $x^* \circ H$ there exist a subsequence, say $(x^* \circ \Phi_n)$, a $\mu \in \text{sca}(\mathscr{A})^+$, and a directed subset $\mathscr{P}$ of $\mathscr{P}(\mathscr{A})$ such that*

$$\lim \{ \|x^* \circ \Phi_n - E(\pi, \mu)(x^* \circ \Phi_n)\| : \pi \in \mathscr{P}\} = 0$$

uniformly in n.

(3) *$H(A)$ is relatively compact in X for all $A \in \mathscr{A}$, $H(\mathscr{A})$ is bounded, there exists $\mu \in \text{sca}(\mathscr{A})^+$ for which H is uniformly μ-continuous, and for any such μ,*

$$\Phi = \lim \{E(\pi, \mu)(\Phi) : \pi \in \mathscr{P}(\mathscr{A})\}$$

uniformly over $\Phi \in H$ in the strong topology.

(4) *$H(A)$ is relatively compact in X for all $A \in \mathscr{A}$, $H(\mathscr{A})$ is bounded, and every sequence in H admits a subsequence, say (Φ_n), for which there exist a directed subset $\mathscr{P}$ of $\mathscr{P}(\mathscr{A})$ and a $\mu \in \text{sca}(\mathscr{A})^+$ such that*

$$\Phi_n = \lim \{E(\pi, \mu)(\Phi_n) : \pi \in \mathscr{P}\}$$

uniformly over n in the strong topology.

(5) *$H(A)$ is relatively compact in X for all $A \in \mathscr{A}$, and there is a neighborhood $\mathscr{U}$ of 0 in $S(\mathscr{A})_c$ ($\mathscr{U} = K^\circ$ is the polar of some compact $K \subseteq \text{sca}(\mathscr{A})$) such that*

$$\left\{ \int f d\Phi : f \in \mathscr{U}, \Phi \in H \right\}$$

is bounded in X.

(6) *There is a neighborhood $\mathscr{U}$ of 0 in $L(\mathscr{A})_c$ ($\mathscr{U} = K^\circ$ for some compact $K \subseteq \text{sca}(\mathscr{A})$) such that $\tilde{H}(\mathscr{U})$ is relatively compact in X.*

If any of these conditions holds and if $\mathscr{A} = \Sigma$ is a σ-algebra, then

$$\left\{ \int f d\Phi : \Phi \in H, f \in \mathscr{U}(H) \right\}$$

is relatively compact in X where $\mathscr{U}(H)$ is the restriction of the $\mathscr{U}$ from (6) *to $B(\Sigma)$ and may be realized by the following process. For each countable*

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partition (A_i) *of* Ω *by members of* Σ, *select a strictly increasing sequence* (N_k) *of positive integers such that*

$$|x^* \circ \Phi| (\cup \{A_i : i \geqq N_k\}) < 2^{-(2k+3)}$$

for all $\Phi \in H$ *and all* $x^* \in X^*$ *with* $\|x^*\| \leqq 1$. *Define* $g \in B(\Sigma)$ *by setting*

$$g(\omega) = 1 \quad \text{for } \omega \in \cup \{A_i : i < N_1\} \quad \text{and}$$

$$g(\omega) = 2^{-k} \quad \text{for} \quad \omega \in \cup \{A_i : N_{k-1} \leqq i < N_k\} \text{ and } k \geqq 2.$$

Now define

$$\mathscr{U}(H, (A_i)) = \{f \in B(\Sigma) : \|fg\| \leqq 1\},$$

the norm being the sup norm. Finally, $\mathscr{U}(H)$ *may be taken to be the* τ*-closed convex hull of the union of the* $\mathscr{U}(H, (A_i))$*'s taken over any cofinal subset of the set of all countable partitions* (A_i) *of* Ω *ordered by refinement.*

Proof. The proof proceeds in stages, the first revealing the connection between strong compactness and equicontinuity in $\mathscr{L}(S(\mathscr{A})_c, X)$. A direct application of Theorem 1.10 of [**29**] to the present circumstance allowed by Theorem 2.1.4 above gives the following lemma:

LEMMA 3.2. *With* B *and* B_X *denoting the closed sup norm unit balls in* $S(\mathscr{A})$ *and* X *respectively, the following are equivalent for any* $\tilde{H} \subseteq \mathscr{K}_b(S(\mathscr{A})_c, X)$.

(1) $\tilde{H}$ *is relatively compact.*

(2) $\tilde{H}(B)$ *is relatively compact in* X, *and* $x^* \circ \tilde{H}$ *is relatively compact in* $S(\mathscr{A})' = \text{sca}(\mathscr{A})$ *for all* $x^* \in X^*$.

(3) $\tilde{H}$ *is equicontinuous in* $\mathscr{L}(S(\mathscr{A})_c, X)$ *and* $\tilde{H}(f)$ *is relatively compact in* X *for every* $f \in S(\mathscr{A})$.

(4) $\tilde{H}(f)$ *is relatively compact in* X *for each* $f \in S(\mathscr{A})$, *and* $\tilde{H}(\mathscr{U})$ *is bounded for some neighborhood* $\mathscr{U}$ *of* 0 *in* $S(\mathscr{A})_c$.

(5) $\tilde{H}(\mathscr{U})$ *is relatively compact in* X *for some neighborhood* $\mathscr{U}$ *of* 0 *in* $S(\mathscr{A})_c$.

