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HYPERSURFACES WITH SPECIAL QUADRIC REPRESENTATIONS

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Let $x : M^n \to E^m$ be an isometric immersion of an n-dimensional Riemannian manifold into the m-dimensional Euclidean space. Then the map $\widetilde{x} = xx^t$ (where t denotes transpose) is called the quadric representation of M^n. In this paper, we study and classify hypersurfaces in the Euclidean space E^m which satisfy $\triangle\widetilde{x} = B\widetilde{x} + C$, where B and C are two constant matrices, and $\triangle$ is the Laplacian operator of M^n. Some classification results are obtained.

1. INTRODUCTION

Let x be an isometric immersion of a smooth manifold M^n into a space form and f be a vector function on M^n. Assume $\triangle f = Bf + C$ for suitable constant matrices B and C. Then the question arises: what is the geometric meaning involved in this algebraic condition? This question has been studied by many authors when f is the position vector function of M^n, for example, [**1, 4, 6**]. In this paper, we try to answer the same question when f is the quadric representation of M^n.

Let $x : M^n \to E^m$ be an isometric immersion of an n-dimensional Riemannian manifold into the m-dimensional Euclidean space, and $SM(m)$ be the $m \times m$ real symmetric matrices space (this space becomes the standard $1/(2m(m+1))$-dimension Eclidean space when equipped with the metric $g(P,Q) = 1/(2tr(PQ))$ [**2**]). We regard x as a column matrix in E^m and denote by x^t the transpose of x. Let $\widetilde{x} = xx^t$, then we obtain a smooth map $\widetilde{x} : M^n \to SM(m)$. Since the coordinates of $\widetilde{x}$ depend on the coordinates of x in a quadric manner, we call $\widetilde{x}$ the quadric representation of M^n [**3**]. $\widetilde{x}$ is an important map, because it has many interesting relations with the geometric properties of the submanifold. In fact, for the hypersphere centred at the origin embedded in the Euclidean space in the standard way, the quadric representation is just the second standard embedding of the sphere. In [**3**], Dimitric established some general results about the quadric representation, in particular those related to the condition of $\widetilde{x}$ being of finite type. In [**5**], the author also gave some related results. In this paper, we shall give some classification theorems for submanifolds in E^m which

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satisfy the condition $\triangle\widetilde{x} = B\widetilde{x}+C$. We prove that the only hypersurfaces with constant mean curvature satisfying $\triangle\widetilde{x} = B\widetilde{x}+C$ are (a piece of) a hyperplane or a hypersphere centred at the origin, but there is no hypersurface satisfying $\triangle\widetilde{x} = B\widetilde{x}$. Especially, we also give a classification theorem for surfaces in the 3-dimensional Euclidean space.

2. Preliminaries

Let us fix the notations first. Let $x : M^n \to E^{n+1}$ be a hypersurface. We denote by H the mean curvature vector of M^n in E^m. Let $e_1, \cdots, e_n, e_{n+1}$ be a local field of orthonormal frames of E^{n+1}, such that when restricted to M^n, $e_1, \cdots, e_n$ are tangent to M^n, and e_{n+1} is normal to M^n. Then $H = \alpha e_{n+1}$. Let $\langle , \rangle$ and $\overline{\nabla}$ be the Euclidean metric and the connection of E^{n+1}, and denote by ∇, h, D, A and $|A|$ respectively, the connection of M^n, the second fundamental form of M^n in E^m, the normal connection of M^n in E^{n+1}, the Weigarten endomorphism relative to the normal direction e_{n+1}, and the length of A.

In this setting, the indices i, j, k always range from 1 to n. At any point $x \in M^n$, for any $V \in T_x(E^{n+1})$, we denote the tangent part to M^n of V by V_T.