If any of these conditions holds, then the $\mathscr{U}$ *in* (4) *and* (5) *may be taken to be*

$$\cap \{nB + \frac{1}{n}(\tilde{H})^{-1}(B_X) : n \in \mathbf{N}\}.$$

According to this result, characterizing equicontinuity in $\mathscr{L}(S(\mathscr{A})_c, X)$ is now a key issue in completing the proof of Theorem 3.1. The next two propositions do this, the first treating the scalar case as a modest generalization of a classical result [**14**, IV.8.18 and IV.13.19] proved without recourse to L_p-theory.

PROPOSITION 3.3. *The following are equivalent for* $H \subseteq \text{sca}(\mathscr{A})$.

(1) *H is relatively compact in variation norm.*

(2) $\tilde{H}$ *is* c*-equicontinuous as a subset of* $S(\mathscr{A})' = (S(\mathscr{A})_c)' = (\mathscr{L}(\mathscr{A})_c)'$.

(3) $H(\mathscr{A})$ *is bounded, there exists* $\mu \in \text{sca}(\mathscr{A})^+$ *such that* $|H|$ *is uniformly* μ*-continuous, and for any such* μ

$$\lim \{ \|\lambda - E(\pi, \mu)(\lambda)\| : \pi \in \mathscr{P}(\mathscr{A}) \} = 0$$

uniformly over $\lambda \in H$.

(4) $H(\mathscr{A})$ *is bounded, and every sequence in H admits a subsequence, say* (λ_n), *for which there exist a* $\mu \in \text{sca}(\mathscr{A})^+$ *and a directed subset* $\mathscr{P}$ *of* $\mathscr{P}(\mathscr{A})$ *such that*

$$\lim \{ \|\lambda_n - E(\pi, \mu)(\lambda_n)\| : \pi \in \mathscr{P}\} = 0$$

uniformly in n.

Proof. The equivalence of (1) and (2) is just a formal translation in view of the definition of the c-topology, the fact that $S(\mathscr{A})$ is c-dense in $L(\mathscr{A})$, and the identification in Theorem 2.1.1 above. To see that (3) follows from (1), first assume that H is relatively compact in variation norm. Then $H(\mathscr{A})$ is bounded and H is weakly relatively compact, and it follows from Theorem 2.1.2 that $|H|$ is uniformly μ-continuous for some $\mu \in \text{sca}(\mathscr{A})^+$. For any such μ, if λ is indefinite integration against some simple function g, $\lambda(A) = \int_A g d\mu$ for $A \in \mathscr{A}$, then

$$\lim \{ \|\lambda - E(\pi, \mu)(\lambda)\| : \pi \in \mathscr{P}(\mathscr{A}) \} = 0$$

since eventually $\lambda = E(\pi, \mu)(\lambda)$. But according to the generalized Radon-Nikodym Theorem 9.5 in [**16**], such λ's are norm dense in the subspace of μ-continuous members of $\text{sca}(\mathscr{A})$. The family

$$\{E(\pi, \mu) : \pi \in \mathscr{P}(\mathscr{A}) \}$$

is uniformly bounded, and so an application of the Banach-Steinhaus Theorem reveals that the above limit is uniform over λ in relatively norm compact H. It is clear that (4) follows from (3). To see that (1) follows from (4), first observe that each operator $E(\pi, \mu)$ has finite-dimensional range and so is a compact operator, for it transforms $\text{sca}(\mathscr{A})$ into the span of $\{\mu_A : A \in \pi\}$ where

$$\mu_A(C) = \mu(A \cap C) \quad \text{for all } C \in \mathscr{A}.$$

The hypotheses of (4) guarantee that every sequence in H admits a subsequence, say (λ_n), such that for any k, there is a relatively norm compact set C_k and a sequence $(\lambda_{k,n})$ from C_k such that

$$\|\lambda_n - \lambda_{k,n}\| < 1/k \quad \text{for all } n.$$

Thus, a standard diagonalization process produces a norm convergent subsequence of (λ_n).

The above result has a vector analogue.

PROPOSITION 3.4. *The following are equivalent for* $H \subseteq \mathrm{sca}(\mathscr{A}, X)$.

(1) $\tilde{H}$ *is equicontinuous in* $\mathscr{L}(S(\mathscr{A})_c, X) = \mathscr{L}(L(\mathscr{A})_c, X)$.

(2) $\tilde{H}(A)$ *is bounded, there exists* $\mu \in \mathrm{sca}(\mathscr{A})^+$ *such that H is uniformly* μ*-continuous, and for any such* μ,

$$\Phi = \lim \{E(\pi, \mu)(\Phi):\pi \in \mathscr{P}(\mathscr{A})\}$$

uniformly over $\Phi \in H$ *in the strong topology.*

(3) $H(\mathscr{A})$ *is bounded, and every sequence in H admits a subsequence, say* (Φ_n), *for which there exist* $\mu \in \mathrm{sca}(\mathscr{A})^+$ *and a directed subset* $\mathscr{P}$ *of* $\mathscr{P}(\mathscr{A})$ *such that*

$$\Phi_n = \lim \{E(\pi, \mu)(\Phi_n):\pi \in \mathscr{P}\}$$

uniformly over n in the strong topology.

Proof. If $\tilde{H}$ is c-equicontinuous, then it is τ-equicontinuous. So according to Theorem 2.1.2, $H(\mathscr{A})$ is bounded while H is uniformly μ-continuous for some $\mu \in \mathrm{sca}(\mathscr{A})^+$. For any such μ, it follows that $\{x^* \circ H:\|x^*\| \leqq 1\}$ is uniformly μ-continuous. But $\{x^* \circ \tilde{H}:\|x^*\| \leqq 1\}$ is c-equicontinuous in $(L(\mathscr{A})_c)'$. Since

$$x^* \circ E(\pi, \mu)(\Phi) = E(\pi, \mu)(x^* \circ \Phi) \quad \text{for } x^* \in X^* \text{ and } \Phi \in \mathrm{sca}(\mathscr{A}, X),$$

an application of Proposition 3.3 reveals that

$$\Phi = \lim \{E(\pi, \mu)(\Phi):\pi \in \mathscr{P}(\mathscr{A})\}$$

uniformly over $\Phi \in H$. So (2) follows from (1). It is clear that (3) follows from (2). To see that (3) implies (1), just observe that $\tilde{H}$ is c-equicontinuous if and only if $\{x^* \circ H:\|x^*\| \leqq 1\}$ is c-equicontinuous and then note, according to Proposition 3.3, that the hypotheses of (3) ensure the c-equicontinuity of the latter.

The proof of Theorem 3.1 is now completed by combining (3.2)-(3.4) above, and by using the idea of the proof of Theorems 10 and 11 of [**17**] to conclude that if $\mathscr{A} = \Sigma$ is a σ-algebra, then $\mathscr{U}(H)$ as described in Theorem 3.1 is a c-neighborhood of 0 in $B(\Sigma)$ and is contained in the neighborhood described in Lemma 3.2.

4. Applications of the main theorem on strong compactness. The statement of Theorem 3.1 is sufficiently general to allow application in several directions. It is at full strength when $\mathscr{A} = \Sigma$ is a σ-algebra, for not only is strong compactness in $\mathrm{cca}(\Sigma, X)$ characterized but a construction is given for the unbounded τ-neighborhood of 0 of bounded measurable functions with relatively compact integrals. Without the assumption that $\mathscr{A}$ is a σ-algebra, this neighborhood can still be constructed, but its

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construction takes place in an abstract completion, $L(\mathscr{A})$, and so is not here given. In this section, Theorem 3.1 is first applied under the assumption that $\mathscr{A}$ is the σ-algebra of all subsets of the positive integers, then is applied with the assumption that $\mathscr{A}$ is a classical measure space and H is a set of indefinite integrals of Pettis integrable functions, and finally is applied with $\mathscr{A}$ being the algebra of clopen subsets of a Stone space. In Theorem 4.1, a full-blown translation of Theorem 3.1 is given. However, in the interest of brevity, we state in Theorems 4.2 and 4.3 only the counterparts of conditions (1) and (6) of Theorem 3.1. Translating conditions (2)-(5) is an easy task.

THEOREM 4.1. *With conventions and notation as in Theorem* 2.4, *the following are equivalent for* $H \subseteq l_1[X]$.

(1) *H is relatively compact in* $l_1[X]$.

(2) *The set*

$$\{ \sum_{n \in A} \Phi_n : A \subseteq \mathbf{N}, \Phi \in H\}$$

is relatively compact in X, and for each $x^* \in X^*$,

$$\lim_k \sum_{n=k}^{\infty} | \langle x^*, \Phi_n \rangle | = 0$$

uniformly over $\Phi \in H$.

(3) *For each n, the set* $\{\Phi_n : \Phi \in H\}$ *is relatively compact in X, and*

$$\lim_k \sup \{ \sum_{n=k}^{\infty} | \langle x^*, \Phi_n \rangle | : \|x^*\| \leqq 1\} = 0$$

uniformly over $\Phi \in H$.

(4) *For each n, the set* $\{\Phi_n : \Phi \in H\}$ *is relatively compact in X, and for some real sequence* $c_n \searrow 0$, *the set*

$$\{\sum \lambda_n \Phi_n : \Phi \in H, \lambda \in m_0, |\lambda_n c_n| \leqq 1 \text{ for all } n\}$$

is bounded in X.

(5) *For some real sequence* $c_n \searrow 0$ *the set*

$$\{\sum \lambda_n \Phi_n : \Phi \in H, \lambda \in l_\infty, |\lambda_n c_n| \leqq 1 \text{ for all } n\}$$

is relatively compact in X.

If any of these conditions holds then the null sequence (c_n) *in* (5) *may be prescribed by first selecting a strictly increasing sequence* (N_k) *such that*

$$\sum \{ | \langle x^*, \Phi_n \rangle | : n \geqq N_k\} < 2^{-(2k+3)}$$

for all $\Phi \in H$ *and all* $x^* \in X^*$ *with* $\|x^*\| \leqq 1$ *and then setting* $c_n = 1$ *for* $n < N_1$ *and* $c_n = 2^{-k}$ *for* $N_{k-1} \leqq n < N_k$ *and* $k \geqq 2$.

Proof. Because the linear span of the unit vectors e_n (with value 1 at n and value 0 elsewhere) is τ-dense in m_0 and l_∞, relative compactness of $\{\Phi_n:\Phi \in H\}$ for all n is equivalent to relative compactness of $\{\sum_{n\in A} \Phi_n:\Phi \in H\}$ for all $A \subseteq \mathbf{N}$. So in view of the fact that all ca measures on $\mathbf{P}$ have relatively compact range, the description in **[16]** of τ-neighborhoods of 0 in m_0 and l_∞, and the fact that every partition of $\mathbf{N}$ into finitely many subsets is refined by one with finitely many singleton sets and finitely many cofinite sets, this result is but a translation of Theorem 3.1.