We define a map $*$ from $E^{n+1} \times E^{n+1}$ into $SM(n+1)$ by $V * W = VW^t + WV^t$ for column vectors V and W in E^{n+1}. Then $V * W = W * V$. Let $\widetilde{\nabla}$ denote the Euclidean connection of $SM(n+1)$. Then we have

$$\widetilde{\nabla}_V(W_1 * W_2) = (\overline{\nabla}_V W_1) * W_2 + W_1 * (\overline{\nabla}_V W_2), \tag{2.1}$$

$$g(V_1 * V_2, W_1 * W_2) = \langle V_1, W_1\rangle\langle V_2, W_2\rangle + \langle V_1, W_2\rangle\langle V_2, W_2\rangle, \tag{2.2}$$

and

$$\triangle(V * W) = (\triangle V) * W + V * (\triangle W) - 2\sum_i (\overline{\nabla}_{e_i} V) * (\overline{\nabla}_{e_i} W), \tag{2.3}$$

where V, W, W_1, W_2, V_1 and V_2 are all vectors in E^m, and $\triangle$ is the Laplacian operator of M^n [**3**].

Using (2.3), by a lengthy but direct compution, we see

$$\triangle\widetilde{x} = -n\alpha e_{n+1} * x - \sum_i e_i * e_i, \tag{2.4}$$

and when $\alpha = 0$,

$$\triangle^2\widetilde{x} = 2\,|A|^2\, e_{n+1} * e_{n+1} - 2\sum_i (Ae_i) * (Ae_i). \tag{2.5}$$

Unless mentioned otherwise, in the following we always denote by X, Y and Z the tangent vectors, of M^n.

3. Hypersurfaces satisfying $\Delta\widetilde{x} = B\widetilde{x} + C$

Let $x : M^n \to E^{n+1}$ be a hypersurface. Suppose its quadric representation satisfies the condition $\Delta\widetilde{x} = B\widetilde{x}+C$. Let $Q(x) = B\widetilde{x}+C-\Delta\widetilde{x}$, then $Q(x) = 0$. Differentiating $Q(x) = 0$ along X, an arbitrary tangent vector of M^n, we have

$$0 = \widetilde{\nabla}_X Q(x) = (BX)x^t + (Bx)X^t + n\alpha e_{n+1} * X + 2(AX) * e_{n+1} - n\alpha(AX) * x. \tag{3.1}$$

Now we find the $e_{n+1} * e_{n+1}$ component of (3.1), that is

$$\{\langle BX, e_{n+1}\rangle + 2nX(\alpha)\}\langle x, e_{n+1}\rangle = 0. \tag{3.2}$$

Thus, at any given point of M^n, we know that $\langle x, e_{n+1}\rangle = 0$ or $\langle BX, e_{n+1}\rangle + 2nX(\alpha) = 0$. We shall discuss both cases.

Case 1. $\langle e_{n+1}, x\rangle = 0$. In this case $x = x_T$ and for any tangent vector Y of M^n,

$$0 = Y\langle e_{n+1}, x\rangle = -\langle Ax, Y\rangle, \tag{3.3}$$

and

$$Ax = 0. \tag{3.4}$$

Finding the $Z * Y$ component of (3.1), we have

$$\begin{aligned}\langle BX, Z\rangle\langle x, Y\rangle + \langle BX, Y\rangle\langle x, Z\rangle + \langle Bx, Z\rangle\langle X, Y\rangle + \langle Bx, Y\rangle\langle X, Z\rangle \\ = 2n\alpha\langle AX, Y\rangle\langle x, Z\rangle + 2n\alpha\langle AX, Z\rangle\langle x, Y\rangle.\end{aligned} \tag{3.5}$$

In (3.5), let $Z = Y = e_j$ and sum on j:

$$\langle BX, x_T\rangle + \langle Bx, X\rangle = 2n\alpha\langle AX, x\rangle. \tag{3.6}$$

In (3.5), letting $X = Y = x$ and using (3.4), we have $\langle BX, x\rangle = 0$. Combining it with (3.6) and using (3.3), we obtain

$$\langle Bx, X\rangle = 0. \tag{3.7}$$

Finding the $e_{n+1} * Y$ component of (3.1), we obtain

$$\begin{aligned}\langle BX, Y\rangle\langle x, e_{n+1}\rangle + \langle Bx, e_{n+1}\rangle\langle X, Y\rangle - 2n\alpha\langle AX, Y\rangle\langle e_{n+1}, x\rangle \\ + \langle BX, e_{n+1}\rangle\langle x, Y\rangle + 2nX(\alpha)\langle x, Y\rangle + 2n\alpha\langle X, Y\rangle + 4\langle AX, Y\rangle = 0.\end{aligned} \tag{3.8}$$