Now assume that $\mathscr{A} = \Sigma$ is a σ-algebra and that $\mu \in \text{ca}(\Sigma)^+$ is fixed. The linear space $P(\mu, X)$ of all Pettis integrable $f:\Omega \to X$ is normed by the Pettis norm

$$||f|| = \sup \left\{ \int |x^* \circ f| d\mu : ||x^*|| \leqq 1 \right\}$$

and, with respect to this norm, isometrically embeds in $\text{ca}(\Sigma, X)$ by $f \mapsto \Phi_f$ where for $A \in \Sigma$,

$$\Phi_f(A) = \int_A f d\mu.$$

It is well known that Φ_f has relatively compact range if f is strongly measurable and it has recently been shown in **[15]** that Φ_f has relatively compact range if (Ω, Σ, μ) is a perfect measure space and f is bounded. This gives the next theorem content. It is proved by translation of Theorem 3.1 via the identifications and notations of this paragraph after noting from **[17]** that $L_\infty(\mu)$ with the Mackey topology of $L_\infty(\mu) - L_1(\mu)$ may be regarded as a τ-complementary subspace of $L(\Sigma)$ and that the neighborhood $\mathscr{U}$ of Theorem 3.1 may in this case be chosen from that subspace, the τ-dual of which may be naturally identified with $L_1(\mu)$. Keep in mind that $P(\mu, X)$ need not be complete.

THEOREM 4.2. *For a σ-algebra Σ and a $\mu \in \text{ca}(\Sigma)^+$ the following are equivalent for any subset H of $P(\mu, X)$ for which Φ_f has relatively compact range for all $f \in H$.*

(1) *H is precompact in Pettis norm.*

(6) *There is a compact subset K of $L_1(\mu)$ such that*

$$\left\{ \int g f d\mu : g \in \mathscr{U}, f \in H \right\}$$

is relatively compact in X where $\mathscr{U} = K^\circ$ is the polar of K in $L_\infty(\mu)$.

If any of these conditions holds, then $\mathscr{U}$ in (6) *may be prescribed by the process in Theorem* 3.1 *after replacing $B(\Sigma)$ by $L_\infty(\mu)$ and measures Φ by measures Φ_f, $f \in H$.*

The next result is the most general result on strong compactness.

THEOREM 4.3. *Let $\mathscr{B}$ be a Boolean algebra with Stone space* Ω. *Let*

$$H \subseteq \text{cba}(\mathscr{B}, X) = \text{c}M(\Omega, X),$$

and let $\tilde{H}$ *be the corresponding subset of* $\mathscr{K}(C(\Omega), X)$. *The following are equivalent.*

(1) *H is strongly relatively compact (or, $\tilde{H}$ is relatively compact in operator norm).*

(6) *There is a strongly compact subset K of* $\text{ba}(\mathscr{B}) = M(\Omega)$ *such that if* $\mathscr{U}$ *is the polar in* $C(\Omega)$ *of K, then*

$$\left\{ \int f d\Phi : \Phi \in H, f \in \mathscr{U} \right\}$$

is relatively compact.

Proof. After noting the identifications of Theorem 2.3, this follows by taking $\mathscr{A}$ in Theorem 3.1 to be the algebra of clopen subsets of Ω.

Remark 4.4. Let Σ be a σ-algebra of sets and $\mu \in \text{ca}(\Sigma)^+$. Through a Stone-Weierstrass isometry, the above result gives a measure-theoretic characterization of relative compactness in operator norm in the space of compact operators on either of the classical function spaces $B(\Sigma)$ or $L_\infty(\mu)$. Simply take $\mathscr{B}$ to be either Σ or $\Sigma/\mu^{-1}(0)$.

5. Compactness of a single measure. In this section, a result of Rosenthal is combined with Theorem 3.1 to see that whether a measure has relatively compact range depends on uniform convergence of some sequences of expectations.

Consider the Cantor space $\Omega = \{-1, 1\}^{\mathbf{N}}$, and, for each n, let

$$B_n = \{\omega \in \Omega : \omega_n = 1\},$$

$$-B_n = \Omega \setminus B_n,$$

$$\sigma_n = \{-1, 1\}^{\{1,2,\dots,n\}}, \quad \text{and}$$

$$\epsilon(B_1, \dots, B_n) = \epsilon_1 B_1 \cap \epsilon_2 B_2 \cap \dots \cap \epsilon_n B_n$$

for each $\epsilon \in \sigma_n$. Then

$$\pi_n = \{\epsilon(B_1, \dots, B_n) : \epsilon \in \sigma_n\}$$

is a partition of Ω (a familiar one under the usual dyadic identification of Ω with $[0, 1]$).

Extend in an obvious way the definition above of $\epsilon(B_1, \dots, B_n)$ to a sequence (B_k) from an arbitrary Boolean algebra $\mathscr{B}$, and let $\pi_n(\mathscr{B})$ be the partition of its Stone space corresponding to π_n above. The sequence (B_k) is *independent* if

$$\epsilon(B_1, \dots, B_n) \neq \emptyset \text{ for each } \epsilon \in \sigma_n \text{ and each } n.$$

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According to [**34**], this is equivalent to the existence of a Boolean isomorphism of the algebra generated by (B_k) onto the algebra of clopen subsets of the Cantor space (the two sets of B_k's being identified by this isomorphism).

THEOREM 5.1. *For any Boolean algebra $\mathscr{B}'$ and any $\Phi \in \mathrm{sb}(\mathscr{B}', X)$, Φ has relatively compact range if and only if for any independent sequence (B_k) from $\mathscr{B}'$,*

$$\Phi_{|\mathscr{B}} = \lim_n E(\pi_n(\mathscr{B}), \mu)(\Phi_{|\mathscr{B}})$$

where $\Phi_{|\mathscr{B}}$ is the restriction of Φ to the Boolean subalgebra $\mathscr{B}$ generated by (B_k), μ is some member of $\mathrm{ba}(\mathscr{B})^+$ for which $\Phi_{|\mathscr{B}}$ is μ-continuous, and the limit is in uniform convergence on $\mathscr{B}$.

Proof. Consider any sequence (B_k) from $\mathscr{B}'$ as a sequence in the algebra $\mathscr{A}'$ of clopen subsets of the Stone space Ω' of $\mathscr{B}'$. According to a result in [**25**], it may be assumed without loss of generality that either

$$\limsup B_k = \liminf B_k \text{ in } \Omega'$$

or that (B_k) is an independent sequence. Employing the identifications

$$\mathrm{sb}(\mathscr{B}', X) = \mathrm{sca}(\mathscr{A}', X) = M(\Omega', X)$$

surrounding the discussion of Theorem 2.3 it is now easy to see that in the first case, $\lim \Phi(B_k)$ exists. In view of the discussion of independence above, the proof will be complete if the following special case is proved.

THEOREM 5.2. *Let Φ be a regular, countably additive Borel measure on the Cantor space $\Omega = \{-1, 1\}^{\mathbf{N}}$ with values in X. Then Φ has relatively compact range if and only if*

$$\Phi = \lim_n E(\pi_n, \mu)(\Phi)$$

for some $\mu \in M(\Omega)^+$ for which Φ is μ-continuous with convergence in norm or, equivalently, uniform on the clopen subsets of Ω.

Proof. First observe that there is a $\mu \in M(\Omega)^+$ for which Φ is μ-continuous [**16**]. Since each $E(\pi_n, \mu)(\Phi)$ has its range in a finite-dimensional subspace of X, the sufficiency of the condition is now clear. To prove necessity, invoke the identification

$$\mathrm{csca}(\mathscr{A}, X) = cM(\Omega, X)$$

of Theorem 2.3, $\mathscr{A}$ being the algebra of clopen subsets of Ω, and note that the result follows from Theorem 3.1 provided that the sequence (π_n) is cofinal in $\mathscr{P}(\mathscr{A})$. But a basic compactness argument reveals that each clopen subset of Ω is a finite disjoint union of sets of the form $\epsilon(B_1, \ldots, B_n)$ and so the proof is complete.

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6. Weak compactness inspaces of compact-range measures. The measure-operator identifications of Section 2 are combined in this section with results from [**10**] on weak compactness in spaces of compact operators to derive characterizations of sequential weak convergence and relative weak compactness in spaces of measures having relatively compact range. The Vitali-Hahn-Saks property (VHS) plays a major role. If $\mathscr{B}$ is σ-complete, then $\mathscr{B}$ has VHS. However, many interesting examples of non-σ-complete $\mathscr{B}$ with VHS can be found in [**19**] and [**31**].

THEOREM 6.1. *Let $\mathscr{B}$ be a Boolean algebra with* VHS, *and let (Φ_n) be a sequence from either* $\text{cba}(\mathscr{B}, X)$ *or* $\text{csca}(\mathscr{B}, X)$.

(1) $\Phi_n \to \Phi$ *weakly if and only if*

$$\langle x^*, \Phi_n(A) \rangle \to \langle x^*, \Phi(A) \rangle \text{ for all } A \in \mathscr{B} \text{ and all } x^* \in X^*.$$

(2) (Φ_n) *is weakly Cauchy if and only if $\langle x^*, \Phi_n(A) \rangle$ converges for all $A \in \mathscr{B}$ and all $x^* \in X^*$.*

Moreover, if $(\Phi_n(A))$ is bounded for each $A \in \mathscr{B}$, then the functionals x^* in (1) *and* (2) *can be restricted to those which are extreme points of the closed unit ball in X^*.*

Proof. Let $\mathscr{A}$ be the algebra of clopen subsets of the Stone space of $\mathscr{B}$ and apply Theorems 2.1 and 2.3 to conclude that

$$L(\mathscr{A})_c \simeq \text{ba}(\mathscr{B})'_c \quad \text{and} \quad \text{cba}(\mathscr{B}, X) \simeq \mathscr{L}_b(L(\mathscr{A})_c, X).$$

Then according to [**10**] $\Phi_n \to \Phi$ weakly if and only if

$$\langle x^*, \tilde{\Phi}_n(F) \rangle \to \langle x^*, \tilde{\Phi}(F) \rangle$$

for all $x^* \in X^*$ and all $F \in L(\mathscr{A})$. Since $\mathscr{B}$ has VHS, the latter convergence is equivalent to

$$\langle x^*, \Phi_n(A) \rangle \to \langle x^*, \Phi(A) \rangle$$

for all $x^* \in X^*$ and all $A \in \mathscr{B}$. This proves (1) for the space $\text{cba}(\mathscr{B}, X)$, and it is easy to derive (2) from (1) for this space. Since $\text{csca}(\mathscr{B}, X)$ is a closed subspace of $\text{cba}(\mathscr{B}, X)$, (1) and (2) are proved for both spaces. The final statement of the theorem now follows from Rainwater's Theorem on the weak convergence of bounded sequences.

The importance of weak convergence on members of $\mathscr{B}$ is now clear. Let the *$M \otimes N$-weak operator topology* ($M \otimes N$-wot) be the topology of the semi-norms

$$P_{F,x^*}(\Phi) = |\langle x^*, \tilde{\Phi}(F) \rangle|$$

where x^* is from a subset N of X^* and F is from a subset M of $L(\mathscr{A})$. Also let B_Y denote the closed unit ball of a Banach space Y, and let ext C denote the set of extreme points of subset C of a linear space. Recall that weak conditional compactness requires a sequence to admit a weak Cauchy subsequence. The second result is a direct consequence of the first and Eberlein's Theorem.