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Using the condition $\langle e_{n+1}, x\rangle = 0$, we have

$$\text{(3.9)} \quad \langle BX, e_{n+1}\rangle\langle x, Y\rangle + \langle Bx, e_{n+1}\rangle\langle X, Y\rangle + 2nX(\alpha)\langle x, Y\rangle + 2n\alpha\langle X, Y\rangle + 4\langle AX, Y\rangle = 0.$$

Letting $X = Y = x$ in (3.9), we have

$$\text{(3.10)} \qquad \langle Bx, e_{n+1}\rangle = -n\alpha - nx(\alpha).$$

Letting $X = Y = e_i$ in (3.9) and summing on i, we obtain

$$\text{(3.11)} \qquad (n+1)\langle Bx, e_{n+1}\rangle + 2nx(\alpha) + 2n^2\alpha + 4n\alpha = 0.$$

Combining (3.10) with (3.11), we get

$$\text{(3.12)} \qquad x(\alpha) = \frac{n+3}{n-1}\alpha.$$

From formula (3.9), since Y is arbitrary, we deduce

$$\text{(3.13)} \qquad (\langle BX, e_{n+1}\rangle + 2nX(\alpha))x + (\langle Bx, e_{n+1}\rangle + 2n\alpha)X + 4AX = 0.$$

Using (3.7), (3.13) becomes

$$X\{(\langle Bx, e_{n+1}\rangle + 2n\alpha)x - 4e_{n+1}\} = 0,$$

and then

$$(\langle Bx, e_{n+1}\rangle + 2n\alpha)x - 4e_{n+1} = X_0,$$

where X_0 is a constant vector. We also deduce

$$\text{(3.14)} \qquad (\langle Bx, e_{n+1}\rangle + 2n\alpha)^2\langle x, x\rangle = C_1,$$

where C_1 is a constant. Combining (3.10) and (3.12) with (3.14), we know that $\alpha^2\langle x, x\rangle = C_2$, where C_2 is also a constant. Differentiating this formula along the tangent vector field x, we have that $\alpha\langle x, x\rangle(x(\alpha) + \alpha) = 0$. Using (3.12) we deduce $\alpha = 0$.

CASE 2. $\langle x, e_{n+1}\rangle \neq 0$. Then $\langle BX, e_{n+1}\rangle + 2nX(\alpha) = 0$, and (3.8) becomes

$$\text{(3.15)} \quad \langle Bx, e_{n+1}\rangle\langle X, Y\rangle + 2n\alpha\langle X, Y\rangle + 4\langle AX, Y\rangle - 2n\alpha\langle AX, Y\rangle\langle e_{n+1}, x\rangle + \langle BX, Y\rangle\langle x, e_{n+1}\rangle = 0.$$

From this formula, we get

$$\text{(3.16)} \qquad \langle BX, Y\rangle = \langle BY, X\rangle.$$

On substituting $Y = x_T$ in (3.15), we have

$$\langle Bx, e_{n+1}\rangle\langle X, x\rangle + 2n\alpha\langle X, x\rangle + 4\langle AX, x\rangle \\ - 2n\alpha\langle AX, x\rangle\langle e_{n+1}, x\rangle + \langle BX, x_T\rangle\langle x, e_{n+1}\rangle = 0.$$

Combining this relation with (3.6), we obtain

$$2n\alpha\langle X, x\rangle + \langle Bx, e_{n+1}\rangle\langle X, x\rangle + 4\langle AX, x\rangle - \langle Bx, X\rangle\langle e_{n+1}, x\rangle = 0, \tag{3.17}$$

that is,

$$(\langle Bx, e_{n+1}\rangle + 2n\alpha)x_T + 4Ax_T - \langle e_{n+1}, x\rangle(Bx)_T = 0. \tag{3.18}$$

In (3.5), letting $X = Y = e_i$ and summing on i, we have

$$(n+1)(Bx)_T = 2n\alpha Ax_T + \Big(2n^2\alpha^2 - \sum_i\langle Be_i, e_i\rangle\Big)x_T - (Bx_T)_T. \tag{3.19}$$

From (3.15) we also have

$$0 = (\langle Bx, e_{n+1}\rangle + 2n\alpha)X + (4 - 2n\alpha\langle e_{n+1}, x\rangle)AX + \langle e_{n+1}, x\rangle(BX)_T.$$