COROLLARY 6.2. *Let $\mathscr{B}$ be a Boolean algebra with* VHS. *A subset H of either* $\text{cba}(\mathscr{B}, X)$ *or* $\text{csca}(\mathscr{B}, X)$ *is weakly relatively (respectively, weakly conditionally) compact if and only if it is* $\mathscr{B} \otimes X^*$-wot *relatively sequentially (respectively, conditionally) compact. If $H(A)$ is bounded for each $A \in \mathscr{B}$, then $\mathscr{B} \otimes X^*$ may be replaced by $\mathscr{B} \otimes \text{ext}(B_{X^*})$ in the preceding statement.*

Note that $\mathscr{B}$ is being identified in $L(\mathscr{A})$ with the characteristic functions of clopen sets. A special case of the corollary is of interest in its own right.

COROLLARY 6.3. *Let Σ be a σ-algebra of sets, and let $\mu \in \text{ca}(\Sigma)^+$. A subset of either of the Banach spaces $\mathscr{K}(B(\Sigma), X)$ and $\mathscr{K}(L_\infty(\mu), X)$ is weakly relatively (respectively, weakly conditionally) compact if and only if it is* $\Sigma \otimes X^*$-wot *relatively sequentially (respectively, conditionally) compact.*

Proof. Let $\mathscr{B}$ be either Σ or $\Sigma/\mu^{-1}(0)$ in Corollary 6.2 and apply Theorem 2.3 along with a Stone-Weierstrass isometry.

Corollary 6.2 can be strengthened by combining results of [**3**] and [**12**] with those of [**10**] and [**30**].

THEOREM 6.4. (1) *A bounded subset of* $\text{csca}(\mathscr{A}, X)$ *is weakly relatively compact if and only if it is* $\text{ext}(B_L) \otimes \text{ext}(B_{X^*})$-wot *relatively countably compact where B_L is the closed unit ball in $L(\mathscr{A}) = \text{sca}(\mathscr{A})^*$.*

(2) *If $\mathscr{A} = \Sigma$ is a σ-algebra, then a convex subset of* $\text{cca}(\Sigma, X)$ *is weakly relatively compact if and only if it is* $\Sigma \otimes \text{ext}(B_{X^*})$-wot *relatively countably compact.*

Proof. Consider the isometric identification

$$\text{csca}(\mathscr{A}, X) = \mathscr{K}_b(L(\mathscr{A}), X)$$

of Theorem 2.2 and apply Theorem 3.2 of [**10**] and Theorem 1.3 of [**30**] to conclude that $\text{ext}(B_L) \otimes \text{ext}(B_{X^*})$ is precisely the set of all extreme points of the closed unit ball of $\text{csca}(\mathscr{A}, X)^*$. Now, (1) follows from Theorem 1 of [**3**]. To prove (2), it remains to show, according to Theorem 5.1 of [**12**], that for any $\Phi \in \text{cca}(\Sigma, X)$, there exist $f \in S(\Sigma)$ and $x^* \in \text{ext}(B_{X^*})$ such that

$$\sup \{T(\Phi): T \in \text{cca}(\Sigma, X)^*, \|T\| \leqq 1\} = \langle x^*, \int f d\Phi \rangle.$$

It is well known that the sup is attained at an extreme point of the closed unit ball in $\text{cca}(\Sigma, X)^*$ and so, according to the first remark of the proof, has the form $\langle x^*, \tilde{\Phi}(F) \rangle$ for some $x^* \in \text{ext}(B_{X^*})$ and some $F \in \text{ext}(B_L)$. An application of (3) of Theorem 2.1 completes the proof.

The following measure-theoretic characterization of weak compactness is the main result of this section.

THEOREM 6.5. *For a σ-algebra* Σ *of sets and a subset* H *of* cca(Σ, X), *the following are equivalent.*

(1) H *is weakly relatively compact.*

(2) $H(A)$ *is weakly relatively compact for each* $A \in \Sigma$, $x^* \circ H$ *is uniformly* sb *for all* $x^* \in X^*$, *and each* Φ *in the* $L(\Sigma) \otimes X^*$-wot *sequential closure of* H *has relatively compact range.*

(3) $H(A)$ *is weakly relatively compact for all* $A \in \Sigma$, $H(\Sigma)$ *is bounded,* $x^* \circ H$ *is uniformly* ca *for all* $x^* \in X^*$, *and each* Φ *in the* $\Sigma \otimes X^*$-wot *sequential closure of* H *has relatively compact range.*

Proof. Assume that H is weakly relatively compact. Since the weak topology of cca(Σ, X) is finer than the $\Sigma \otimes X^*$-wot, it follows that the last condition of (3) holds, that $H(A)$ is weakly relatively compact for all $A \in \Sigma$, and that $x^* \circ H^*$ is relatively $\sigma(\text{ca}(\Sigma), S(\Sigma))$-compact for each $x^* \in X^*$. So, according to Theorem 2.1, $H(\Sigma)$ is bounded, and $x^* \circ H$ is uniformly ca for all $x^* \in X^*$. Hence, (3) follows from (1). Clearly, (3) implies (2). Assume now that (2) holds. Then H is weakly conditionally compact by Theorem 3.2 of [**22**]. So a sequence (Φ_n) from H may now be assumed, without loss of generality, to be weakly Cauchy and must therefore, according to Theorem 7 of [**17**], cluster in the topology of pointwise convergence on $L(\Sigma)$ on the space $\mathscr{L}(L(\Sigma), X_\sigma)$ at some

$$\tilde{\Phi} \in \mathscr{L}(L(\Sigma), X) = \mathscr{L}(L(\Sigma), X_\sigma).$$

But (Φ_n), being weakly Cauchy, is $L(\Sigma) \otimes X^*$-wot Cauchy and so must converge to $\tilde{\Phi}$ in the topology of pointwise convergence in $\mathscr{L}(L(\Sigma), X_\sigma)$. H is then weakly relatively compact since the last condition of (2) insures that $\Phi \in$ cca(Σ, X) while (1) of Theorem 6.1 then guarantees that (Φ_n) weakly converges to Φ.

Remark 6.6. The last condition in (3) above is quite strong as a necessary condition for relative weak compactness of H, for it reveals that the weak closure of H in cca(Σ, X) is $\Sigma \otimes X^*$-wot sequentially closed in all of ca(Σ, X). On the other hand, the last condition in (2) is quite weak as one of several conditions collectively sufficient for weak relative compactness of H, for it is, in the presence of the other conditions, only the requirement that the weak closure of H in cca(Σ, X) be $L(\Sigma) \otimes X^*$-wot sequentially closed in all of ca(Σ, X) (or ca$(\Sigma)^* \otimes X^*$-wot sequentially closed, since $L(\Sigma) = \text{ca}(\Sigma)^*$).

Remark 6.7. If Σ is a σ-algebra of sets and $\mu \in \text{ca}(\Sigma)^+$, then the space of all strongly measurable Pettis integrable functions embeds isometrically in cca(Σ, X). Thus, Theorem 6.1 and Corollary 6.2 have counterparts for this space.

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The following example illustrates that, as suggested by inclusion of the last convergence conditions in (2) and (3) of Theorem 6.5, there is a difference between weak conditional and weak relative compactness in cca(Σ, X).

Example 6.8. Let $X = c_0$, $\Sigma =$ the Borel subsets of [0, 1], and $\mu =$ Lesbesgue measure on Σ. Let (r_n) be the sequence of Rademacher functions. For each n and $A \in \Sigma$, let $H = \{\Phi_n\}$ with

$$\Phi_n(A) = \sum_{k=1}^{n} \left(\int_A r_k d\mu \right) e_k$$

where (e_k) is the sequence of standard unit vectors in c_0. Let

$$\Phi(A) = \sum_{k=1}^{\infty} \left(\int_A r_k d\mu \right) e_k \quad \text{for } A \in \Sigma.$$

Since

$$\int f r_k d\mu \to 0 \quad \text{for all } f \in L_\infty(\mu),$$

it is easy to see (1) that $H \subseteq$ cca(Σ, X) and $\Phi \in$ ca(Σ, X), (2) that $H(A)$ is weakly (indeed, strongly) relatively compact for all $A \in \Sigma$ and $H(\Sigma)$ is bounded, (3) that $x^* \circ H$ is uniformly ca for all $x^* \in X^*$ (indeed, H is uniformly μ-continuous), (4) that

$$\langle x^*, \tilde{\Phi}_n(F) \rangle \to \langle x^*, \tilde{\Phi}(F) \rangle$$

(indeed, $\tilde{\Phi}_n(F) \to \tilde{\Phi}(F)$) for all $F \in L(\Sigma)$ and $x^* \in X^*$, (5) but that Φ does not have relatively compact range in $X = c_0$ (since for any n, there exists $A \in \Sigma$ with $\int_A r_n d\mu = \frac{1}{2}$). H is weakly conditionally compact by Theorem 3.2 of **[22]** but is not, according to Theorem 6.5, weakly relatively compact in cca(Σ, X). Notice also that this example displays the vital role of expectation operators in the characterization in Theorem 3.1 of strong compactness in cca(Σ, X), for with the exception of uniform convergence of expectations, the conditions of (3) of that theorem are met here by H, yet H is not even strongly precompact.

7. Duals of spaces of compact-range measures. The duals of spaces of compact-range measures can be represented provided that the dual of the range space has the Radon-Nikokym property (RNP).

THEOREM 7.1. *If X^* has* RNP *and $\mathscr{B}$ is any Boolean algebra, then there exist natural isometrical isomorphisms*:

(1) cba($\mathscr{B}$, X) $\simeq$ ba($\mathscr{B}$) $\tilde{\otimes}_\epsilon X$, csca($\mathscr{B}$, X) $\simeq$ sca($\mathscr{B}$) $\tilde{\otimes}_\epsilon X$.

(2) cba($\mathscr{B}$, X)* $\simeq$ ba($\mathscr{B}$)* $\tilde{\otimes}_\pi X^*$ $\simeq N_b$(ba($\mathscr{B}$), X^*),

$$\text{csca}(\mathscr{B}, X)^* \simeq \text{sca}(\mathscr{B})^* \tilde{\otimes}_\pi X^* \simeq N_b(\text{sca}(\mathscr{B}), X^*).$$

(3) $\text{cba}(\mathscr{B}, X)^{**} \simeq \mathscr{L}_b(\text{ba}(\mathscr{B})^*, X^{**}),$

$$\text{csca}(\mathscr{B}, X)^{**} \simeq \mathscr{L}_b(\text{sca}(\mathscr{B})^*, X^{**}).$$