But by using (3.16), we can obtain from (3.6)

$$(Bx_T)_T + (Bx)_T = 2n\alpha Ax_T. \tag{3.21}$$

By combining (3.21) with (3.19) and (3.18), we obtain

$$4nAx_T + \Big\{n\langle Bx, e_{n+1}\rangle + 2n^2\alpha + \Big(\sum_i\langle Be_i, e_i\rangle - 2n^2\alpha^2\Big)\langle e_{n+1}, x\rangle\Big\}x_T = 0. \tag{3.22}$$

In (3.15), letting $X = Y = e_i$ and summing on i, we obtain

$$n\langle Bx, e_{n+1}\rangle + 2n^2\alpha + 4n\alpha + \Big(\sum_i\langle Be_i, e_i\rangle - 2n^2\alpha^2\Big)\langle e_{n+1}, x\rangle = 0. \tag{3.23}$$

By combining (3.23) with (3.22) we deduce $Ax_T = \alpha x_T$. Hence, we have proved the following result:

Proposition 3.1. *Let $x : M^n \to E^{n+1}$ be a hypersurface. If its quadric representation satisfies the condition $\Delta\widetilde{x} = B\widetilde{x} + C$, then $\alpha = 0$ or $Ax_T = \alpha x_T$.*

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4. The Main Results

First, let us give some examples satisfying the condition $\Delta\widetilde{x} = B\widetilde{x} + C$.

EXAMPLE 4.1. The hyperplane in E^{n+1} : for the standard hyperplane $E^n = \{x = (x_1, \cdots, x_n, x_{n+1})^t\,;\, x_{n+1} = 0\}$, we can easily obtain

$$\Delta\widetilde{x} = \begin{pmatrix} -2 & & & \\ & \ddots & & \\ & & -2 & \\ & & & 0 \end{pmatrix},$$

and for any hyperplane $M^n = \{x = Q(x_1, \cdots, x_n, x_{n+1})^t + x_0\,;\, x_{n+1} = 0,\ QQ^t = I$, x_0 is a constant vector $\}$, then we have

$$\Delta\widetilde{x} = Q \begin{pmatrix} -2 & & & \\ & \ddots & & \\ & & -2 & \\ & & & 0 \end{pmatrix} Q^t.$$

EXAMPLE 4.2. The hypersphere in E^{n+1} centred at the origin: $S(r) = \{x\,;\, \langle x, x\rangle = r^2\}$. In this case, $\Delta\widetilde{x} = \big(2(n+1)/r^2\big)\widetilde{x} - 2I$.

We remark that a hypersphere not centred at the origin does not satisfy the condition $\Delta\widetilde{x} = B\widetilde{x} + C$; this can be obtained by a direct computation.

THEOREM 4.1. *Let $x : M^n \to E^{n+1}$ be a hypersurface with constant mean curvature. Then its quadric representation satisfies the condition $\Delta\widetilde{x} = B\widetilde{x} + C$ if and only if it is (a piece of) a hyperplane or a hypersphere centred at the origin.*

PROOF: Let $x : M^n \to E^{n+1}$ be a hypersurface satisfying the given condition. Then from Proposition 3.1 we know that $\alpha = 0$ or $Ax_T = \alpha x_T$.

If $\alpha = 0$, let $p(x) = \Delta^2\widetilde{x} - B(\Delta\widetilde{x})$. Obviously, we have $p(x) = 0$. From (2.4) and (2.5) we obtain that $p(x) = 2\,|A|^2\, e_{n+1} * e_{n+1} - 2\sum_i (Ae_i) * (Ae_i) - \sum_i B(e_i * e_i)$ and $g(p, e_{n+1} * e_{n+1}) = 4\,|A|^2 = 0$, then M^n is (a piece of) a totally geodesic hypersurface of E^{n+1}, that is to say, a hyperplane.

If $x_T = 0$, then for any tangent vector field X, we have $\langle X, x\rangle = 0$, that is, $\langle x, x\rangle =$constant, and M^n is (a piece of) a hypersphere centred at the origin.