Proof. The tensor products above are, in order, the complete injective and the complete projective tensor products while $N(\cdot, \cdot)$ denotes a space of nuclear operators. If μ is a scalar measure, $x \in X$, and $A \in \mathscr{B}$, then the isometries in (1) arise from treating $\mu \otimes x$ as an X-valued measure by setting

$$(\mu \otimes x)(A) = \mu(A)x.$$

In (2), every functional T on compact-range X-valued measures will be seen to have the form

$$T = \sum \lambda_n F_n \otimes x_n^*$$

where (x_n^*) is a null sequence in X^*, $(\lambda_n) \in l_1$, and (F_n) is a null sequence from the dual of the given space of scalar measures. The form of the isometry in (3) will then be a consequence of the form of the representation in (2). First consider the three asserted isometries for the case of ba measures on $\mathscr{B}$. That the indicated natural maps are indeed isometries onto is a consequence of the measure-operator identifications surrounding Theorem 2.3, of Theorem 3.1 of [**11**], and of the fact that $\text{ba}(\mathscr{B})^*$ has the approximation property. To treat the case of sca measures on $\mathscr{B}$, first recall that there is a projection of $\text{ba}(\mathscr{B})$ onto its closed subspace $\text{sca}(\mathscr{B})$ via the Hewitt-Yosida decomposition. It is easy to verify that the adjoint of this projection is τ-continuous on $L(\mathscr{A}) = \text{ba}(\mathscr{B})^*$ where $\mathscr{A}$ is the algebra of clopen subsets of the Stone space of $\mathscr{B}$. The resulting τ-closed subspace of $L(\mathscr{A})$ represents all vector-valued sca measures on $\mathscr{B}$, and the proof for this case proceeds just as above.

The next result follows from (2) of the above theorem.

THEOREM 7.2. *With assumptions as in Theorem* 7.1, *the weak topology and the* $\text{ba}(\mathscr{B})^* \otimes X^*$-wot (*respectively,* $\text{sca}(\mathscr{B})^* \otimes X^*$-wot) *coincide on bounded subsets of* $\text{cba}(\mathscr{B}, X)$ (*respectively,* $\text{csca}(\mathscr{B}, X)$).

An interesting example can also be viewed as a corollary to Theorem 7.1. Let Σ be a σ-algebra of sets, $\mu \in \text{ca}(\Sigma)^+$, and $\text{ca}(\Sigma, \mu, X)$ the space of μ-continuous X-valued measures on Σ. Set $\mathscr{B} = \Sigma/\mu^{-1}(0)$ in Theorem 7.1 and apply classical $L_1 - L_\infty$ duality to derive the next result.

COROLLARY 7.3. *If* X^* *has* RNP *and* $\mu \in \text{ca}(\Sigma)^+$ *for some* σ*-algebra* Σ *of sets, then there are (natural) isometrical isomorphisms:*

(1) $\text{cca}(\Sigma, \mu, X) \simeq L_1(\mu) \tilde{\otimes}_\epsilon X.$

(2) $\operatorname{cca}(\Sigma, \mu, X)^* \simeq L_\infty(\mu) \tilde{\bigotimes}_\pi X^* \simeq N_b(L_1(\mu), X^*)$.

(3) $\operatorname{cca}(\Sigma, \mu, X)^{**} \simeq \mathscr{L}_b(L_\infty(\mu), X^{**})$.

Moreover, a bounded net (Φ_α) *from* $\operatorname{cca}(\Sigma, \mu, X)$ *converges weakly to* $\Phi \in \operatorname{cca}(\Sigma, \mu, X)$ *if and only if*

$$\langle x^*, \Phi_\alpha(A) \rangle \rightarrow \langle x^*, \Phi(A) \rangle$$

for all $A \in \Sigma$ *and all* x^* *from the set* $\operatorname{sexp}(B_{X^*})$ *of strongly exposed points of the unit ball* B_{X^*}.

Proof. The isometries have already been explained. The remaining statement is just the analogue of Corollary 7.2 taking into account that the image of $S(\Sigma)$ in $L_\infty(\mu)$ is norm dense and that the convex circled hull of $\operatorname{sexp}(B_{X^*})$ is norm dense in B_{X^*} when X^* has RNP [**13**, VII.3.3].

Remark 7.4. Since $l_1 = c_0^*$ has RNP, Example 6.8 reveals that the last condition in (2) of Theorem 6.5 cannot be avoided even with the representation of $\operatorname{cca}(\Sigma, X)^*$ given above when X^* has RNP.

Question 7.5. Do the representations of this section hold under the assumption that X^* has weak RNP? At the base of the representations of this section is the fact that a Banach space Z has RNP if and only if all integral operators from Y to Z are nuclear for every Banach space Y (which ultimately depends on the existence of Bochner derivatives of Z-valued measures rather than Pettis derivatives (cf. Chapter I, Theorems 3 and 6 of [**2**]). In the special case of Corollary 7.3, the question thus amounts to asking whether for a given X^* with weak RNP but without RNP (such as the odd duals of the James tree space [**23**, Corollary 11]) at least all integral operators from just $L_1(\mu)$ into X^* are nuclear.

8. Generalizations. The main results of this paper, Theorem 3.1 and its applications in Sections 4 and 5 and Theorems 6.1-6.5, are valid when X is a Fréchet space. Of course, their statements must be modified accordingly. Because Theorem 2.1 with a minor modification to statement (2) holds for any quasi-complete X as do Theorems 2.2 and 2.3 in slightly modified forms, many of the main results hold for any quasi-complete, locally convex X. For example, conditions (1)-(4) of Theorem 3.1 are equivalent in this generality, and Theorem 6.1 remains valid. The strong topology becomes a locally convex topology of semi-variation semi-norms and the set $\operatorname{ext}(B_{X^*})$ must be replaced by the union of sets of extreme points of polars of basic neighborhoods of 0 in X.

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