If $x_T \neq 0$, let $e_1, \cdots, e_n$ be the principal directions of M^n with e_1 in the direction of x_T, let $x_T = ve_1$, and let $\mu_1, \cdots, \mu_n$ be the corresponding principal curvatures with $\mu_1 = \alpha$. Since $X = \overline{\nabla}_X x = \nabla_X x_T + h(X, x_T) - \langle x, e_{n+1}\rangle AX + D_X x_N$, by comparing those parts tangent to M^n, we obtain

$$\nabla_X x_T = X + \langle x, e_{n+1}\rangle AX. \tag{4.1}$$

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By substituting $X = x_T = ve_1$ into this relation, we obtain $\nabla_{e_i} e_i = 0$ and $e_1(v) = 1 + \alpha\langle x, e_{n+1}\rangle$. Putting $X = e_k$, $k \geqslant 2$, into (4.1), we obtain $\omega_1^l(e_k) = 0$ for $l \neq k$ and $v\omega_1^k(e_k) = 1 + \langle x, e_{n+1}\rangle\mu_k$. By using the above relations and the Codazzi equation $(\nabla_{e_i} A)e_k = (\nabla_{e_i} A)e_1$, for $k \geqslant 2$, and comparing the e_k components we have

$$e_1(\mu_k) = (\alpha - \mu_k)\omega_1^k(e_k) = \frac{1}{v}(\alpha - \mu_k)(1 + \langle x, e_{n+1}\rangle\mu_k). \tag{4.2}$$

From (4.2), by addition over k, we obtain

$$ne_1(\alpha) = \frac{1}{v}\left(n\alpha^2 - |A|^2\right)\langle x, e_{n+1}\rangle = 0;$$

this implies $n\alpha^2 = |A|^2$. But $n|A|^2 = n(n\alpha^2) + \sum_{i<j}(\mu_i - \mu_j)^2$ holds for any hypersurface. Thus we get that $\mu_k = \alpha$, and then the submanifold is (a piece of) a hypersphere, and from the remark at the beginning of the section, we know that it must centre at the origin. ∎

THEOREM 4.2. *Let $M^2 \to E^3$ be a surface. Then its quadric representation satisfies $\Delta\tilde{x} = B\tilde{x} + C$ if and only if M^2 is (a piece of) a plane or a sphere centred at the origin.*

PROOF: Following the same argument as in the proof of Theorem 4.1, we can prove that if $\alpha = 0$, M^2 must be (a piece of) a plane, and that if $x_T = 0$, M^2 must be (a piece of) a sphere centred at the origin. But if $x_T \neq 0$, we have $\mu_2 = \alpha = \mu_1$, without the assumption that α is constant. Then the surface is (a piece of) a sphere. Obviously it can centre only at the origin. ∎

THEOREM 4.3. *There is no hypersurface in E^{n+1} whose quadric representation satisfies $\Delta\tilde{x} = B\tilde{x}$.*

PROOF: Obviously a hypersurface in E^{n+1} satisfying $\Delta\tilde{x} = B\tilde{x}$ can not be minimal. Then $\alpha \neq 0$ and

$$B\tilde{x} + \sum_i e_i * e_i + n\alpha e_{n+1} * x = 0. \tag{4.3}$$

From the proof of Proposition 3.1, we know that (3.2) and (3.6) hold.

If $\langle e_{n+1}, x\rangle = 0$, then $x = x_T$, and (3.7) holds. By finding the $e_j * e_j$ component of the above equation and summing on j, we have $\langle Bx, x_T\rangle + 2n = 0$. This is a contradiction to (3.7).

If $\langle e_{n+1}, x\rangle \neq 0$, then $2nX(\alpha) + \langle Bx, e_{n+1}\rangle = 0$. By finding the $e_{n+1} * e_{n+1}$ and $e_{n+1} * X$ components of (4.3) respectively, we obtain

$$\langle Bx, e_{n+1}\rangle + 2n\alpha = 0, \tag{4.4}$$

and

$$\langle Bx, e_{n+1}\rangle\langle X, x\rangle + \langle x, e_{n+1}\rangle\langle Bx, X\rangle + 2n\alpha\langle x, X\rangle = 0. \tag{4.5}$$

Combining the last two relations, we get $\langle x, e_{n+1}\rangle\langle Bx, X\rangle = 0$. Thus $\langle Bx, X\rangle = 0$. This contradicts $\langle Bx, x_T\rangle + 2n = 0$. ∎

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